CS 105: DIC on Discrete Structures

Graph theory

Connectedness in graphs

Lecture 32
Oct 17 2024

Last topic: Graph theory

Recap

1. Basic definitions: graphs, paths, cycles, walks, trails; connected graphs.
2. Eulerian graphs and a characterization in terms of degrees of vertices.
3. Bipartite graphs and a characterization in terms of odd length cycles.
4. Graph representation (as matrices, lists, etc.)
5. Graph isomorphisms and automorphisms

Reference: Sections 1.1-1.3 of Chapter 1 from Douglas West.

Recall: Graph Automorphisms

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- How many automorphisms does $K_{r,s}$ have?

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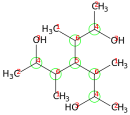
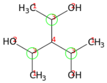
Automorphisms are a measure of symmetry.

Practical applications in graph drawing, visualization, molecular symmetry, structured boolean satisfiability, formal verification



Symmetry Classes
Stereogenic Atoms
(rule $_2$b')



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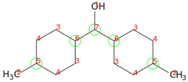

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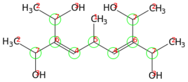

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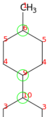
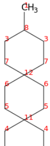
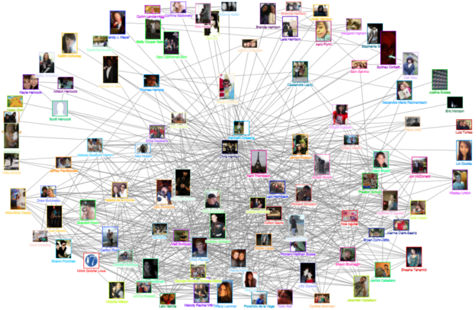


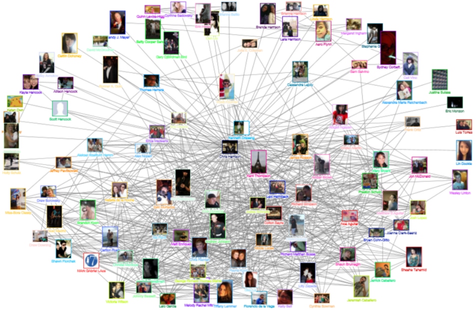


Cliques and independent sets



- Consider a large social network graph where friends are linked by an edge.
- What is the largest clique of friends?
- If we want to spread a youtube video, how many people should we send it to so that we are guaranteed everyone will see it (assuming friends forward to each other)?

Cliques and independent sets



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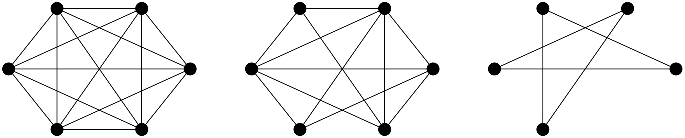

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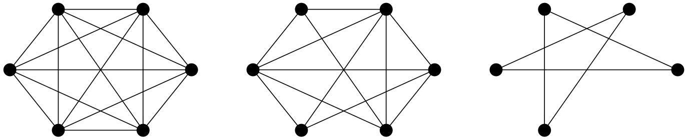

- Thus, a clique in a graph G is a complete subgraph of G.

Subgraphs of a graph G

A subgraph H of a graph G is a graph H such that

- $V(H) \subseteq V(G)$ and
- $E(H) \subseteq E(G)$
  & assignment of endpoints to edges in H is same as in G.

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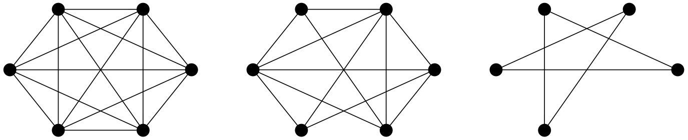

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- An independent set in G is a complete subgraph of $\overline{G}$, where $\overline{G}$ is the complement of G obtained by making all adjacent vertices non-adjacent and vice versa.

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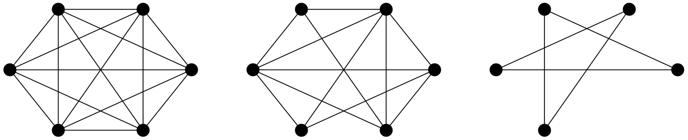

Questions:

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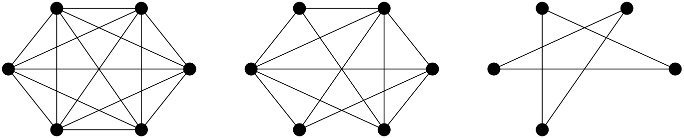

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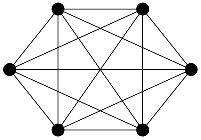
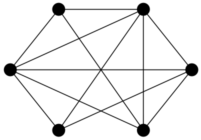
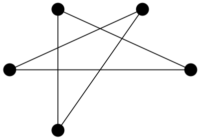

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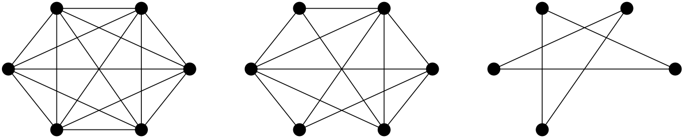

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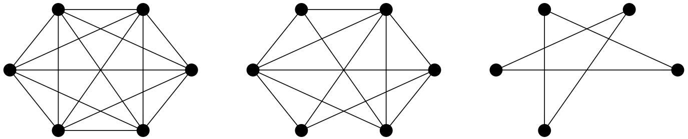

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Ramsey's theorem - restated

In any graph with $R(k,\ell)$ vertices, there exists either a clique of size k or an independent set of size ℓ.

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Definition

A (connected) component of G is a maximal connected subgraph, i.e., a subgraph that is connected and is not contained in any other connected subgraph of G.

Thus, equivalence classes of P are the vertex sets of the components of G.

Recall: Difference between maximal and maximum

- Is every maximal path maximum, i.e., have maximum length?
- A maximal structure is a structure that is not contained in a larger structure, i.e., increasing the structure will violate some property.
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Exercises!

1. Give a path which is maximal but not maximum.
2. Give a subgraph of a graph which is maximally connected, but not maximum (i.e., does not have maximum # edges).
3. How many maximal/maximum independent sets does $K_{r,s}$ have?

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Properties of components

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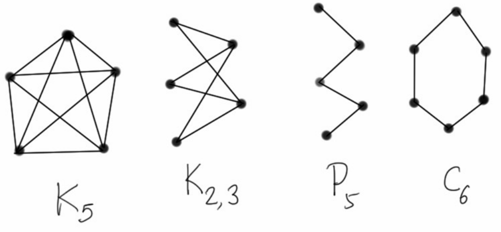

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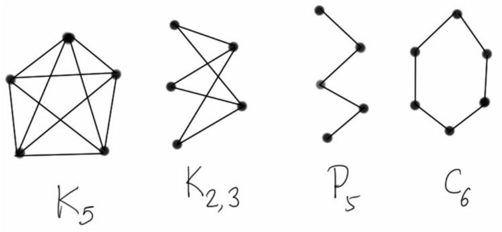

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H.W. Exercise! An edge is a cut-edge iff it belongs to no cycle.