CS 105: DIC on Discrete Structures

Graph theory

Basic terminology, Eulerian walks

Lecture 28

Oct 08 2024

Topic 3: Graph theory

Last topic of this course

Graphs and their properties!

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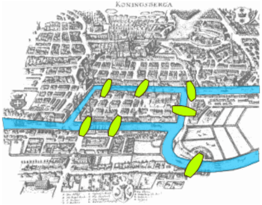
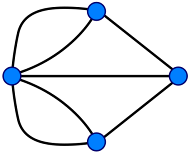
Graphs and their properties!

Textbook Reference

- Introduction to Graph Theory, 2^{nd} Ed., by Douglas West.
- Low cost Indian edition available, published by PHI Learning Private Ltd.

Defining graphs





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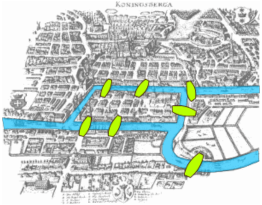

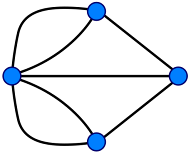

Definition

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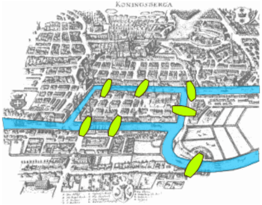

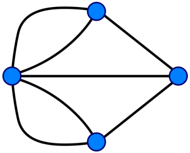

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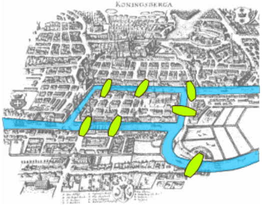

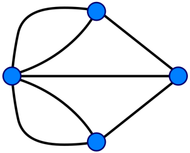

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We will consider only finite graphs (i.e., $|V|, |E|$ are finite) and often simple graphs. **Also, we assume** $|V| > 0$.

Basic terminology

- The degree $d(v)$ of a vertex v (in an undirected loopless graph) is the number of edges incident to it, i.e., $|\{vw \in E \mid w \in V\}|$. A vertex of degree 0 is called an isolated vertex.

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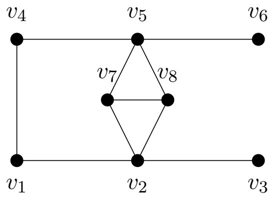
- **The length of a walk is the number of edges in it.**

Basic terminology: Early Morning Quiz Part 1



1. Give examples for each of the following in the above graph:
   1.1 All vertices of degree 3
   1.2 A walk of length 5
   1.3 A closed walk of length 6

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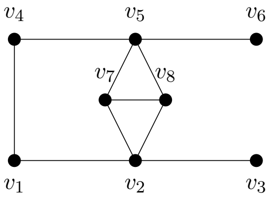

1. Give examples for each of the following in the above graph:
   1.1 All vertices of degree 3
   1.2 A walk of length 5
   1.3 A closed walk of length 6
2. True or False: (a) If a graph is not connected, it must have an isolated vertex.
   (b) A graph is connected iff some vertex has an edge to every other vertex.

Early morning Quiz Part 2

More definitions

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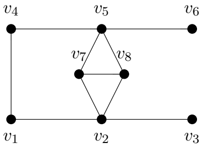

3. Give examples for each of the following in the above graph:
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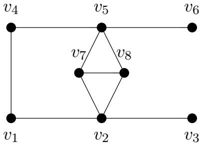

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A final exercise!

5. Prove or disprove

If every vertex of a graph G has degree at least 2, then G contains a cycle.

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A path P is said to be maximal in G if it is not contained in any longer path. In a finite graph maximal paths exist.

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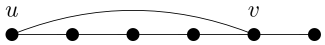

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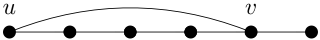

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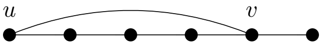

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- Thus we have the cycle $u \dots vu$. $\square$

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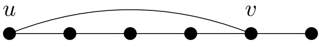

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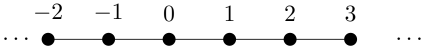
No! Consider $V = \mathbb{Z}$, $E = \{ij : |i - j| = 1\}$.



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- Any two edges are in the same walk implies graph is connected (unless it has isolated vertices).

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- Base case: $m = 1$.

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    - Traverse along cycle C in G and when some G_i is entered for first time, detour along an Eulerian walk of G_i.
    - This walk ends at vertex where we started detour.
    - When we complete traversal of C in this way, we have completed an Eulerian walk on G. $\square$