CS 105: DIC on Discrete Structures

Instructor : S. Akshay

Sept 26, 2024

Lecture 23 – Counting and Combinatorics

Some Applications of Generating functions, Principle of Inclusion-Exclusion

Last few weeks

Basic counting techniques and applications

1. Sum and product, bijection, double counting principles
2. Binomial coefficients and binomial theorem, Pascal's triangle
3. Permutations and combinations with/without repetitions
4. Counting subsets, relations, Handshake lemma
5. Stirling's approximation: Estimating $n!$
6. Recurrence relations and one method to solve them.
7. Solving recurrence relations via generating functions.

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Reading assignment

Read examples/generalizations from Sections 6.1 and 6.2 from Rosen's book (7th Indian Edition). In International 7th version its Sec 8.2 and 8.4?

Properties of generating functions

Definition

The (ordinary) generating function for a sequence $a_0, a_1, \ldots \in \mathbb{R}$ is the infinite series $\phi(x) = \sum_{k=0}^{\infty} a_k x^k$.

- Let $f(x) = \sum_{k=0}^{\infty} a_k x^k$, $g(x) = \sum_{k=0}^{\infty} b_k x^k$. Then
  1. If $f(x) = g(x)$, then $a_k = b_k$ for all k.
  2. $f(x) + g(x) = \sum_{k=0}^{\infty} (a_k + b_k) x^k$,
  3. $f(x)g(x) = \sum_{k=0}^{\infty} (\sum_{j=0}^{k} a_j b_{k-j}) x^k$,
  4. $\frac{d}{dx}(\sum_{k=0}^{\infty} a_k x^k) = \sum_{k=1}^{\infty} (k a_k) x^{k-1}$

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- Let $u \in \mathbb{R}$, $k \in \mathbb{Z}^{\geq 0}$, Then extended binomial coefficient $\binom{u}{k}$ is defined as $\binom{u}{k} = \frac{u(u-1)\ldots(u-k+1)}{k!}$ if $k > 0$ and $= 1$ if $k = 0$.
- What if $u = -n$ for $n \in \mathbb{N}$?

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If you don't like this, take $x \in \mathbb{R}, |x| < 1$.

Simple examples using generating functions

Standard identities:

- $\frac{1}{1-ax} = \sum_{k=0}^{\infty} a^k x^k$
- $\frac{1}{1-x^r} = \sum_{k=0}^{\infty} x^{rk}$
- $e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$

Forming and solving recurrences

- **A coding problem:** A cryptographer builds a system where a string of decimals is codeword if it contains an even number of 0s. E.g., 1023038 is valid but not 10244.

  Let a_n be the number of n-digit codewords. Empty string is also valid.

  1. Find a recurrence relation for a_n. What is a_0, a_1?
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- Now can you solve it?

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- (H.W) Write a recurrence for the number of derrangements. That is, no. of ways to arrange n letters into n addressed envelopes such that no letter goes to the correct envelope.
- (H.W) How many ways can a convex n-sided polygon be cut into triangles by adding non-intersecting diagonals (i.e., connecting vertices with non-crossing lines)? Write a recurrence and solve it!

Solving Catalan numbers using generating functions

Catalan Numbers

$$C(n) = \sum_{i=1}^{n-1} C(i)C(n-i) \text{ for } n > 1,\ C(0) = 0, C(1) = 1.$$

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- Now consider $\phi(x)^2$.
- $\phi(x)^2 = (\sum_{k=1}^{\infty} C(k)x^k)(\sum_{k=1}^{\infty} C(k)x^k)$
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  $= (\sum_{k=2}^{\infty} C(k)x^k) = \phi(x) - x$
- Solving for $\phi(x)$ we get, $\phi(x) = \frac{1}{2}(1 \pm (1-4x)^{1/2})$
- But since $\phi(0) = 0$, we have
  $\phi(x) = \frac{1}{2}(1 - (1-4x)^{1/2}) = \frac{1}{2} + (-\frac{1}{2}(1-4x)^{1/2})$.

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Recall: Extended binomial theorem

Let $\alpha \in \mathbb{R}, (1+x)^\alpha = \sum_{n=0}^{\infty} \binom{\alpha}{n} x^n$, where $\binom{\alpha}{n} = \frac{\alpha(\alpha-1)\dots(\alpha-n+1)}{n!}$.

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- $C(k) = \frac{(-1)^k(-4)^k}{2^{k+1}k!} \cdot 1 \cdot 3 \cdots (2k-3)$
- $C(k) = \frac{1 \cdot 4^k}{2^{k+1} \cdot k!} \cdot \frac{1.2....(2k-3)(2k-2)}{2^{k-1}(k-1)!} = \frac{(2k-2)!}{k!(k-1)!}$.

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- $C(k) = \frac{(-1)^k(-4)^k}{2^{k+1}k!} \cdot 1 \cdot 3 \cdots (2k-3)$
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Thus, the n^{th} Catalan number is given by

$C(n) = \frac{(2n-2)!}{n!(n-1)!} = \frac{1}{n}\binom{2n-2}{n-1}$

Principle of Inclusion-Exclusion (PIE)

A simple example:

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Theorem: Principle of Inclusion-Exclusion (PIE)

Let $A_1, A_2, \ldots, A_n$ be finite sets. Then,

$$|A_1 \cup \ldots \cup A_n| = \sum_{1 \leq i \leq n} |A_i| - \sum_{1 \leq i < j \leq n} |A_i \cap A_j|$$
$$+ \sum_{1 \leq i < j < k \leq n} |A_i \cap A_j \cap A_k| - \ldots + (-1)^{n+1} |A_1 \cap \ldots \cap A_n|$$

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- # surjections = total #functions - those that miss some element in range.
- Let $A_i = \{f : [n] \to [m] \mid i \notin Range(f)\}$
- Then, # surjections $= m^n - | \cup_{i \in [m]} A_i|$.