CS 105: DIC on Discrete Structures

Graph theory

Graph Isomorphism

Lecture 31
Oct 15 2024

Part 4: Graph theory

Recap of last four lectures:

1. Basics: graphs, paths, cycles, walks, trails; connected graphs.
2. Eulerian graphs and a characterization in terms of degrees of vertices.
3. Bipartite graphs and a characterization in terms of odd length cycles.

Reference: Sections 1.1-1.3 of Chapter 1 from Douglas West.

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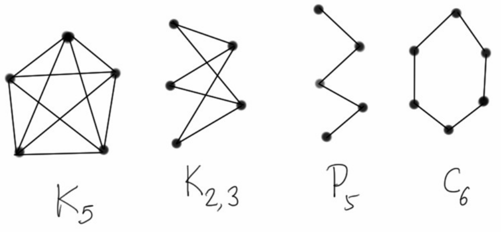
Today

Graph representations and naming

Some special graphs and notations



- Complete graphs K_n
- Complete bipartite graphs $K_{i,j}$
- Paths P_n
- Cycles C_n

Some special graphs and notations

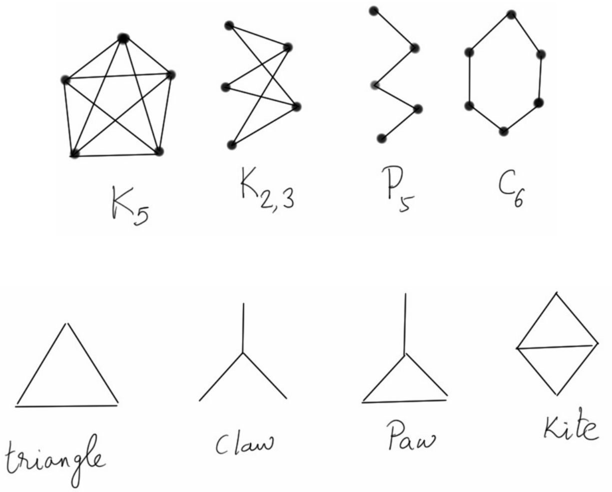


Figure: A whole graph zoo!

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- To compare graphs, we need to name them!

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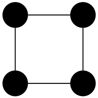

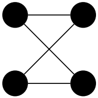

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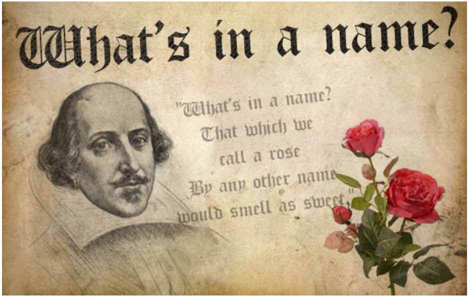

Representing and comparing graphs

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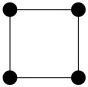

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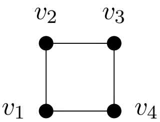

To represent it, we need to name the vertices...

- As an adjacency list:

v_1	v_2, v_4
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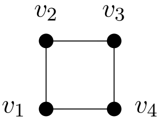

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- As an adjacency matrix:

$$
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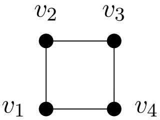

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- Are two given graphs the “same”, wrt these properties?

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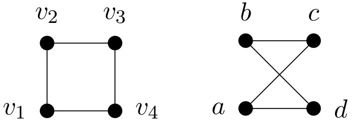

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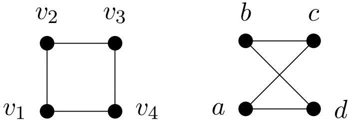

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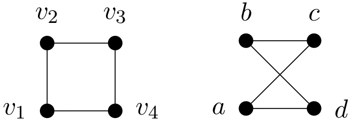

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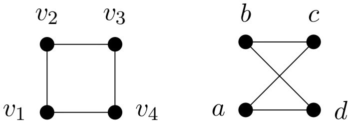

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- So, it seems two graphs are "same" if by reordering and renaming the vertices we get the same graph/matrix.

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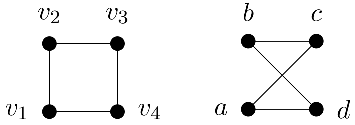

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- So, it seems two graphs are "same" if by reordering and renaming the vertices we get the same graph/matrix.
- How do we formalize this?

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- When we talked about an "unlabeled" graph till now, we actually meant the isomorphism class of that graph!

Properties of isomorphic graphs

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- Are C_5 and $P_5 \cup \{e\}$ isomorphic?

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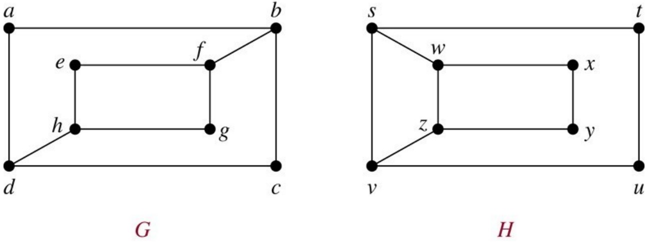

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5. G is bipartite iff H is bipartite.
6. $\ldots$

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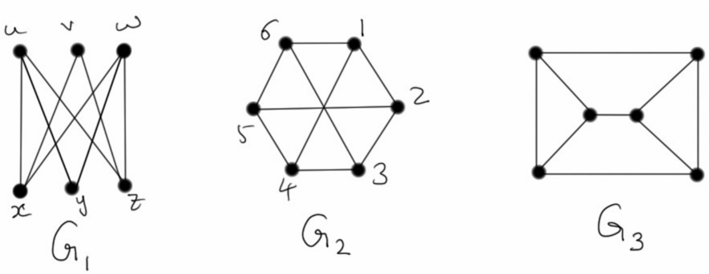

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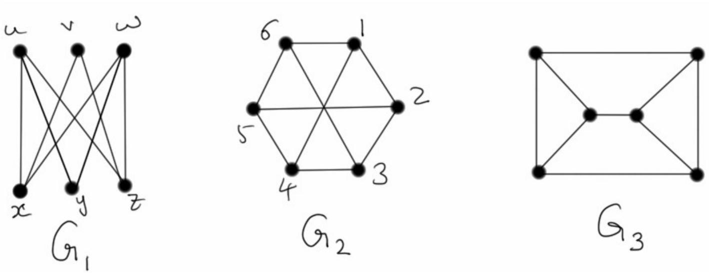

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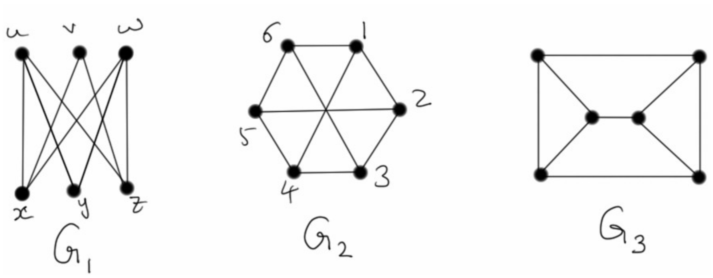

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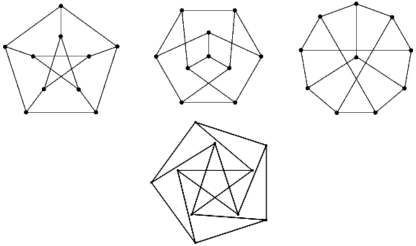
- To show that two graphs are isomorphic, you have to
  1. give names to vertices
  2. specify a bijection
  3. check that it preserves the adjacency relation
- To show that two graphs are non-isomorphic, find a structural property that is different.

Is checking graph isomorphism easy?



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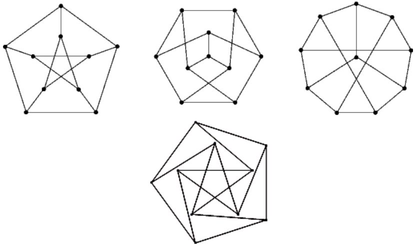

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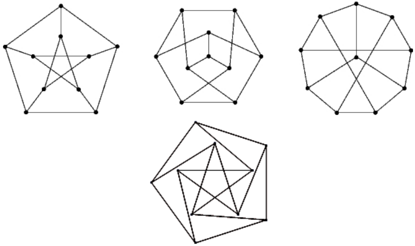

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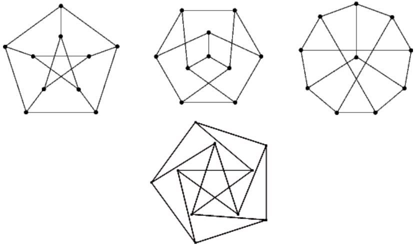

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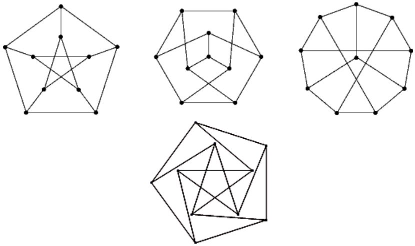

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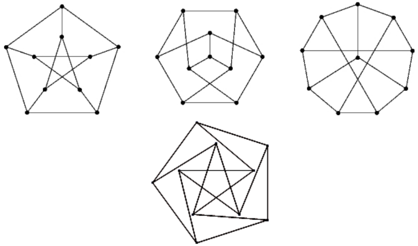

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Further reading: Graph and sub-graph isomorphism problems.

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- How many automorphisms does $K_{r,s}$ have?

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Automorphisms are a measure of symmetry.

Practical applications in graph drawing, visualization, molecular symmetry, structured boolean satisfiability, formal verification