CS 105: DIC on Discrete Structures

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Lecture 25 – Counting and Combinatorics
Pigeon-Hole Principle (PHP) and its extensions

Recap: Topics in Combinatorics

Counting techniques and applications

1. Basic counting techniques, double counting
2. Binomial theorem, permutations and combinations, Estimating $n!$
3. Recurrence relations and generating functions
4. Principle of Inclusion-Exclusion (PIE) and its applications.
   - Hand-shake Lemma
   - Counting the number of surjections on $[n]$.

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4. Principle of Inclusion-Exclusion (PIE) and its applications.
   - Hand-shake Lemma
   - Counting the number of surjections on $[n]$.
5. Pigeon-Hole Principle (PHP) and its applications.

Another application of PHP

Theorem

Every sequence of $n^2 + 1$ distinct real numbers contains an $n + 1$-length increasing or decreasing subsequence.

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2. For each $k \in \{1 \ldots n^2 + 1\}$, let (i_k, d_k) denote a pair:

$$i_k = \text{ length of longest increasing subsequence starting from } a_k$$
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Every sequence of n^2+1 distinct real numbers contains an $n+1$-length increasing or decreasing subsequence.

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   - Case 1: $a_\ell < a_m$. Then $i_m \geq i_\ell + 1$, a contradiction.

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6. All a_i's are distinct so this completes the proof. □

Different variants of PHP

Simplest formulation (Variant 0)

Let $k \in \mathbb{N}$. If $k + 1$ (or more) objects are to be placed in k boxes, then at least one box will have 2 (or more) objects.

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PHP (Variant 1)

If N objects are placed into k boxes then there is at least one box with at least $\lceil N/k \rceil$ objects.

PHP (Variant 2)

If there are $n \geq 1 + r(\ell - 1)$ objects colored with r different colors, then there exist ℓ objects all with the same color.

Applications of PHP

1. Does there exist an injective function from a set of $k + 1$ elements to a set with k elements? Why or why not?

2. How many cards must be selected from a pack of 52 cards so that at least three cards of the same suit are chosen?

3. Prove or disprove

   3.1 For every $n \in \mathbb{Z}^+$, there exists a multiple of n whose decimal expansion only has $0's$ and $1's$.

   3.2 Every sequence of $n^2 + 1$ distinct real numbers contains a subsequence of length $n + 1$ which is either increasing or decreasing.

   3.3 (H.W.) If there are $n \geq 1 + r(\ell - 1)$ objects which are colored with r different colors, then there exist ℓ objects all with the same color.

This lecture

Pigeon-Hole Principle (PHP) and its extensions

Let's play a game

The coloring game

- There are six points on board and two colored chalks.

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- Can you ever have a draw?

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Lemma

Any 2-coloring of edges of a graph on 6 nodes has a monochromatic triangle.

- 2-coloring of edges: coloring all edges of the graph using atmost 2 colors.
- monochromatic (triangle): all edges have the same color.

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Proof:

- Let $1, \ldots, 6$ be the points, and red/blue the colors.
- Consider the edges $16, 26, 36, 46, 56$.

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- Let $1, \ldots, 6$ be the points, and red/blue the colors.
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- By PHP at least 3 of them must be same color, say $16, 26, 36$ are red.

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  - Either one of $12, 23, 31$ is red (then we have a red triangle).

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Optimality: 6 is the smallest such number

For any graph on 5 or less nodes the above lemma does not hold.

Another coloring problem...

Theorem

Any 2-coloring (say red and blue) of a graph on 10 nodes has either a red triangle or a blue complete graph on 4 nodes.

- complete: all pairs of edges are present.
- How do you prove this? Any ideas?
- How is this different from the previous problem?

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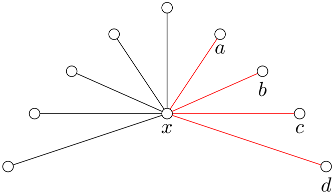
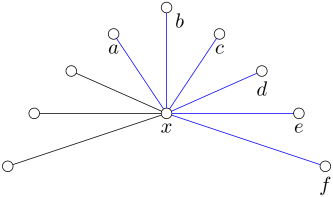
Proof:





- Consider all edges from some node x.
- Either ≥ 4 edges have red color or ≥ 6 have blue (why?).

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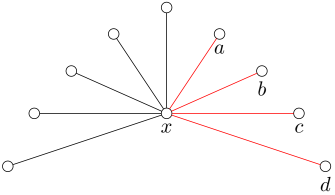

- Case 1: ≥ 4 red edges
  - Either one of edges between a, b, c, d is red or all are blue. So, we are done.

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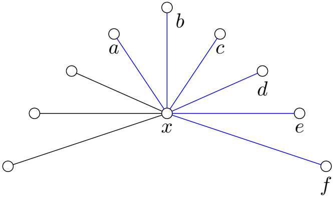

- Case 2: ≥ 6 blue edges
  - But this means that there are 6 nodes $a, \ldots f$.

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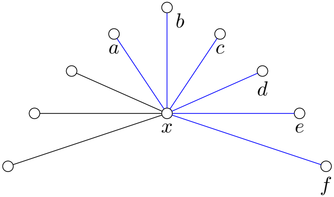

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  - Any 2-coloring on 6 vertices has a red or blue triangle.
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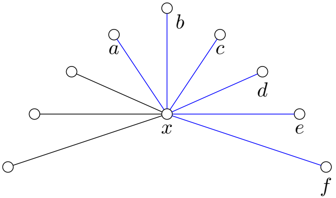

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  - But this means that there are 6 nodes $a, \ldots f$.
  - Any 2-coloring on 6 vertices has a red or blue triangle.
  - Thus we are done again.
- And this completes the proof. □

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Thus, we have showed...

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- But, is this optimal?
- That is, does this fail for a graph on 9 nodes?
- Can you find 2-coloring on a graph of 9 nodes such that the statement above does NOT hold?