CS 105: DIC on Discrete Structures

Graph theory

Basic terminology, Applications of Eulerian graphs,
Bipartite graphs

Lecture 29
Oct 10 2024

Two points to note in last class's proof

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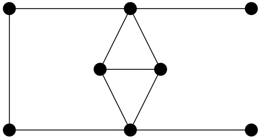
Proof: (H.W) or read from Douglas West's book!

Yet Another Pop Quiz (and Puzzle!)



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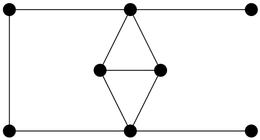

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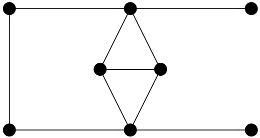

Question

1. Can the above graph be drawn without lifting your pen from paper? No segment should be drawn twice.
2. How many times must you lift your pen to draw the graph?
3. For any connected graph G, what is the number of times we need to lift our pen to draw it?

Application of Eulerian graphs

Another application of Eulerian graphs

If we want to draw a given connected graph G on paper, how many times must we stop and move the pen? No segment should be drawn twice.

- This is the number of walks with no repeated edges into which it can be decomposed.
- Walks with no repeated edges are called trails.

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- Thus, we have shown that at least $\max\{k, 1\}$ trails are required.

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- If $k = 0$, one trail suffices (i.e., an Eulerian walk by previous Thm)
- If $k > 0$ we need to prove that k trails suffice.
  - Pair up odd vertices in G (in any order) and form G' by adding an edge between them.
  - G' is connected, by previous Thm has an Eulerian walk C.
  - Traverse C in G' and for each time we cross an edge of G' not in G, start a new trail (lift pen!).
  - Thus, we get k trails decomposing G. $\square$

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- We have already seen some: connected graphs.
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- Are there other interesting classes of graphs?

Bipartite graphs

Definition

A graph is called bipartite, if the vertices of the graph can be partitioned into $V = X \cup Y$, $X \cap Y = \emptyset$ s.t., $\forall e = (u, v) \in E$,

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- How can we check if a graph is bipartite?
- Can we characterize bipartite graphs?

Characterizing bipartite graphs using cycles.

- Recall: A path or a cycle has length n if the number of edges in it is n.
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Theorem, Konig, 1936

A graph is bipartite iff it has no odd cycle.