CS 105: DIC on Discrete Structures

Graph theory

Characterizing maximum matchings
via augmenting paths
Guest Lecture: Rohit Gurjar

Lecture 34
Oct 22 2024

Topic 3: Graph theory

Basic concepts

- Basics: graphs, paths, cycles, walks, trails, …
- Cliques and independent sets.
- Graph representations, isomorphisms and automorphisms.
- Matchings: perfect, maximal and maximum.

Topic 3: Graph theory

Basic concepts

- Basics: graphs, paths, cycles, walks, trails, ...
- Cliques and independent sets.
- Graph representations, isomorphisms and automorphisms.
- Matchings: perfect, maximal and maximum.

Characterizations

1. Eulerian graphs: Using degrees of vertices.
2. Bipartite graphs: Using odd length cycles.
3. Connected components: Using cycles.

Topic 3: Graph theory

Basic concepts

- Basics: graphs, paths, cycles, walks, trails, ...
- Cliques and independent sets.
- Graph representations, isomorphisms and automorphisms.
- Matchings: perfect, maximal and maximum.

Characterizations

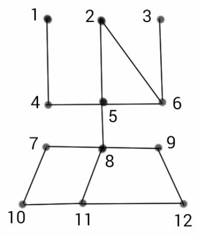
1. Eulerian graphs: Using degrees of vertices.
2. Bipartite graphs: Using odd length cycles.
3. Connected components: Using cycles.
4. Maximum matchings: Using augmenting paths.

Matchings

Definitions

- A matching in a graph G is a set of (non-loop) edges with no shared end-points. The vertices incident to edges in a matching are called matched or saturated. Others are unsaturated.
- A perfect matching in a graph is a matching that saturates every vertex.
- A maximal matching in a graph is a matching that cannot be enlarged by adding an edge.
- A maximum matching is a matching of maximum size (# edges) among all matchings in a graph.

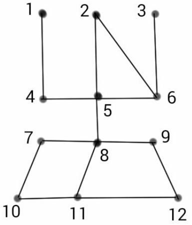
Matchings



Give an example of the following, if possible:

1. A maximal matching in G which is not a maximum matching.
2. A maximum matching in G. How do you know it is maximum?
3. Can there be more than one maximum matching in a graph?
4. A graph which has no perfect matching but has a maximum matching. Is G such a graph?

Matchings



- Perfect matching $\implies$ maximum matching $\implies$ maximal matching
- The reverse directions in the above implications do not hold.

Alternating and Augmenting paths

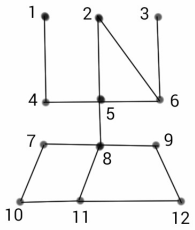
Definition

- Given a matching M, an M-alternating path is a path that alternates between edges in M and edges not in M.
- An M-alternating path whose endpoints are unmatched by M is an M-augmenting path.

Alternating and Augmenting paths

Definition

- Given a matching M, an M-alternating path is a path that alternates between edges in M and edges not in M.
- An M-alternating path whose endpoints are unmatched by M is an M-augmenting path.

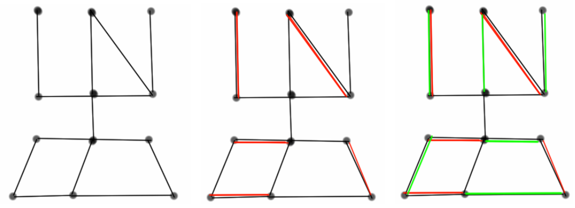


- Ex 1: Give an example of a matching M in G and
  1. a M-alternating path which is an M-augmenting path and
  2. a M-alternating path which is not an M-augmenting path

Alternating and Augmenting paths

Definition

- Given a matching M, an M-alternating path is a path that alternates between edges in M and edges not in M.
- An M-alternating path whose endpoints are unmatched by M is an M-augmenting path.



Characterization of Maximum matchings

- Clearly, a maximum matching cannot have an M-augmenting path.

Characterization of Maximum matchings

- Clearly, a maximum matching cannot have an M-augmenting path.
- In fact, this is the characterization!

Characterization of Maximum matchings

- Clearly, a maximum matching cannot have an M-augmenting path.
- In fact, this is the characterization!

Theorem

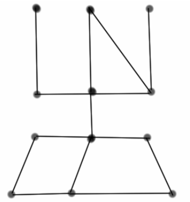
A matching M in G is a maximum matching iff G has no M-augmenting path.

Characterizing maximum matchings

We need a definition and a lemma.

Definition

If M, M' are matchings in a graph H, the symmetric difference $M \triangle M'$ is the set of edges which are either in M or in M' but not both, i.e., $M \triangle M' = (M \setminus M') \cup (M' \setminus M)$.

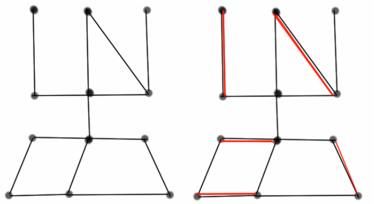


Characterizing maximum matchings

We need a definition and a lemma.

Definition

If M, M' are matchings in a graph H, the symmetric difference $M \triangle M'$ is the set of edges which are either in M or in M' but not both, i.e., $M \triangle M' = (M \setminus M') \cup (M' \setminus M)$.

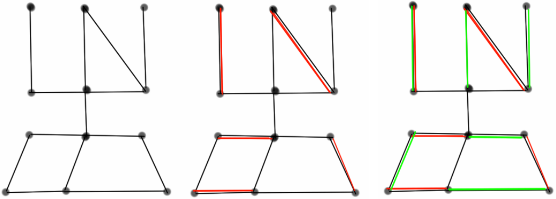


Characterizing maximum matchings

We need a definition and a lemma.

Definition

If M, M' are matchings in a graph H, the symmetric difference $M \triangle M'$ is the set of edges which are either in M or in M' but not both, i.e., $M \triangle M' = (M \setminus M') \cup (M' \setminus M)$.



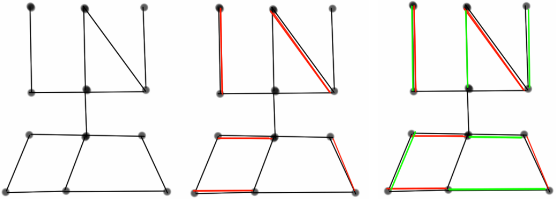
Ex 2: What is the symmetric difference of M (red) and M' (green) in the above graph?

Characterizing maximum matchings

We need a definition and a lemma.

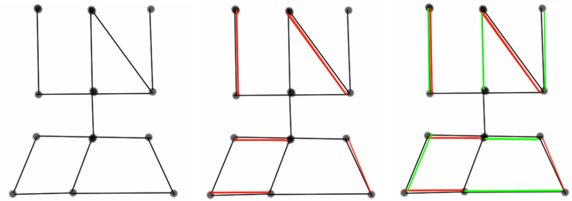
Definition

If M, M' are matchings in a graph H, the symmetric difference $M \triangle M'$ is the set of edges which are either in M or in M' but not both, i.e., $M \triangle M' = (M \setminus M') \cup (M' \setminus M)$.



Ex 2: What is the symmetric difference of M (red) and M' (green) in the above graph? Can you generalize this?

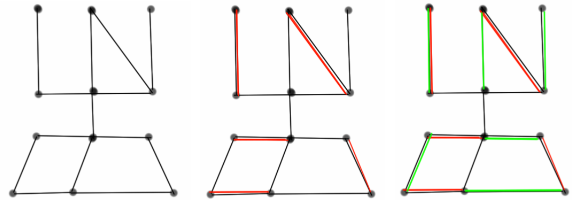
Characterizing maximum matchings



Lemma

Every component of the symmetric difference of two matchings is either a path or an even cycle.

Characterizing maximum matchings

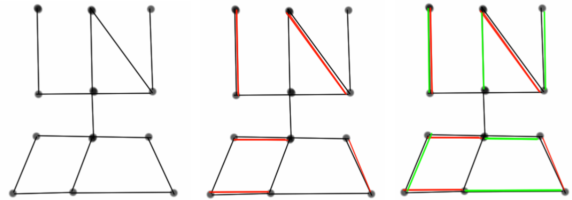


Lemma

Every component of the symmetric difference of two matchings is either a path or an even cycle.

- Let $F = M \triangle M'$. F has at most 2 edges at each vertex, hence every component is a path or a cycle.

Characterizing maximum matchings

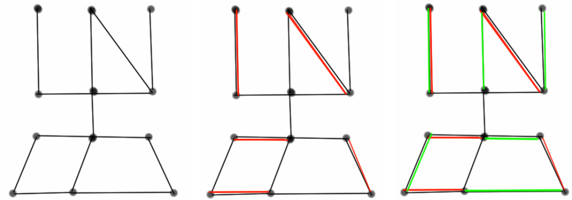


Lemma

Every component of the symmetric difference of two matchings is either a path or an even cycle.

- Let $F = M \triangle M'$. F has at most 2 edges at each vertex, hence every component is a path or a cycle.
- Further every path/cycle alternates between edges of $M \setminus M'$ and $M' \setminus M$.

Characterizing maximum matchings



Lemma

Every component of the symmetric difference of two matchings is either a path or an even cycle.

- Let $F = M \triangle M'$. F has at most 2 edges at each vertex, hence every component is a path or a cycle.
- Further every path/cycle alternates between edges of $M \setminus M'$ and $M' \setminus M$.
- Thus, each cycle has even length with equal edges from M and M'. $\square$

Characterizing maximum matchings

Theorem (Berge'57)

A matching M in G is a maximum matching iff G has no M-augmenting path.

Proof:

Characterizing maximum matchings

Theorem (Berge'57)

A matching M in G is a maximum matching iff G has no M-augmenting path.

Proof:

- One direction is trivial (which one?!).

Characterizing maximum matchings

Theorem (Berge'57)

A matching M in G is a maximum matching iff G has no M-augmenting path.

Proof:

- One direction is trivial (which one?!).
- $(\Longleftarrow)$ For the other, we will show the contrapositive.

Characterizing maximum matchings

Theorem (Berge'57)

A matching M in G is a maximum matching iff G has no M-augmenting path.

Proof:

- One direction is trivial (which one?!).
- ($\Longleftarrow$) For the other, we will show the contrapositive.
- i.e., if $\exists$ matching M' larger than M, we will construct an M-augmenting path.

Characterizing maximum matchings

Theorem (Berge'57)

A matching M in G is a maximum matching iff G has no M-augmenting path.

Proof:

- One direction is trivial (which one?!).
- ($\Longleftarrow$) For the other, we will show the contrapositive.
- i.e., if $\exists$ matching M' larger than M, we will construct an M-augmenting path.
- Let $F = M \triangle M'$. By Lemma, F has only paths and even cycles with equal no. of edges from M and M'.

Characterizing maximum matchings

Theorem (Berge'57)

A matching M in G is a maximum matching iff G has no M-augmenting path.

Proof:

- One direction is trivial (which one?!).
- ($\Longleftarrow$) For the other, we will show the contrapositive.
- i.e., if $\exists$ matching M' larger than M, we will construct an M-augmenting path.
- Let $F = M \triangle M'$. By Lemma, F has only paths and even cycles with equal no. of edges from M and M'.
- But then since $|M'| > |M|$ it must have a component with more edges in M' than M.

Characterizing maximum matchings

Theorem (Berge'57)

A matching M in G is a maximum matching iff G has no M-augmenting path.

Proof:

- One direction is trivial (which one?!).
- ($\Longleftarrow$) For the other, we will show the contrapositive.
- i.e., if $\exists$ matching M' larger than M, we will construct an M-augmenting path.
- Let $F = M \triangle M'$. By Lemma, F has only paths and even cycles with equal no. of edges from M and M'.
- But then since $|M'| > |M|$ it must have a component with more edges in M' than M.
- This component can only be a path that starts and ends with an edge of M'; i.e., it is an M-augmenting path in G. $\square$