CS 105: DIC on Discrete Structures

Graph theory

Basic terminology, Eulerian walks

Lecture 27
Oct 07 2024

Topic 3: Graph theory

Topics covered in the last three lectures

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- A glimpse of Ramsey theory

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Next topic

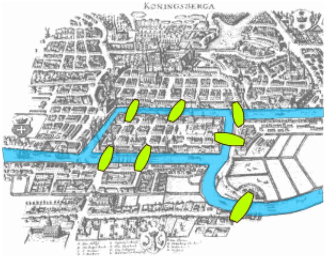
Graphs and their properties!

Topic 3: Graph theory

Textbook Reference

- Introduction to Graph Theory, 2^{nd} Ed., by Douglas West.
- Low cost Indian edition available, published by PHI Learning Private Ltd.

Königsberg Bridge problem



- In 18th century Prussia, the city on river Pregel...
- Find a walk from home, crossing every bridge exactly once and returning home.

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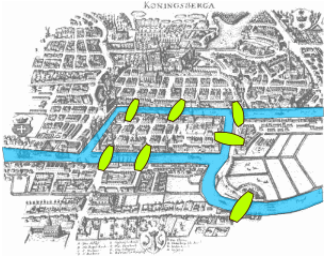

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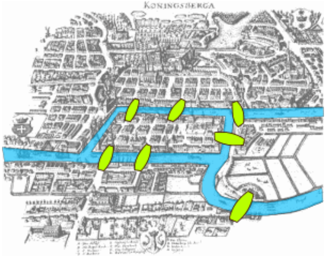

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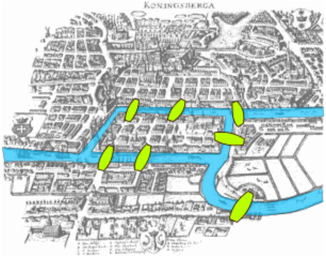

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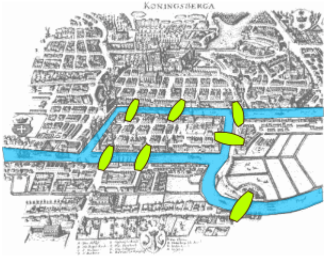

- In 18th century Prussia, the city on river Pregel...
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- "*This question is so banal, but seemed to me worthy of attention in that [neither] geometry, nor algebra, nor even the art of counting was sufficient to solve it.*"
- Still, he wrote a paper showing that this is impossible!
- Thus, he "gave birth" to the area of graph theory.

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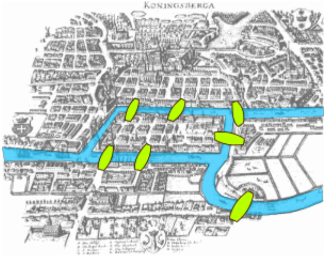

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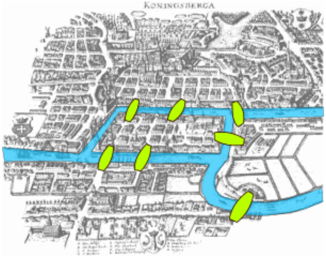

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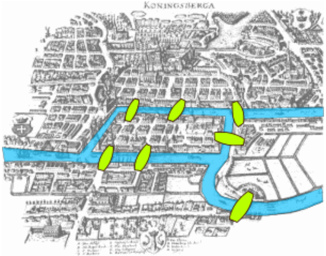

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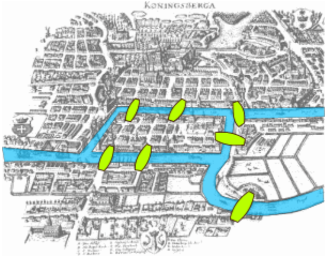

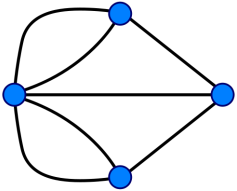

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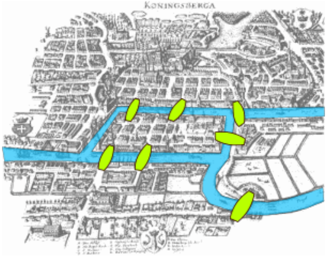

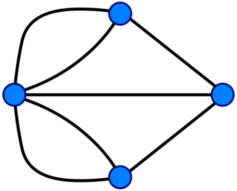

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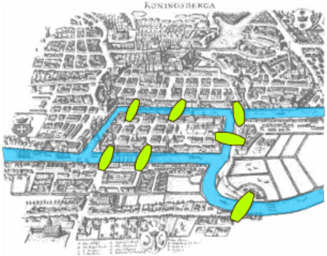

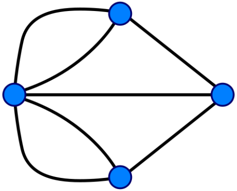

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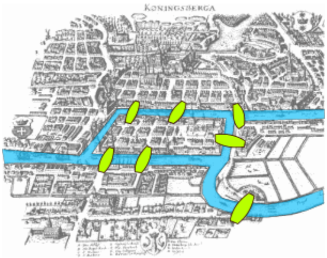

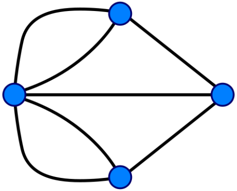

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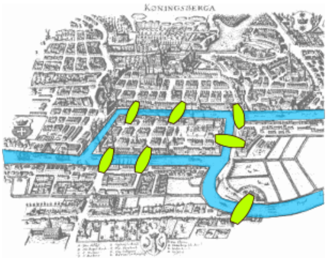

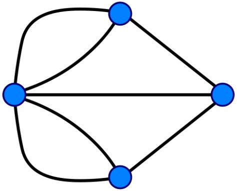

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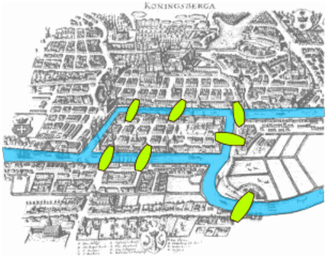

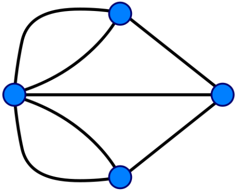

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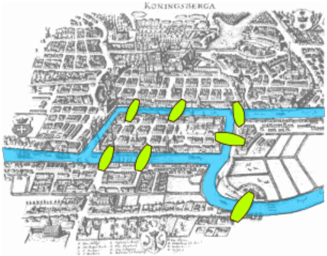

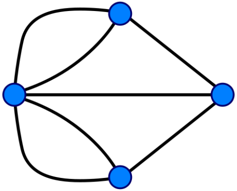

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What are graphs

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Definition

A simple graph G is a pair (V, E), where

- V is a set of vertices/nodes, and,
- E is a set of edges between unordered pairs of vertices (i.e., end-points): $e = vu$ means e is edge between v, u ($u \neq v$)

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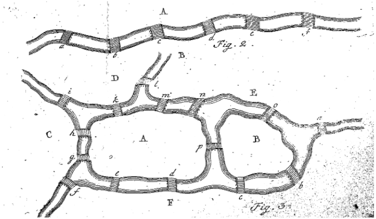
General Definition

A graph G is a triple V, E, R where

- V is a set of vertices,
- E is a set of edges and
- $R \subseteq E \times V \times V$ is a relation that associates each edge with two vertices called its end-points.

We will consider only finite graphs (i.e., $|V|, |E|$ are finite) and often deal with simple graphs.

Basic terminology



Ex. Draw the graphs!

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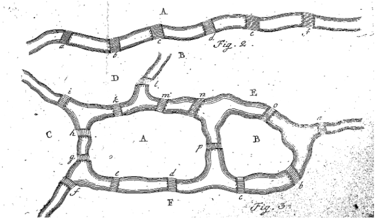

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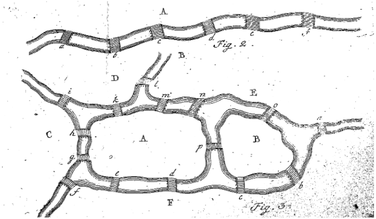

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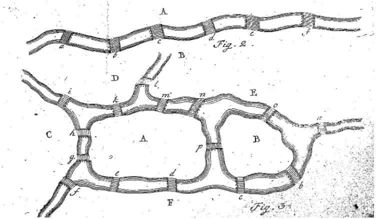

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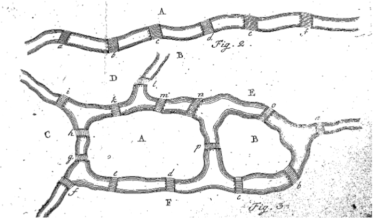

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Graph is connected if there is a walk between any two vertices.

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- Any two edges are in the same walk implies graph is connected (unless it has isolated vertices).