CS 105: DIC on Discrete Structures

Graph theory

Basic terminology, Bipartite graphs and a characterization

Lecture 30
Oct 14 2024

Course content

Part 1. Proofs and Reasoning

Part 2. Basic discrete structures

Part 3. Counting and combinatorics

Part 4. Introduction to Graph theory

Course content

Part 1. Proofs and Reasoning

- Propositions and Predicates
- Types of proofs: contradiction and contrapositive
- Induction, strong Induction and WOP
- 6 lectures, 2 problemsheets.

Part 2. Basic discrete structures

Part 3. Counting and combinatorics

Part 4. Introduction to Graph theory

Course content

Part 1. Proofs and Reasoning

Part 2. Basic discrete structures

- Finite and infinite sets
- Functions: injections, surjections, bijections
- Countability and Uncountability, Diagonalization
- Equivalence relations and partitions
- Posets, chains, anti-chains, lattices
- Applications: Topological sorting, Task scheduling, defining new objects, comparing infinite sets.
- 10 lectures, 4 problemsheets

Part 3. Counting and combinatorics

Part 4. Introduction to Graph theory

Course content

Part 1. Proofs and Reasoning

Part 2. Basic discrete structures

Part 3. Counting and combinatorics

- Basic counting: product, sum and bijection principles
- Double counting, handshake lemma
- Binomial theorem, Pascal's triangle, Estimating factorial
- Recurrence relations and how to solve them, generating functions
- PIE and PHP and their applications
- A tiny bit of Ramsey theory
- 10 lectures, 3 problemsheets

Part 4. Introduction to Graph theory

Course content

Part 1. Proofs and Reasoning

Part 2. Basic discrete structures

Part 3. Counting and combinatorics

Part 4. Introduction to Graph theory

- What are graphs and why?!
- Konigsberg Bridge problem
- Eulerian graphs and a characterization using even degrees
- 3 lectures done, $\sim$ 10 more left.

Course content

Part 1. Proofs and Reasoning

Part 2. Basic discrete structures

Part 3. Counting and combinatorics

Part 4. Introduction to Graph theory

- What are graphs and why?!
- Konigsberg Bridge problem
- Eulerian graphs and a characterization using even degrees
- 3 lectures done, $\sim$ 10 more left.
- Next: Bipartite graphs and a characterization

Logistics

Evaluation

- Quiz 1 (15%) and Midsem (25%) done.
- 2 Popquizzes over, more to come? (5%)

Logistics

Evaluation

- Quiz 1 (15%) and Midsem (25%) done.
- 2 Popquizzes over, more to come? (5%)
- Upcoming Quiz 2 (15%) this Friday, Endsem (40%)

Logistics

Evaluation

- Quiz 1 (15%) and Midsem (25%) done.
- 2 Popquizzes over, more to come? (5%)
- Upcoming Quiz 2 (15%) this Friday, Endsem (40%)
- So about 45% done, 55% left!

Logistics

Evaluation

- Quiz 1 (15%) and Midsem (25%) done.
- 2 Popquizzes over, more to come? (5%)
- Upcoming Quiz 2 (15%) this Friday, Endsem (40%)
- So about 45% done, 55% left!

Quiz 2 logistics

- Syllabus: Combinatorics and last wk of graph theory
- Syllabus: Lectures 18 - 29 (both included!)
- Venue
  - LH 101 (Roll numbers: 24B0901 to 24B0999)
  - LH 102 (Roll numbers: 24B1000 to 24B1101)
  - LT 105 (All other roll numbers and PwD students)
  - Time: 8:15am

Logistics

Evaluation

- Quiz 1 (15%) and Midsem (25%) done.
- 2 Popquizzes over, more to come? (5%)
- Upcoming Quiz 2 (15%) this Friday, Endsem (40%)
- So about 45% done, 55% left!

Quiz 2 logistics

- Syllabus: Combinatorics and last wk of graph theory
- Syllabus: Lectures 18 - 29 (both included!)
- Venue
  - LH 101 (Roll numbers: 24B0901 to 24B0999)
  - LH 102 (Roll numbers: 24B1000 to 24B1101)
  - LT 105 (All other roll numbers and PwD students)
  - Time: 8:15am

Feedback? Complaints?

Some simple types of Graphs

- We have already seen some: connected graphs, complete graphs.

Some simple types of Graphs

- We have already seen some: connected graphs, complete graphs.
- paths, cycles.

Some simple types of Graphs

- We have already seen some: connected graphs, complete graphs.
- paths, cycles.
- Are there other interesting classes of graphs?

Bipartite graphs

Definition

A (simple) graph is called bipartite, if the vertices of the graph can be partitioned into $V = X \cup Y$, $X \cap Y = \emptyset$ s.t., $\forall e = (u, v) \in E$,

- either $u \in X$ and $v \in Y$
- or $v \in X$ and $u \in Y$

Example: m jobs and n people, k courses and ℓ students.

- How can we check if a graph is bipartite?
- Can we characterize bipartite graphs?

Bipartite graphs

Definition

A (simple) graph is called bipartite, if the vertices of the graph can be partitioned into $V = X \cup Y$, $X \cap Y = \emptyset$ s.t., $\forall e = (u, v) \in E$,

- either $u \in X$ and $v \in Y$
- or $v \in X$ and $u \in Y$

Example: m jobs and n people, k courses and ℓ students.

- How can we check if a graph is bipartite?
- Can we characterize bipartite graphs?

Exercises

1. What is largest n s.t n-vertex complete graph is bipartite?
2. Can a n-vertex bipartite graph have $n-1$ degree vertex?
3. A graph is bipartite iff we can color vertices with 2 colors s.t no 2 adjacent vertices have same color.

Characterizing bipartite graphs using cycles.

- Recall: A path or a cycle has length n if the number of edges in it is n.
- A path (or cycle) is call odd (or even) if its length is odd (or even, respectively).

Exercise: Prove or Disprove:

Every closed odd walk contains an odd cycle.

Characterizing bipartite graphs using cycles.

Exercise: Prove or Disprove:

Every closed odd walk contains an odd cycle.

Proof: By induction on the length of the given closed odd walk.
Exercise!

Characterizing bipartite graphs using cycles.

Lemma

Every closed odd walk contains an odd cycle.

Theorem, Konig, 1936

A graph is bipartite iff it has no odd cycle.

Proof:

- ($\Longrightarrow$) direction is easy.

Characterizing bipartite graphs using cycles.

Lemma

Every closed odd walk contains an odd cycle.

Theorem, Konig, 1936

A graph is bipartite iff it has no odd cycle.

Proof:

- $(\implies)$ direction is easy.
- Let G be bipartite with $(V = X \cup Y)$. Then, every walk in G alternates between X, Y.

Characterizing bipartite graphs using cycles.

Lemma

Every closed odd walk contains an odd cycle.

Theorem, Konig, 1936

A graph is bipartite iff it has no odd cycle.

Proof:

- ($\Longrightarrow$) direction is easy.
- Let G be bipartite with $(V = X \cup Y)$. Then, every walk in G alternates between X, Y.

$\Longrightarrow$ if we start from X, each return to X can only happen after an even number of steps.

$\Longrightarrow$ G has no odd cycles.

Characterizing bipartite graphs using cycles.

Lemma

Every closed odd walk contains an odd cycle.

Theorem, Konig, 1936

A graph is bipartite iff it has no odd cycle.

Proof:

- ($\Longleftarrow$) Suppose G has no odd cycle, then let us construct the bipartition. Wlog assume G is connected.

Characterizing bipartite graphs using cycles.

Lemma

Every closed odd walk contains an odd cycle.

Theorem, Konig, 1936

A graph is bipartite iff it has no odd cycle.

Proof:

- ($\Longleftarrow$) Suppose G has no odd cycle, then let us construct the bipartition. Wlog assume G is connected.
- Let $u \in V$. Break V into

$$\begin{aligned} X &= \{v \in V \mid \text{length of shortest path } P_{uv} \text{ from } u \text{ to } v \text{ is even}\}, \\ Y &= \{v \in V \mid \text{length of shortest path } P_{uv} \text{ from } u \text{ to } v \text{ is odd}\}, \end{aligned}$$

Characterizing bipartite graphs using cycles.

Lemma

Every closed odd walk contains an odd cycle.

Theorem, Konig, 1936

A graph is bipartite iff it has no odd cycle.

Proof:

- ($\Longleftarrow$) Suppose G has no odd cycle, then let us construct the bipartition. Wlog assume G is connected.
- Let $u \in V$. Break V into

$$\begin{aligned} X &= \{v \in V \mid \text{length of shortest path } P_{uv} \text{ from } u \text{ to } v \text{ is even}\}, \\ Y &= \{v \in V \mid \text{length of shortest path } P_{uv} \text{ from } u \text{ to } v \text{ is odd}\}, \end{aligned}$$

- If there is an edge vv' between two vertices of X or two vertices of Y, this creates a closed odd walk: $uP_{uv}vv'P_{v'u}u$.

Characterizing bipartite graphs using cycles.

Lemma

Every closed odd walk contains an odd cycle.

Theorem, Konig, 1936

A graph is bipartite iff it has no odd cycle.

Proof:

- ($\Longleftarrow$) Suppose G has no odd cycle, then let us construct the bipartition. Wlog assume G is connected.
- Let $u \in V$. Break V into
  $$X = \{v \in V \mid \text{length of shortest path } P_{uv} \text{ from } u \text{ to } v \text{ is even}\},$$
  $$Y = \{v \in V \mid \text{length of shortest path } P_{uv} \text{ from } u \text{ to } v \text{ is odd}\},$$
- If there is an edge vv' between two vertices of X or two vertices of Y, this creates a closed odd walk: $uP_{uv}vv'P_{v'u}u$.
- By Lemma, it must contain an odd cycle: contradiction.
- This along with $X \cap Y = \emptyset$ and $X \cup Y = V$, implies X, Y is a bipartition. $\square$