CS 105: DIC on Discrete Structures

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Oct 03, 2024

Lecture 26 – Counting and Combinatorics

Searching for order in chaos!

A second coloring problem...

Theorem

Any 2-coloring (say red and blue) of a graph on 10 nodes has either a red triangle or a blue complete graph on 4 nodes.

- complete: all pairs of edges are present.
- How do you prove this? Any ideas?
- How is this different from the previous problem?

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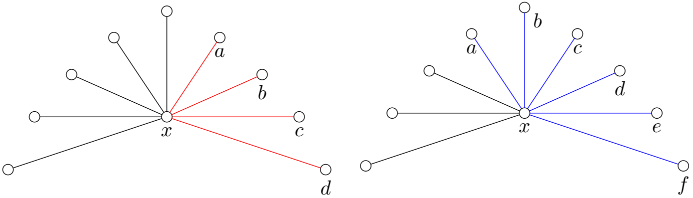
Proof:



- Consider all edges from some node x.
- Either ≥ 4 edges have red color or < 4 edges have red color, i.e., ≥ 6 have blue.

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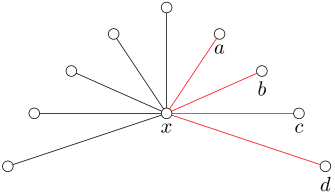

- Case 1: ≥ 4 red edges
  - Either one of edges between a, b, c, d is red or all are blue. So, we are done.

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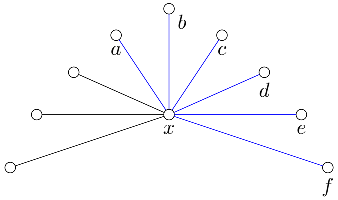

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  - But this means that there are 6 nodes $a, \ldots f$.

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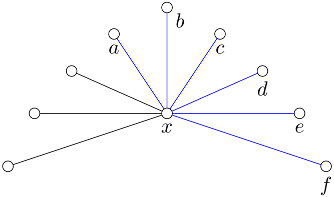

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  - Any 2-coloring on 6 vertices has a red or blue triangle.
  - Thus we are done again.

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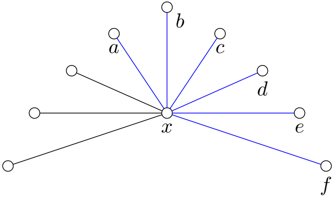

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  - But this means that there are 6 nodes $a, \ldots f$.
  - Any 2-coloring on 6 vertices has a red or blue triangle.
  - Thus we are done again.
- And this completes the proof. $\square$

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Thus, we have showed...

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- But, is this optimal?
- That is, does this fail for a graph on 9 nodes?
- Can you find 2-coloring on a graph of 9 nodes such that the statement above does NOT hold?

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Answer: No! In fact, it does hold on 9 nodes!

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- The only new case, where the previous proof does not work, is if all nodes have 3 red edges and 5 blue edges. (Why?)

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- But is this case possible?

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- But is this case possible?
- Recall the Handshake lemma!
  - In any graph, the number of nodes having odd degree is even.

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Any 2-coloring (say red and blue) of a graph on 9 nodes has either a red triangle or a blue complete graph on 4 nodes.

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- But is this case possible?
- Recall the Handshake lemma!
  - In any graph, the number of nodes having odd degree is even.
- Thus, this case is impossible and we are done. □

Edge coloring problems

Summary of results till now

1. Any 2-coloring of a graph on 6 nodes has either a red triangle or a blue triangle.
   - 6 is the optimal such number.

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1. Any 2-coloring of a graph on 6 nodes has either a red triangle or a blue triangle.
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3. Any 2-coloring of a graph on 9 nodes has either a red triangle or a blue complete graph on 4 nodes.
   - Is 9 the optimal such number?
   - (H.W?) Prove that it is!

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Summary of results till now

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3. Any 2-coloring of a graph on 9 nodes has either a red triangle or a blue complete graph on 4 nodes.
   - Is 9 the optimal such number?
   - (H.W?) Prove that it is!

- (H.W) Prove that any 2-coloring of a graph on 18 nodes has a monochromatic complete graph on 4 nodes.
  (hint: you may use any of the above results)

Can we generalize the above?

In general,

How many nodes should a (complete) graph have so that any 2 coloring of its edges has

- either, a k-sized complete graph with all red edges
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- We have seen that $R(3, 3) = 6$.
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What about $R(k, \ell)$ in general?

Ramsey's theorem

Figure: Frank Plumpton Ramsey (1903-1930)

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Ramsey's theorem (simplified version)

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For any $k, \ell \in \mathbb{N}$, there exists $R(k, \ell) \in \mathbb{N}$ such that any 2-coloring of a (complete) graph on $R(k, \ell)$ nodes has

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Moreover, we have

$$R(k, \ell) \leq \binom{k + \ell - 2}{k - 1}$$

Ramsey theory: A search for order in disorder!

Every structure no matter how disordered must contain some regular sub-part!

E.g., any 2-coloring on a complete graph of 10 nodes contains either a complete graph of 3 nodes of one color or a complete graph of 4 nodes of the other color.

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- Suppose in a group of people any two are friends or enemies.
- In any set of 10 people there must be either 3 mutual friends or 4 mutual enemies.

Proof of Ramsey's theorem

- What is $R(n, 2) = R(2, n)$?

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Proof:

- By strong induction on $k + \ell$.
- Base case: $R(2, 2) = 2$.
- Suppose it is true for all k, ℓ such that $k + \ell < N$. We will show that $R(k, \ell)$ is finite by showing

  $$R(k, \ell) \leq R(k - 1, \ell) + R(k, \ell - 1)$$

  Indeed, we can use induction to conclude that $R(k, \ell)$ is finite, since by induction hypothesis $R(k - 1, \ell)$ and $R(k, \ell - 1)$ exist (i.e., are finite) as $k + \ell - 1 < N$.

Proof of Ramsey's theorem contd.

Claim: $R(k, \ell) \leq R(k-1, \ell) + R(k, \ell - 1)$

Proof of Ramsey's theorem contd.

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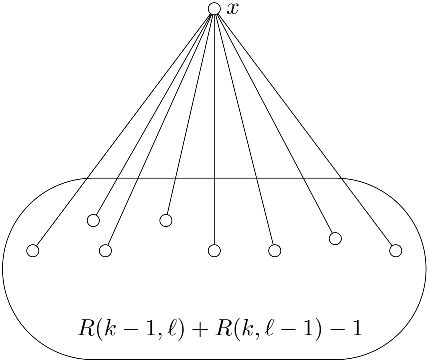
- i.e, we just need to show:
  Any 2-colored complete graph of $R(k-1, \ell) + R(k, \ell - 1)$ nodes, has a complete red graph of k nodes or a complete blue graph of ℓ nodes.

Proof of Ramsey's theorem contd.

Consider complete graph with $R(k-1,\ell)+R(k,\ell-1)$ nodes.



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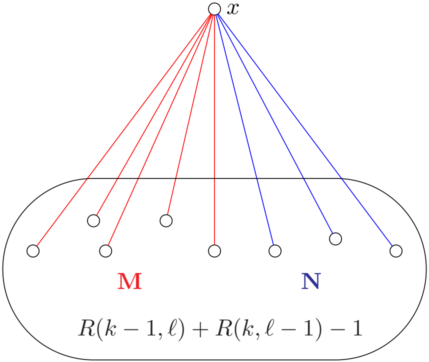
WTS It has k-node complete red or ℓ node complete blue graph



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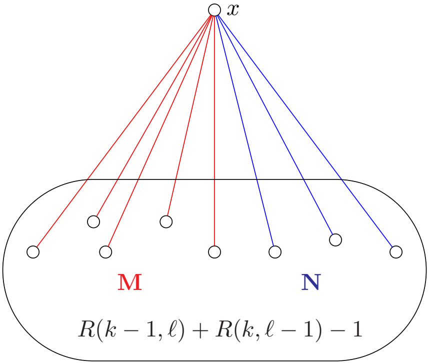

- Clearly $M+N=R(k-1,\ell)+R(k,\ell-1)-1$.

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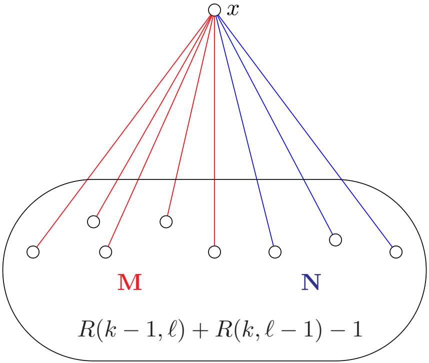

- Clearly $M+N=R(k-1,\ell)+R(k,\ell-1)-1$.
- Either $M \geq R(k-1,\ell)$ or $N \geq R(k,\ell-1)$ (PHP!)

Proof of Ramsey's theorem contd.

Consider complete graph with $R(k-1,\ell)+R(k,\ell-1)$ nodes.
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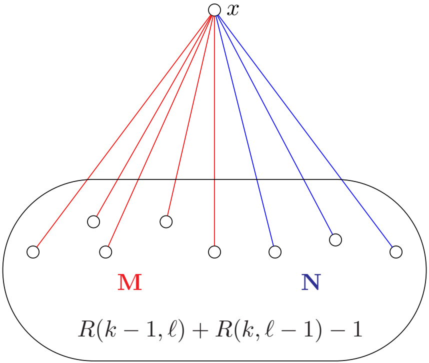

- Case 1: $M \geq R(k-1,\ell)$.

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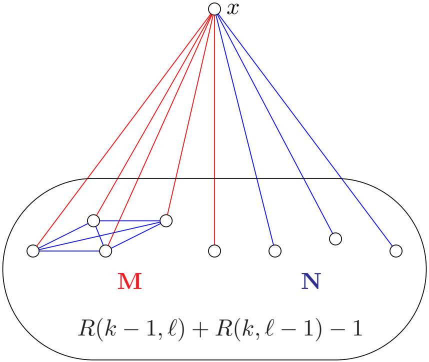

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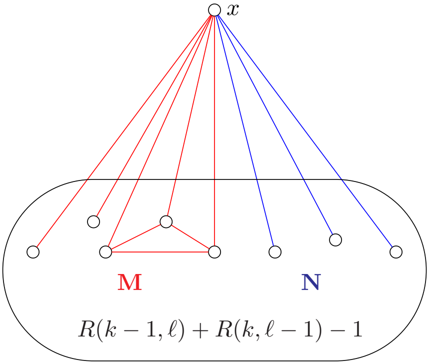

- Case 1: $M \geq R(k-1,\ell)$. Either complete blue graph on ℓ nodes or complete red graph on $k-1$ nodes $+\ x$

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Consider complete graph with $R(k-1,\ell) + R(k,\ell-1)$ nodes.
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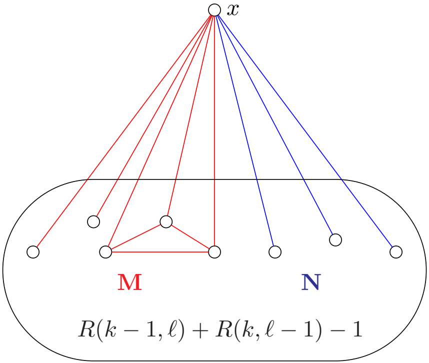

- Case 1: $M \geq R(k-1,\ell)$. ✓
- Case 2: $N \geq R(k,\ell-1)$ leads to same argument.(Do it!) ✓

Proof of Ramsey's theorem contd.

Consider complete graph with $R(k-1,\ell)+R(k,\ell-1)$ nodes.
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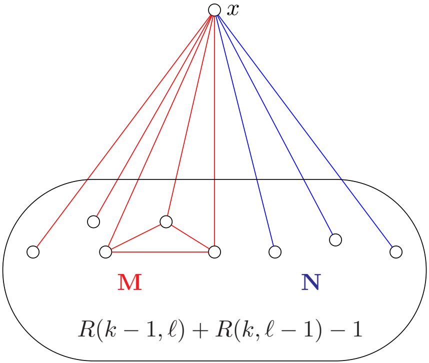

Thus in all cases, we have $R(k,\ell) \leq R(k-1,\ell)+R(k,\ell-1)$. $\square$

Proof of Ramsey's theorem

Ramsey's theorem (simplified version)

For all $k, \ell \geq 2$, $R(k, \ell)$ exists, i.e., it is finite. Further,

$$R(k, \ell) \leq \binom{k + \ell - 2}{k - 1}$$

Proof:

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Proof: Now, this should be trivial!

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- By induction on $k + \ell$ as before.

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Proof:

- By induction on $k + \ell$ as before.
- Base case for $k = \ell = 2$ is done.
- By what we just showed and induction hypothesis we have:

$$\begin{aligned} R(k,\ell) &\leq R(k-1,\ell) + R(k,\ell-1) \\ &\leq \binom{k+\ell-3}{k-2} + \binom{k+\ell-3}{k-1} = \binom{k+\ell-2}{k-1} \end{aligned}$$

$\square$

Ramsey theory

Some interesting facts

- The general Ramsey theorem extends this to any finite number of colors (not just 2).
- Several applications, vast research area!
- Exact values are known only for 6 or so entries: $R(3,3) = 6$, $R(3,4) = 9$, $R(4,4) = 18$,.... $R(3,8) = 28$ or 29...
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So how hard is it? Paul Erdös is supposed to have said:

> *Suppose an evil alien would tell mankind "Either you tell me the value of $R(5,5)$ or I will exterminate the human race." ... It would be best to try to compute it, both by mathematics and with a computer. If he would ask for the value of $R(6,6)$, the best thing would be to destroy him before he destroys us, because we couldn't.*