CS 105: DIC on Discrete Structures

Instructor : S. Akshay

Sept 24, 2024
Lecture 22 – Counting and Combinatorics
Solving Recurrence relations via generating functions

Last few weeks

Basic counting techniques and applications

1. Sum and product, bijection, double counting principles
2. Binomial coefficients and binomial theorem, Pascal's triangle
3. Permutations and combinations with/without repetitions
4. Counting subsets, relations, Handshake lemma
5. Stirling's approximation: Estimating $n!$
6. Recurrence relations and one method to solve them.

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Today

Solving recurrence relations via generating functions.

Solving general linear recurrence sequences

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- Recall the recurrence for Catalan Numbers:
  $$C(n) = \sum_{i=1}^{n-1} C(i)C(n-i) \text{ for } n > 1$$
  $C(0) = 0, C(1) = 1$ (by convention) & $C(2) = 1, C(3) = 2...$
  No. of ways to bracket a sum of n terms s.t. it can be computed by adding two numbers at a time?

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Reading assignment

Read examples/generalizations from Sections 6.1 and 6.2 from Rosen's book (6th Edition).

Recap: Solving recurrence relations

By solving, we mean give a closed-form expression for n^{th} term.

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1. $\alpha^n = \alpha^{n-1} + \alpha^{n-2}$ implies $\alpha^{n-2}(\alpha^2 - \alpha - 1) = 0$.
2. So if $\alpha^2 - \alpha - 1 = 0$, the recurrence holds for all n.
3. Solving, $\alpha = \frac{1+\sqrt{5}}{2}, \beta = \frac{1-\sqrt{5}}{2}$
4. Thus, general solution is $F_n = a(\frac{1+\sqrt{5}}{2})^n + b(\frac{1-\sqrt{5}}{2})^n$.
5. Use F_0 and F_1 – initial conditions: $a = \frac{\sqrt{5}+1}{2\sqrt{5}}$, $b = \frac{\sqrt{5}-1}{2\sqrt{5}}$

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We next consider a method of much wider applicability...

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thus, as before

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Properties of generating functions

Definition

The (ordinary) generating function for a sequence $a_0, a_1, \ldots \in \mathbb{R}$ is the infinite series $\phi(x) = \sum_{k=0}^{\infty} a_k x^k$.

- Let $f(x) = \sum_{k=0}^{\infty} a_k x^k$, $g(x) = \sum_{k=0}^{\infty} b_k x^k$. Then
  1. If $f(x) = g(x)$, then $a_k = b_k$ for all k.
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  3. $f(x)g(x) = \sum_{k=0}^{\infty} (\sum_{j=0}^{k} a_j b_{k-j}) x^k$,
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The extended binomial theorem

Let $u \in \mathbb{R}$, $(1+x)^u = \sum_{k=0}^{\infty} \binom{u}{k} x^k$.
If you don't like this, take $x \in \mathbb{R}, |x| < 1$.

Simple examples using generating functions

Standard identities:

- $\frac{1}{1-ax} = \sum_{k=0}^{\infty} a^k x^k$
- $\frac{1}{1-x^r} = \sum_{k=0}^{\infty} x^{rk}$
- $e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$

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Class work/Pop Quiz!:

1. Solve the recurrence $a_k = 4a_{k-1}$ with $a_0 = 3$.

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1. Solve the recurrence $a_k = 4a_{k-1}$ with $a_0 = 3$.
2. **A coding problem:** A cryptographer builds a system where a string of decimals is codeword if it contains an even number of 0s. E.g., 1023038 is valid but not 10244.

   Let a_n be the number of n-digit codewords. Empty string is also valid.

   2.1 Find a recurrence relation for a_n. What is a_0, a_1?
   2.2 Solve the recurrence using generating functions.