CS 105: DIC on Discrete Structures

Graph theory

Hall's theorem and its applications

Guest Lecture: Rohit Gurjar

Lecture 35

Oct 24 2024

Topic 3: Graph theory

Basic definitions and concepts

Characterizations

1. Eulerian graphs: Using degrees of vertices.
2. Bipartite graphs: Using odd length cycles.
3. Connected components: Using cycles.
4. Maximum matchings: Using augmenting paths.

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4. Maximum matchings: Using augmenting paths.
5. Perfect matchings in bipartite graphs: Using neighbour sets. – Hall's theorem

Recap: Matchings

Definitions

- Matching: set of edges with no shared end-points.
- The vertices incident to edges in a matching are called saturated. Others are unsaturated.
- Perfect matching: saturates every vertex in graph.
- Maximum matching: matching of maximum size (# edges).
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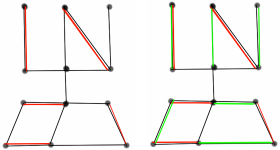

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Theorem

A matching M in G is a maximum matching iff G has no M-augmenting path.

Perfect matchings in bipartite graphs

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Perfect matchings in bipartite graphs

- If there are n women and n men, and each woman is compatible with exactly k men and each man compatible with exactly k women, can they be perfectly matched?
- If there are m jobs and n applicants, when can we find a perfect matching where all m jobs are saturated?

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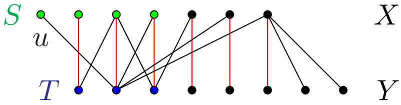

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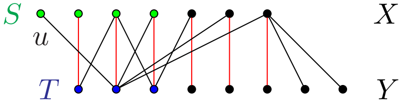

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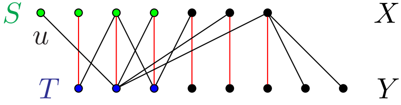
- That is, $\forall S \subseteq X, |N(S)| \geq |S|$ (Hall's Condition).



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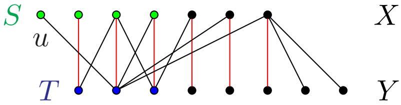

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Theorem (Hall'35)

A bipartite graph G with bipartitions X, Y has a matching that saturates X iff for all $S \subseteq X$, $|N(S)| \geq |S|$.

Characterizing perfect matchings in bipartite graphs

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- For $v \in V$, its neighbour-set $N(v) = \{u \in V \mid (u, v) \in E\}$.
- For $S \subseteq V$, $N(S) = \{u \in V \mid (u, v) \in E \text{ for some } v \in S\}$.

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Proof: ($\implies$) is straightforward:

- Let M be a matching.
- Then for any $S \subseteq X$, each vertex of S is matched to a distinct vertex in $N(S)$
- So $|N(S)| \geq |S|$.

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- If G does not have any matching that saturates X, then surely any maximum matching of G does not saturate X.
- Let M be such a maximum matching. Then, we will construct $S \subseteq X$ s.t. $|N(S)| < |S|$.

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A bipartite graph G with bipartitions X, Y has a matching that saturates X iff for all $S \subseteq X$, $|N(S)| \geq |S|$.

Proof: $(\Longleftarrow)$ Thus, starting from a maximum matching M which does not saturate X, we construct $S \subseteq X, |N(S)| < |S|$.

- Let $u \in X$ be any unsaturated vertex of M.

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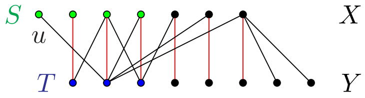
- Consider vertices V_u from u by M-alternating paths in G and let $S = V_u \cap X$ and $T = V_u \cap Y$.



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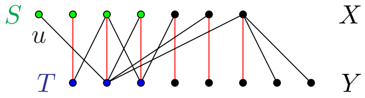

Claim: M matches T with $S \setminus \{u\}$ and $|N(S)| = |T|$.

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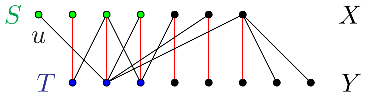

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- Every vertex of $S \setminus \{u\}$ has an edge in M to a vertex in T.

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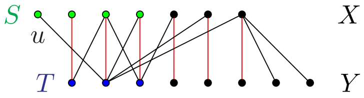

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- Every vertex of $S \setminus \{u\}$ has an edge in M to a vertex in T.
- Every vertex of T extends via M to a unique vertex of S.

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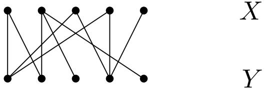

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- Every vertex of $S \setminus \{u\}$ has an edge in M to a vertex in T.
- Every vertex of T extends via M to a unique vertex of S.
- Thus, there is a bijection between T and $S \setminus \{u\}$.

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- $T \subseteq N(S)$ (from T any M-alternating path will reach S).
- Conversely, if $v \in S$ has edge to $y \in Y \setminus T$, then path from u to v via M to y is an M-alternating path, implies $y \in T$.

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Claim: M matches T with $S \setminus \{u\}$ and $|N(S)| = |T|$.
Thus, $|N(S)| = |T| = |S| - 1 < |S|$ $\square$.

Applications of Hall's condition

Theorem (Hall'35)

A bipartite graph G with bipartitions X, Y has a matching that saturates X iff for all $S \subseteq X$, $|N(S)| \geq |S|$.

The Marriage Theorem (1917)

- In a group of n women and n men, if every woman is compatible with k men and every man compatible with k women, then a perfect matching must exist!
- What is the formal statement?

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- For $k > 0$, every k-regular bipartite graph (i.e, every vertex has degree exactly k) has a perfect matching. Ex. Prove this!

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- If G is a k-regular X, Y bipartite graph, then $|X| = |Y|$. (why?)

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- Can you now verify Hall's condition?

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