CS 105: DIC on Discrete Structures

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Lecture 21 – Counting and Combinatorics

Summary and what's next

Part 1: Proofs and basic mathematical structures

Part 2: Counting and Combinatorics

- Basics of counting
  - Product principle
  - Sum principle
  - Bijection principle
  - Double counting
- Subsets, partitions, Permutations and combinations
  1. Binomial coefficients and Binomial theorem
  2. Pascal's triangle
  3. Permutations and combinations with repetitions
  4. Estimating $n!$

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  4. Estimating $n!$
- Recurrence relations and generating functions
- Pigeonhole Principle and its extensions

Next: Recurrence relations and generating functions

Definition

- A recurrence relation for a sequence is an equation that expresses its n^{th} term using one or more of the previous terms of the sequence.
- A special case is the linear recurrence relation, which is of the form

$$u_n = a_{k-1}u_{n-1} + \ldots + a_1 u_{n-k+1} + a_0 u_{n-k}$$

where $a_0, \ldots, a_{k-1} \in \mathbb{R}, k \in \mathbb{N}$ are constants.

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- k is called the degree/depth of the sequence.
- The first few (e.g., k elements $u_0, \ldots, u_{k-1}$) are initial conditions and they determine the whole sequence.

Examples of Recurrences

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- $n = 4 : (((a+b)+c)+d), ((a+b)+(c+d)), ((a+(b+c))+d), \ldots$
  In general, let $C(n)$ be the number of ways of doing this.

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- Initial conditions are $C(0) = 0$, $C(1) = 1$ (by convention).
- Note that $C(2) = 1, C(3) = 2, C(4) = 5$.
- This sequence are called Catalan numbers...

How do we solve such recurrences? We start with the Fibonacci sequence.

An aside: find the Fibonacci sequence!

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

1 6 15 20 15 6 1

1 7 21 35 35 21 7 1

1 8 28 56 70 56 28 8 1

1 9 36 84 126 126 84 36 9 1

1 10 45 120 200 252 200 120 45 10 1

- $F(n) = F(n-1) + F(n-2)$.
- $1, 1, 2, 3, 5, 8, 13, ...$
- Can you observe the sum of which terms in the Pascal's triangle gives rise to the terms of the Fibonacci sequence?

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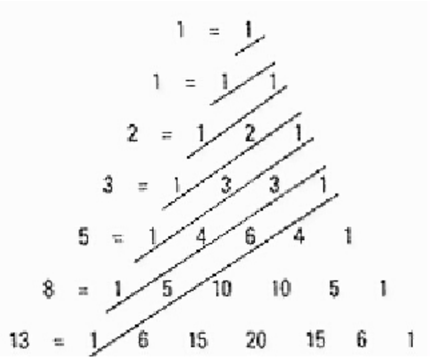

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How to “solve” the Fibonacci recurrence?

By solving, we mean give a closed-form expression for n^{th} term.

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2. i.e., if $F_n = F_{n-1} + F_{n-2}$, $G_n = G_{n-1} + G_{n-2}$ and $H_n = aF_n + bG_n$, then what is a recurrence for H_n?

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6. How do we get a and b?

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Thus, $F_n = \frac{\sqrt{5}+1}{2\sqrt{5}}(\frac{1+\sqrt{5}}{2})^n + \frac{\sqrt{5}-1}{2\sqrt{5}}(\frac{1-\sqrt{5}}{2})^n$.

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- Recall the recurrence for Catalan Numbers:
  $C(n) = \sum_{i=1}^{n-1} C(i)C(n-i)$ for $n > 1$, $C(0) = 0$, $C(1) = 1$.
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This method does not work if we have repeated roots (this can be fixed!) and non-linear recurrences.