CS 207: Discrete Structures

Lecture 15 – Counting and Combinatorics
Recurrence relations

Aug 18 2016
Last three classes

Basic counting techniques and applications

1. Sum and product, bijection, double counting principles
2. Counting no. of (ordered) subsets, relations...
3. Handshake lemma

4. Binomial coefficients and binomial theorem
5. Pascal’s triangle and its applications
6. Permutations and combinations with/without repetitions
7. Estimating $n!$ and Stirling’s approximation
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Recall: No. of subsets of a set of $n$ elements

How many subsets does a set $A$ of $n$ elements have?

- **Induction**
- **Product principle**: two choices for each element, hence $2 \cdot 2 \cdots 2 \cdot 2$ ($n$-times).
- **Bijection**: between $P(X)$ and $n$-length $\{0, 1\}$-sequences.
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Next: Recurrence relations and generating functions
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But how do you solve it?
Another example of recurrence: The Fibonacci Sequence

- Fibonacci sequence: 1, 1, 2, 3, 5, 8, 13, 21, \ldots

- Recurrence relation: $u_n = u_{n-1} + u_{n-2}$ where $u_1 = u_0 = 1$

- But rabbits die!

- Consider $u_n = u_{n-1} + u_{n-2} - u_{n-3}$ where $u_2 = 2$, $u_1 = u_0 = 1$
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A recurrence relation for a sequence is an equation that expresses its $n^{th}$ term using one or more of the previous terms of the sequence.

A linear recurrence relation is of the form

$$u_n = a_{k-1}u_{n-1} + \ldots + a_1u_{n-k+1} + a_0u_{n-k}$$

where $a_0, \ldots, a_{k-1} \in \mathbb{R}, k \in \mathbb{N}$ are constants.

$k$ is called the degree/depth of the sequence.

The first few (e.g., $k$ elements $u_0, \ldots, u_{k-1}$) are initial conditions and they determine the whole sequence.
Some more examples of recurrences

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- Let \( C(n) \) be the number of ways of doing this.
- If outermost bracketing \((A + B)\) appears between \( x_k \) and \( x_{k+1} \), then there are \( C(k) \cdot C(n - k) \) ways of bracketing it.

Thus, \( C(n) = \sum_{i=1}^{n-1} C(i) \cdot C(n - i) \) for \( n > 1 \).

Initial conditions are \( C(0) = C(1) = 1 \).

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How do we solve such recurrences? We start with the Fibonacci sequence.
An aside: find the Fibonacci sequence!

\[
F(n) = F(n - 1) + F(n - 2).
\]

1, 1, 2, 3, 5, 8, 13, ....

Can you observe the sum of which terms in the Pascal’s triangle gives rise to the terms of the Fibonacci sequence?