CS 105: DIC on Discrete Structures

Instructor : S. Akshay

Sept 12, 2024

Lecture 20 – Counting and Combinatorics

Logistics: Midsem

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- Date and Time: Sept 21st, Saturday, at 16:00 hrs, 4pm

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First 17 lectures of course, till & incl Thursday Sept 05.

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First 17 lectures of course, till & incl Thursday Sept 05.

- Basic reasoning: Propositions, proofs, induction,
- Basic structures: sets, functions, (un)countability, relations, posets, chains, anti-chains, lattices.
- Basic counting: counting principles, double counting, permutations & combinations.

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- Only Basic material, NOT advanced.
- Please follow piazza and ask, if necessary.

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- Pattern of exam similar to quiz, some easy/basic, some hard.
- Solve more questions from Kenneth Rosen etc.
- Few more extra/advanced questions may be released (no solutions, but can discuss on piazza).

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  3. Permutations and combinations with repetitions
  4. Estimating $n!$

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  4. Estimating $n!$
- Recurrence relations and generating functions
- Pigeonhole Principle and its extensions

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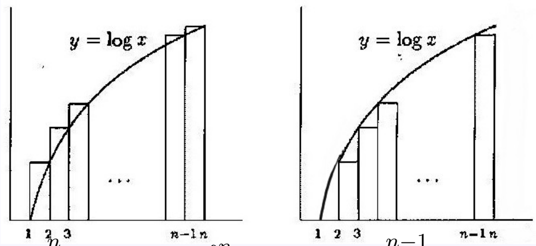
Now, we relate it to natural log function as shown in the figure.



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- r.h.s. $n! \le e^{(n+1)\log(n)-n+1} = n^{n+1}/e^{n-1} = ne(n/e)^n$. □

Next: Recurrence relations and generating functions

Recall: No. of subsets of a set of n elements

How many subsets does a set A of n elements have?

- Induction
- Product principle: two choices for each element, hence $2 \cdot 2 \cdots 2 \cdot 2$ (n-times).
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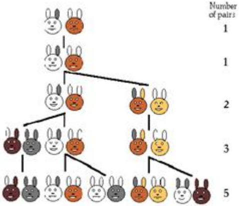
How do we form recurrences and how do we solve them?

Another example of recurrence: The Fibonacci Sequence



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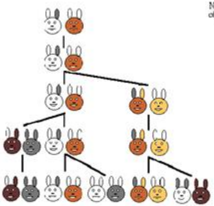

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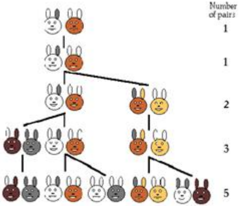

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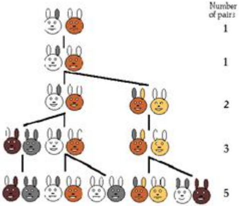

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- Consider $u_n = u_{n-1} + u_{n-2} - u_{n-3}$ where $u_2 = 2, u_1 = u_0 = 1$

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- Once the initial conditions and recurrence relation are given, the entire sequence is determined!

Applications of recurrences

Analysis of running time of algorithms

- How many comparisons is done by a binary search algorithm on a sorted list?

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Macroeconomics

- Time series data
- Mathematical models of economic systems

Exercise!

How many bit strings of length n are there that do not have two consecutive 0's?

- Find a recurrence relation for this
- Give the initial conditions
- How many such bit strings are there of length 7?

Some more examples of recurrences

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An aside: find the Fibonacci sequence!

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

1 6 15 20 15 6 1

1 7 21 35 35 21 7 1

1 8 28 56 70 56 28 8 1

1 9 36 84 126 126 84 36 9 1

1 10 45 120 200 252 200 120 45 10 1

- $F(n) = F(n-1) + F(n-2)$.
- $1, 1, 2, 3, 5, 8, 13, \ldots.$
- Can you observe the sum of which terms in the Pascal's triangle gives rise to the terms of the Fibonacci sequence?