CS 105: DIC on Discrete Structures

Graph theory

More Applications of Hall's theorem and minmax theorems

Lecture 37

Oct 29 2024

Topic 3: Graph theory

Topics in Graph theory

1. Basics concepts and definitions.
2. Eulerian graphs: Using degrees of vertices.
3. Bipartite graphs: Using odd length cycles.
4. Connected components: Using cycles.
5. Maximum matchings: Using augmenting paths.
6. Perfect matchings in bipartite graphs: Using neighbour sets. – Hall's theorem

Topic 3: Graph theory

Topics in Graph theory

1. Basics concepts and definitions.
2. Eulerian graphs: Using degrees of vertices.
3. Bipartite graphs: Using odd length cycles.
4. Connected components: Using cycles.
5. Maximum matchings: Using augmenting paths.
6. Perfect matchings in bipartite graphs: Using neighbour sets. – Hall's theorem
7. Today: Applications of Hall's theorem. Matchings in bipartite graphs: Minimum vertex covers. – Konig-Egervary's theorem

Applications of Hall's condition

- Matching: set of edges with no shared end-points.
- The vertices incident to edges in a matching are called saturated. Others are unsaturated.
- Perfect matching: saturates every vertex in graph.

Theorem (Hall'35)

A bipartite graph G with bipartitions X, Y has a matching that saturates X iff for all $S \subseteq X$, $|N(S)| \geq |S|$.

Applications of Hall's condition

- Matching: set of edges with no shared end-points.
- The vertices incident to edges in a matching are called saturated. Others are unsaturated.
- Perfect matching: saturates every vertex in graph.

Theorem (Hall'35)

A bipartite graph G with bipartitions X, Y has a matching that saturates X iff for all $S \subseteq X$, $|N(S)| \geq |S|$.

Application 1: Marriage theorem

- Consider n women and n men. If every woman is compatible with k men and every man compatible with k women, then a perfect matching must exist.
- For $k > 0$, every k-regular bipartite graph (i.e, every vertex has degree exactly k) has a perfect matching.

Application 2

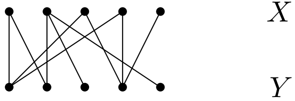
A two player game on a graph

1. Given a graph G, two players will alternatively choose distinct vertices.
2. One player starts by choosing any vertex.
3. Subsequent move must be adjacent to preceding choice (of other player).
4. Thus, we draw a path.
5. Last player who can move wins.

Application 2

A two player game on a graph

1. Given a graph G, two players will alternatively choose distinct vertices.
2. One player starts by choosing any vertex.
3. Subsequent move must be adjacent to preceding choice (of other player).
4. Thus, we draw a path.
5. Last player who can move wins.



Application 2

A two player game on a graph

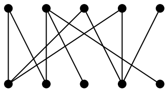
1. Given a graph G, two players will alternatively choose distinct vertices.
2. One player starts by choosing any vertex.
3. Subsequent move must be adjacent to preceding choice (of other player).
4. Thus, we draw a path.
5. Last player who can move wins.

Theorem (Proof: Qn 3.1.18 from Douglas West)

If G has a perfect matching, then player 2 has a winning strategy; otherwise, player 1 has a winning strategy.

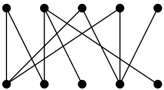
Application 3: Another game

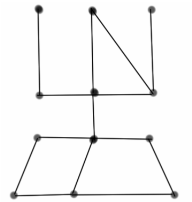
Consider the graph of a road network in a city. When a minister is visiting, the Chief of Police wants to place a policeman to watch every road. What is the minimum number of policemen required?



Application 3: Another game

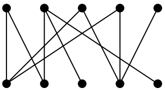
Consider the graph of a road network in a city. When a minister is visiting, the Chief of Police wants to place a policeman to watch every road. What is the minimum number of policemen required?

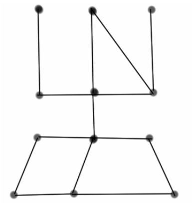




Application 3: Another game

Consider the graph of a road network in a city. When a minister is visiting, the Chief of Police wants to place a policeman to watch every road. What is the minimum number of policemen required?





Definition

A vertex cover of a graph G is a set $Q \subseteq V$ that contains at least one endpoint of every edge. Vertices in Q are said to cover E.

Matchings and vertex covers

So, what is the link between matchings and vertex covers?

Matchings and vertex covers

So, what is the link between matchings and vertex covers?

First, does a graph always have vertex cover?

Matchings and vertex covers

So, what is the link between matchings and vertex covers?

First, does a graph always have vertex cover?

Examples and properties of vertex covers

Matchings and vertex covers

So, what is the link between matchings and vertex covers?

First, does a graph always have vertex cover?

Examples and properties of vertex covers

- The set of all vertices is always a vertex cover.

Matchings and vertex covers

So, what is the link between matchings and vertex covers?

First, does a graph always have vertex cover?

Examples and properties of vertex covers

- The set of all vertices is always a vertex cover.
- The end-points of a maximal matching

Matchings and vertex covers

So, what is the link between matchings and vertex covers?

First, does a graph always have vertex cover?

Examples and properties of vertex covers

- The set of all vertices is always a vertex cover.
- The end-points of a maximal matching form a vertex cover.

Matchings and vertex covers

So, what is the link between matchings and vertex covers?

First, does a graph always have vertex cover?

Examples and properties of vertex covers

- The set of all vertices is always a vertex cover.
- The end-points of a maximal matching form a vertex cover.
- Size of any vertex cover *vs* size of any matching?

Matchings and vertex covers

So, what is the link between matchings and vertex covers?

First, does a graph always have vertex cover?

Examples and properties of vertex covers

- The set of all vertices is always a vertex cover.
- The end-points of a maximal matching form a vertex cover.
- Size of any vertex cover *vs* size of any matching?

Exercise/Pop quiz?!

1. What is the size of the minimum vertex cover in K_m, $K_{m,n}$?
2. If ℓ is size of any matching and k is size of vertex cover,
   2.1 how are ℓ, k related?
   2.2 what if ℓ' is size of maximum matching?
   2.3 Give an example of a graph where $k \neq \ell' = \ell$.

Matchings and vertex covers

So, what is the link between matchings and vertex covers?

First, does a graph always have vertex cover?

Examples and properties of vertex covers

- The set of all vertices is always a vertex cover.
- The end-points of a maximal matching form a vertex cover.
- Size of any vertex cover *vs* size of any matching?

Exercise/Pop quiz?!

1. What is the size of the minimum vertex cover in K_m, $K_{m,n}$?
2. If ℓ is size of any matching and k is size of vertex cover,
   2.1 how are ℓ, k related?
   2.2 what if ℓ' is size of maximum matching?
   2.3 Give an example of a graph where $k \neq \ell' = \ell$.

So, how do you compute min no. of policemen required, i.e., the size of the minimum vertex covers?

Matchings and vertex covers

So, what is the link between matchings and vertex covers?

First, does a graph always have vertex cover?

Examples and properties of vertex covers

- The set of all vertices is always a vertex cover.
- The end-points of a maximal matching form a vertex cover.
- Size of any vertex cover *vs* size of any matching?

Exercise/Pop quiz?!

1. What is the size of the minimum vertex cover in K_m, $K_{m,n}$?
2. If ℓ is size of any matching and k is size of vertex cover,
   2.1 how are ℓ, k related?
   2.2 what if ℓ' is size of maximum matching?
   2.3 Give an example of a graph where $k \neq \ell' = \ell$.

So, how do you compute min no. of policemen required, i.e., the size of the minimum vertex covers?

Let's consider bipartite graphs...

A min-max theorem

Theorem (Konig '31, Egervary '31)

If G is a bipartite graph, then the size of the maximum matching of G equals the size of the minimum vertex cover of G.

A min-max theorem

Theorem (Konig '31, Egervary '31)

If G is a bipartite graph, then the size of the maximum matching of G equals the size of the minimum vertex cover of G.

Proof.

- Suffices to show that we can achieve a matching which has size equal to min vertex cover.

A min-max theorem

Theorem (Konig '31, Egervary '31)

If G is a bipartite graph, then the size of the maximum matching of G equals the size of the minimum vertex cover of G.

Proof.

- Suffices to show that we can achieve a matching which has size equal to min vertex cover.
- Take min vertex cover Q, partition into $R = Q \cap X$ and $T = Q \cap Y$.

A min-max theorem

Theorem (Konig '31, Egervary '31)

If G is a bipartite graph, then the size of the maximum matching of G equals the size of the minimum vertex cover of G.

Proof.

- Suffices to show that we can achieve a matching which has size equal to min vertex cover.
- Take min vertex cover Q, partition into $R = Q \cap X$ and $T = Q \cap Y$.
- Consider subgraphs H, H' induced by $R \cup (Y \setminus T)$, $T \cup (X \setminus R)$.

A min-max theorem

Theorem (Konig '31, Egervary '31)

If G is a bipartite graph, then the size of the maximum matching of G equals the size of the minimum vertex cover of G.

Proof.

- Suffices to show that we can achieve a matching which has size equal to min vertex cover.
- Take min vertex cover Q, partition into $R = Q \cap X$ and $T = Q \cap Y$.
- Consider subgraphs H, H' induced by $R \cup (Y \setminus T)$, $T \cup (X \setminus R)$.
- Show that H has a matching that saturates $Q \cap X$ into $Y \setminus T$, H' has a matching saturating T.

A min-max theorem

Theorem (Konig '31, Egervary '31)

If G is a bipartite graph, then the size of the maximum matching of G equals the size of the minimum vertex cover of G.

Proof.

- Suffices to show that we can achieve a matching which has size equal to min vertex cover.
- Take min vertex cover Q, partition into $R = Q \cap X$ and $T = Q \cap Y$.
- Consider subgraphs H, H' induced by $R \cup (Y \setminus T)$, $T \cup (X \setminus R)$.
- Show that H has a matching that saturates $Q \cap X$ into $Y \setminus T$, H' has a matching saturating T.
- Together this forms the desired matching (since H, H' are disjoint).

□

Many min max theorems exist

Dual problems

- Minimizing one thing is same as maximizing another!
- Very useful in optimization problems.

Many min max theorems exist

Dual problems

- Minimizing one thing is same as maximizing another!
- Very useful in optimization problems.

Prefinal course feedback: fill by today!

`https://forms.gle/tDLsYD7F7dnuFsaS8`