# Topics in Combinatorics

## Basic counting techniques and applications

1. Basic counting techniques, double counting
2. Binomial coefficients and binomial theorem, permutations and combinations.
3. Estimating $n!$
4. Recurrence relations
5. Generating functions and its applications
Topics in Combinatorics

Basic counting techniques and applications

1. Basic counting techniques, double counting
2. Binomial coefficients and binomial theorem, permutations and combinations.
3. Estimating $n!$
4. Recurrence relations
5. Generating functions and its applications
Applications of generating functions

Solving recurrence relations

- Number of subsets of a set of size $n$,
- Fibonacci sequence,
- Catalan numbers: $C(n) = \frac{(2n-2)!}{n!(n-1)!} = \frac{1}{n} \binom{2n-2}{n-1}$

Some more practice problems:

1. (H.W) How many ways can a convex $n$-sided polygon be cut into triangles by adding non-intersecting diagonals (i.e., connecting vertices with non-crossing lines)? Write a recurrence and solve it!

2. (H.W) Write a recurrence for the number of derangements. That is, number of ways to arrange $n$ letters into $n$ addressed envelopes such that no letter goes to the correct envelope.
Applications of generating functions

Solving recurrence relations

- Number of subsets of a set of size \( n \),
- Fibonacci sequence,
- Catalan numbers: \( C(n) = \frac{(2n-2)!}{n!(n-1)!} = \frac{1}{n} \binom{2n-2}{n-1} \)

Some more practice problems:

1. (H.W) How many ways can a convex \( n \)-sided polygon be cut into triangles by adding non-intersecting diagonals (i.e., connecting vertices with non-crossing lines)? Write a recurrence and solve it!
Applications of generating functions

Solving recurrence relations
- Number of subsets of a set of size \( n \),
- Fibonacci sequence,
- Catalan numbers: 
  \[
  C(n) = \frac{(2n-2)!}{n!(n-1)!} = \frac{1}{n} \binom{2n-2}{n-1}
  \]

Some more practice problems:
1. (H.W) How many ways can a convex \( n \)-sided polygon be cut into triangles by adding non-intersecting diagonals (i.e., connecting vertices with non-crossing lines)? Write a recurrence and solve it!
2. (H.W) Write a recurrence for the number of derrangements. That is, no. of ways to arrange \( n \) letters into \( n \) addressed envelopes such that no letter goes to the correct envelope.
Other applications of generating functions

- What is the number of ways $a_k$ of selecting $k$ elements from an $n$ element set if repetitions are allowed?

Let $\phi(x) = \sum_{k=0}^{\infty} a_k x^k$.

Observe that $\phi(x) = (1 + x + x^2 + \ldots)^n = (1 - x)^{-n}$.

Expand this by the extended binomial theorem and compare coefficients of $x^k$.

$a_k = (-n)_k (-1)^k (n + k - 1)_k (-1)^k$.

(H.W) What if there must be $\geq 1$ element of each type?

Proving binomial identities: Show that $\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}$.

Compare coefficients of $x^n$ in $(1 + x)^n = ((1 + x)^n)^2$. 

$4$
Other applications of generating functions

- What is the number of ways $a_k$ of selecting $k$ elements from an $n$ element set if repetitions are allowed?
  
  - Let $\phi(x) = \sum_{k=0}^{\infty} a_k x^k$. 

(H.W) What if there must be $\geq 1$ element of each type?

Proving binomial identities: Show that $\sum_{n,k=0}^{\infty} (n^k)^2 = (2n^n)^n$. Compare coefficients of $x^n$ in $(1 + x)^2^n = ((1 + x)^n)^n$. 

4
Other applications of generating functions

- What is the number of ways $a_k$ of selecting $k$ elements from an $n$ element set if repetitions are allowed?
  - Let $\phi(x) = \sum_{k=0}^{\infty} a_k x^k$.
  - Observe that $\phi(x) = (1 + x + x^2 + \ldots)^n = (1 - x)^{-n}$. 

(H.W) What if there must be $\geq 1$ element of each type?

Proving binomial identities: Show that $\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}$. Compare coefficients of $x^n$ in $(1 + x + x^2 + \ldots)^2 = ((1 + x + x^2 + \ldots)^n)^2$. 

4
Other applications of generating functions

▷ What is the number of ways $a_k$ of selecting $k$ elements from an $n$ element set if repetitions are allowed?

▷ Let $\phi(x) = \sum_{k=0}^{\infty} a_k x^k$.
▷ Observe that $\phi(x) = (1 + x + x^2 + \ldots)^n = (1 - x)^{-n}$
▷ Expand this by the extended binomial theorem and compare coefficients of $x^k$. 

$\begin{align*}
\sum_{k=0}^{\infty} \binom{n+k-1}{k} &= \sum_{k=0}^{\infty} \binom{n+k-1}{n} \\
\sum_{k=0}^{n} \binom{n+k}{k} &= \binom{2n}{n}
\end{align*}$
Other applications of generating functions

- What is the number of ways \( a_k \) of selecting \( k \) elements from an \( n \) element set if repetitions are allowed?
  - Let \( \phi(x) = \sum_{k=0}^{\infty} a_k x^k \).
  - Observe that \( \phi(x) = (1 + x + x^2 + \ldots)^n = (1 - x)^{-n} \).
  - Expand this by the extended binomial theorem and compare coefficients of \( x^k \).
  - \( a_k = \binom{-n}{k}(-1)^k = (-1)^k \binom{n+k-1}{k} (-1)^k = \binom{n+k-1}{k}. \)
Other applications of generating functions

▶ What is the number of ways $a_k$ of selecting $k$ elements from an $n$ element set if repetitions are allowed?

▶ Let $\phi(x) = \sum_{k=0}^{\infty} a_k x^k$.
▶ Observe that $\phi(x) = (1 + x + x^2 + \ldots)^n = (1 - x)^{-n}$
▶ Expand this by the extended binomial theorem and compare coefficients of $x^k$.
▶ $a_k = (-n)^k (-1)^k = (-1)^k \binom{n+k-1}{k} (-1)^k = \binom{n+k-1}{k}$.
▶ (H.W) What if there must be $\geq 1$ element of each type?
Other applications of generating functions

► What is the number of ways \( a_k \) of selecting \( k \) elements from an \( n \) element set if repetitions are allowed?

► Let \( \phi(x) = \sum_{k=0}^{\infty} a_k x^k \).
► Observe that \( \phi(x) = (1 + x + x^2 + \ldots)^n = (1 - x)^{-n} \)
► Expand this by the extended binomial theorem and compare coefficients of \( x^k \).
► \( a_k = \binom{-n}{k} (-1)^k = (-1)^k \binom{n+k-1}{k}(-1)^k = \binom{n+k-1}{k} \).
► \( \text{(H.W)} \) What if there must be \( \geq 1 \) element of each type?

► Proving binomial identities: Show that \( \sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n} \).
Other applications of generating functions

- What is the number of ways \(a_k\) of selecting \(k\) elements from an \(n\) element set if repetitions are allowed?
  - Let \(\phi(x) = \sum_{k=0}^{\infty} a_k x^k\).
  - Observe that \(\phi(x) = (1 + x + x^2 + \ldots)^n = (1 - x)^{-n}\)
  - Expand this by the extended binomial theorem and compare coefficients of \(x^k\).
  - \(a_k = \binom{-n}{k} (-1)^k = (-1)^k \binom{n+k-1}{k} (-1)^k = \binom{n+k-1}{k}\).
  - (H.W) What if there must be \(\geq 1\) element of each type?

- Proving binomial identities: Show that \(\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}\).
  - Compare coefficients of \(x^n\) in \((1 + x)^{2n} = ((1 + x)^n)^2\).
Principle of Inclusion-Exclusion (PIE)

A simple example:

- If in a class \( n \) students like python, \( m \) students like C and \( k \) students who like both, and \( \ell \) like neither, then how many students are there in the class?
Principle of Inclusion-Exclusion (PIE)

A simple example:

- If in a class $n$ students like python, $m$ students like C and $k$ students who like both, and $\ell$ like neither, then how many students are there in the class?
- Of course, this also counts the no. who were too lazy to lift their hands!
Principle of Inclusion-Exclusion (PIE)

A simple example:

- If in a class $n$ students like python, $m$ students like C and $k$ students who like both, and $\ell$ like neither, then how many students are there in the class?

- Of course, this also counts the no. who were too lazy to lift their hands!

Theorem: Principle of Inclusion-Exclusion (PIE)

Let $A_1, A_2, \ldots, A_n$ be finite sets. Then,

$$|A_1 \cup \ldots \cup A_n| = \sum_{1 \leq i \leq n} |A_i| - \sum_{1 \leq i < j \leq n} |A_i \cap A_j|$$

$$+ \sum_{1 \leq i < j < k \leq n} |A_i \cap A_j \cap A_k| - \ldots + (-1)^{n+1}|A_1 \cap \ldots \cap A_n|$$
Number of surjections

- How many surjections are there from \([n] = \{1, \ldots, n\}\) to \([m] = \{1, \ldots, m\}\)?
Number of surjections

- How many surjections are there from $[n] = \{1, \ldots, n\}$ to $[m] = \{1, \ldots, m\}$?
- $\#$ surjections $= \text{total } \#\text{functions} - \text{those that miss some element in range}$. 

Number of surjections

- How many surjections are there from \([n] = \{1, \ldots, n\}\) to \([m] = \{1, \ldots, m\}\)?

- \# surjections = total \# functions - those that miss some element in range.

- Let \(A_i = \{f : [n] \to [m] \mid i \not\in \text{Range}(f)\}\)

- Then, \# surjections = \(m^n - |\bigcup_{i \in [m]} A_i|\).
Number of surjections

- How many surjections are there from \([n] = \{1, \ldots, n\}\) to \([m] = \{1, \ldots, m\}\)?
- \# surjections = total \# functions - those that miss some element in range.
- Let \(A_i = \{f : [n] \to [m] \mid i \not\in \text{Range}(f)\}\)
- Then, \# surjections = \(m^n - |\cup_{i \in [m]} A_i|\).

**Theorem: Principle of Inclusion-Exclusion (PIE)**

Let \(A_1, A_2, \ldots, A_n\) be finite sets. Then,

\[
|A_1 \cup \ldots \cup A_n| = \sum_{1 \leq i \leq n} |A_i| - \sum_{1 \leq i < j \leq n} |A_i \cap A_j| + \sum_{1 \leq i < j < k \leq n} |A_i \cap A_j \cap A_k| - \ldots + (-1)^{n+1}|A_1 \cap \ldots \cap A_n|
\]
Number of surjections

- How many surjections are there from \([n] = \{1, \ldots, n\}\) to \([m] = \{1, \ldots, m\}\)?

- \# surjections = total \#functions - those that miss some element in range.

- Let \(A_i = \{f : [n] \rightarrow [m] \mid i \not\in \text{Range}(f)\}\)

- Then, \# surjections = \(m^n - |\bigcup_{i \in [m]} A_i|\).

- \(|\bigcup_{i \in [m]} A_i| = \sum_{1 \leq i \leq m} |A_i| - \sum_{1 \leq i < j \leq m} |A_i \cap A_j| + \sum_{1 \leq i < j < k \leq m} |A_i \cap A_j \cap A_k| - \ldots + (-1)^{m+1}|A_1 \cap \ldots \cap A_m|\)
Number of surjections

- How many surjections are there from \([n] = \{1, \ldots, n\}\) to \([m] = \{1, \ldots, m\}\)?

- \# surjections = total \#functions - those that miss some element in range.

- Let \(A_i = \{f : [n] \to [m] \mid i \not\in \text{Range}(f)\}\)

- Then, \# surjections = \(m^n - |\bigcup_{i \in [m]} A_i|\).

- \( |\bigcup_{i \in [m]} A_i| = \sum_{1 \leq i \leq m} |A_i| - \sum_{1 \leq i < j \leq m} |A_i \cap A_j| + \sum_{1 \leq i < j < k \leq m} |A_i \cap A_j \cap A_k| - \ldots + (-1)^{m+1} |A_1 \cap \ldots \cap A_m| \)

- But now what is \(|A_i|, |A_i \cap A_j|, |A_i \cap A_j \cap A_k|, \ldots|\)?
Number of surjections

- How many surjections are there from \([n] = \{1, \ldots, n\}\) to \([m] = \{1, \ldots, m\}\)?

- # surjections = total #functions - those that miss some element in range.

- Let \(A_i = \{f : [n] \to [m] \mid i \not\in \text{Range}(f)\}\).

- Then, # surjections = \(m^n - |\bigcup_{i \in [m]} A_i|\).

- \(|\bigcup_{i \in [m]} A_i| = \sum_{1 \leq i \leq m} |A_i| - \sum_{1 \leq i < j \leq m} |A_i \cap A_j| + \sum_{1 \leq i < j < k \leq m} |A_i \cap A_j \cap A_k| - \ldots + (-1)^{m+1}|A_1 \cap \ldots \cap A_m|\)

- But now what is \(|A_i|, |A_i \cap A_j|, |A_i \cap A_j \cap A_k|, \ldots|\)?

- \(|A_i| = (m - 1)^n, |A_i \cap A_j| = (m - 2)^n\)...
Number of surjections

- How many surjections are there from \([n] = \{1, \ldots, n\}\) to \([m] = \{1, \ldots, m\}\)?
- \(\#\) surjections = total \(\#\) functions - those that miss some element in range.
- Let \(A_i = \{f : [n] \to [m] \mid i \not\in \text{Range}(f)\}\)
- Then, \(\#\) surjections = \(m^n - |\bigcup_{i \in [m]} A_i|\).

\[|\bigcup_{i \in [m]} A_i| = \sum_{1 \leq i \leq m} |A_i| - \sum_{1 \leq i < j \leq m} |A_i \cap A_j| + \sum_{1 \leq i < j < k \leq m} |A_i \cap A_j \cap A_k| - \ldots + (-1)^{m+1}|A_1 \cap \ldots \cap A_m|\]
- But now what is \(|A_i|, |A_i \cap A_j|, |A_i \cap A_j \cap A_k|, \ldots|\)?
- \(|A_i| = (m - 1)^n, |A_i \cap A_j| = (m - 2)^n\...\)
- What about the summation? terms \(1 \leq i < j \leq m = \)
Number of surjections

- How many surjections are there from $[n] = \{1, \ldots, n\}$ to $[m] = \{1, \ldots, m\}$?
- $\#$ surjections = total $\#$functions - those that miss some element in range.
- Let $A_i = \{f : [n] \rightarrow [m] \mid i \not\in \text{Range}(f)\}$
- Then, $\#$ surjections = $m^n - |\cup_{i \in [m]} A_i|.$

$$|\cup_{i \in [m]} A_i| = \sum_{1 \leq i \leq m} |A_i| - \sum_{1 \leq i < j \leq m} |A_i \cap A_j| + \sum_{1 \leq i < j < k \leq m} |A_i \cap A_j \cap A_k| - \ldots + (-1)^{m+1}|A_1 \cap \ldots \cap A_m|$$
- But now what is $|A_i|, |A_i \cap A_j|, |A_i \cap A_j \cap A_k|, \ldots$?
- $|A_i| = (m - 1)^n, |A_i \cap A_j| = (m - 2)^n$
- What about the summation? terms $1 \leq i < j \leq m = \binom{m}{2}$

Thus, we have $\#$ surjections from $[n]$ to $[m] =$
$$m^n - \binom{m}{1}(m - 1)^n + \binom{m}{2}(m - 2)^n - \ldots + (-1)^{m-1}\binom{m}{m-1} \cdot 1^n.$$
Proof of PIE

Theorem: Principle of Inclusion-Exclusion (PIE)

Let $A_1, A_2, \ldots, A_n$ be finite sets. Then,

$$|A_1 \cup \ldots \cup A_n| = \sum_{1 \leq i \leq n} |A_i| - \sum_{1 \leq i < j \leq n} |A_i \cap A_j|$$

$$+ \sum_{1 \leq i < j < k \leq n} |A_i \cap A_j \cap A_k| - \ldots + (-1)^{n+1}|A_1 \cap \ldots \cap A_n|$$

Proof: (H.W): Prove PIE by induction.
Proof of PIE

Theorem: Principle of Inclusion-Exclusion (PIE)

Let $A_1, A_2, \ldots, A_n$ be finite sets. Then,

$$|A_1 \cup \ldots \cup A_n| = \sum_{1 \leq i \leq n} |A_i| - \sum_{1 \leq i < j \leq n} |A_i \cap A_j|$$

$$+ \sum_{1 \leq i < j < k \leq n} |A_i \cap A_j \cap A_k| - \ldots + (-1)^{n+1}|A_1 \cap \ldots \cap A_n|$$

Proof:

- We will show that each element in the union is counted exactly once in the r.h.s
Proof of PIE

Theorem: Principle of Inclusion-Exclusion (PIE)

Let $A_1, A_2, \ldots, A_n$ be finite sets. Then,

$$|A_1 \cup \ldots \cup A_n| = \sum_{1 \leq i \leq n} |A_i| - \sum_{1 \leq i < j \leq n} |A_i \cap A_j|$$

$$+ \sum_{1 \leq i < j < k \leq n} |A_i \cap A_j \cap A_k| - \ldots + (-1)^{n+1}|A_1 \cap \ldots \cap A_n|$$

Proof:

- We will show that each element in the union is counted exactly once in the r.h.s.
- Let $a$ belong to exactly $r$ of the sets $A_1, \ldots, A_n$. 

Proof of PIE

Theorem: Principle of Inclusion-Exclusion (PIE)

Let $A_1, A_2, \ldots, A_n$ be finite sets. Then,

$$|A_1 \cup \ldots \cup A_n| = \sum_{1 \leq i \leq n} |A_i| - \sum_{1 \leq i < j \leq n} |A_i \cap A_j| + \sum_{1 \leq i < j < k \leq n} |A_i \cap A_j \cap A_k| - \ldots + (-1)^{n+1}|A_1 \cap \ldots \cap A_n|$$

Proof:

- We will show that each element in the union is counted exactly once in the r.h.s.
- Let $a$ belong to exactly $r$ of the sets $A_1, \ldots, A_n$.
- Then $a$ is counted $\binom{r}{1}$ times by $\sum |A_i|$, etc.
Proof of PIE

Theorem: Principle of Inclusion-Exclusion (PIE)

Let $A_1, A_2, \ldots, A_n$ be finite sets. Then,

$$|A_1 \cup \ldots \cup A_n| = \sum_{1 \leq i \leq n} |A_i| - \sum_{1 \leq i < j \leq n} |A_i \cap A_j|$$

$$+ \sum_{1 \leq i < j < k \leq n} |A_i \cap A_j \cap A_k| - \ldots + (-1)^{n+1}|A_1 \cap \ldots \cap A_n|$$

Proof:

- We will show that each element in the union is counted exactly once in the r.h.s.
- Let $a$ belong to exactly $r$ of the sets $A_1, \ldots, A_n$.
- Then $a$ is counted $\binom{r}{1}$ times by $\sum |A_i|$, etc.
- Thus, overall count $= \binom{r}{1} - \binom{r}{2} + \ldots (-1)^{r+1}\binom{r}{r}$. 

What is this number?! $= 1!$
Proof of PIE

Theorem: Principle of Inclusion-Exclusion (PIE)

Let $A_1, A_2, \ldots, A_n$ be finite sets. Then,

$$|A_1 \cup \ldots \cup A_n| = \sum_{1 \leq i \leq n} |A_i| - \sum_{1 \leq i < j \leq n} |A_i \cap A_j| + \sum_{1 \leq i < j < k \leq n} |A_i \cap A_j \cap A_k| - \ldots + (-1)^{n+1}|A_1 \cap \ldots \cap A_n|$$

Proof:

- We will show that each element in the union is counted exactly once in the r.h.s.
- Let $a$ belong to exactly $r$ of the sets $A_1, \ldots, A_n$.
- Then $a$ is counted $\binom{r}{1}$ times by $\sum |A_i|$, etc.
- Thus, overall count $= \binom{r}{1} - \binom{r}{2} + \ldots (-1)^{r+1}\binom{r}{r}$.
- What is this number?!
Proof of PIE

Theorem: Principle of Inclusion-Exclusion (PIE)

Let $A_1, A_2, \ldots, A_n$ be finite sets. Then,

$$|A_1 \cup \ldots \cup A_n| = \sum_{1 \leq i \leq n} |A_i| - \sum_{1 \leq i < j \leq n} |A_i \cap A_j| + \sum_{1 \leq i < j < k \leq n} |A_i \cap A_j \cap A_k| - \ldots + (-1)^{n+1}|A_1 \cap \ldots \cap A_n|$$

Proof:

- We will show that each element in the union is counted exactly once in the r.h.s.
- Let $a$ belong to exactly $r$ of the sets $A_1, \ldots, A_n$.
- Then $a$ is counted $\binom{r}{1}$ times by $\sum |A_i|$, etc.
- Thus, overall count $= \binom{r}{1} - \binom{r}{2} + \ldots (-1)^{r+1}\binom{r}{r}$.
- What is this number?! $=1!$
Applications of PIE

- How many integral solutions does $x_1 + x_2 + x_3 = 11$ have where $0 \leq x_1 \leq 3, 0 \leq x_2 \leq 4, 0 \leq x_3 \leq 6$?
- Number of derangements of a set with $n$ elements
  - That is, no. of ways to arrange $n$ letters into $n$ addressed envelopes such that no letter goes to the correct envelope.
Number of derangements

Formally, a derangement is a permutation of objects that leaves no object in its original position.
Number of derangements

Formally, a derangement is a permutation of objects that leaves no object in its original position.

Theorem

Let $D_n$ denote the number of derangements of $[n]$.

$$D_n = n! \sum_{i=0}^{n} \frac{(-1)^i}{i!}$$
Number of derangements

Theorem

Let $D_n$ denote the number of derangements of $[n]$.

$$D_n = n! \sum_{i=0}^{n} \frac{(-1)^i}{i!}$$

- $D_n = (\text{total \# permutations of } [n]) - (\text{\# permutations of } [n] \text{ that fix at least 1 element})$
Number of derangements

Theorem

Let $D_n$ denote the number of derangements of $[n]$.

$$D_n = n! \sum_{i=0}^{n} \frac{(-1)^i}{i!}$$

- $D_n = \text{(total # permutations of } [n]) - \text{(# permutations of [n] that fix at least 1 element)}$
- Apply PIE on latter term, lets call it $P$, let $P(i, j)$ denote permutations which fix $i, j$ and so on.
Number of derangements

Theorem

Let $D_n$ denote the number of derangements of $[n]$.

$$D_n = n! \sum_{i=0}^{n} \frac{(-1)^i}{i!}$$

- $D_n = (\text{total } \# \text{ permutations of } [n]) - (\# \text{ permutations of } [n] \text{ that fix at least } 1 \text{ element})$
- Apply PIE on latter term, lets call it $P$, let $P(i, j)$ denote permutations which fix $i, j$ and so on.
- $P = \sum_{1 \leq i \leq n} P(i) - \sum_{1 \leq i < j \leq n} P(i, j) + \ldots + (-1)^n P(1, \ldots, n)$. 


Number of derangements

**Theorem**

Let $D_n$ denote the number of derangements of $[n]$.

$$D_n = n! \sum_{i=0}^{n} \frac{(-1)^i}{i!}$$

- $D_n = \text{(total # permutations of } [n]) - \text{(# permutations of } [n] \text{ that fix at least 1 element)}$
- Apply PIE on latter term, lets call it $P$, let $P(i, j)$ denote permutations which fix $i, j$ and so on.
- $P = \sum_{1 \leq i \leq n} P(i) - \sum_{1 \leq i < j \leq n} P(i, j) + \ldots + (-1)^n P(1, \ldots, n)$.
- But $P(i) = (n - 1)!$, $P(i, j) = (n - 2)!$, $\ldots$
Theorem

Let $D_n$ denote the number of derangements of $[n]$.

$$D_n = n! \sum_{i=0}^{n} \frac{(-1)^i}{i!}$$

- $D_n = ($total # permutations of $[n]$) - (# permutations of $[n]$ that fix at least 1 element)
- Apply PIE on latter term, let $P$, let $P(i, j)$ denote permutations which fix $i, j$ and so on.
- $P = \sum_{1 \leq i \leq n} P(i) - \sum_{1 \leq i < j \leq n} P(i, j) + \ldots + (-1)^n P(1, \ldots, n)$.
- But $P(i) = (n - 1)!$, $P(i, j) = (n - 2)!$, \ldots
- $P = \binom{n}{1} (n - 1)! - \binom{n}{2} (n - 2)! \ldots + (-1)^{n+1} \binom{n}{n} (n - n)!$
Number of derangements

**Theorem**

Let $D_n$ denote the number of derangements of $[n]$.

$$D_n = n! \sum_{i=0}^{n} \frac{(-1)^i}{i!}$$

- $D_n = \text{(total \# permutations of } [n]) - \text{(\# permutations of } [n] \text{ that fix at least 1 element})$
- Apply PIE on latter term, lets call it $P$, let $P(i, j)$ denote permutations which fix $i, j$ and so on.
- $P = \sum_{1 \leq i \leq n} P(i) - \sum_{1 \leq i < j \leq n} P(i, j) + \ldots + (-1)^n P(1, \ldots, n)$.
- But $P(i) = (n - 1)!$, $P(i, j) = (n - 2)!$, $\ldots$
- $P = \binom{n}{1}(n - 1)! - \binom{n}{2}(n - 2)! \ldots + (-1)^{n+1} \binom{n}{n}(n - n)!$
- Thus, $D_n = n!(1 - \frac{1}{1!} + \frac{1}{2!} \ldots + (-1)^n \frac{1}{n!})$
Thus we have,

**Theorem**

Let $D_n$ denote the number of derangements of $[n]$.

$$D_n = n! \sum_{i=0}^{n} \frac{(-1)^i}{i!}$$
Number of derangements

Thus we have,

**Theorem**

Let $D_n$ denote the number of derangements of $[n]$. 

\[ D_n = n! \sum_{i=0}^{n} \frac{(-1)^i}{i!} \]

- Now it is easy to see that $\lim_{n \to \infty} \frac{D_n}{n!} = \frac{1}{e}$.
- In other words, $\forall \delta > 0, \exists N_\delta \in \mathbb{N}$, such that for all $n > N_\delta$, 
  \[ |\frac{D(n)}{n!} - \frac{1}{e}| \leq \delta. \]
Number of derangements

Thus we have,

**Theorem**

Let \( D_n \) denote the number of derangements of \([n]\).

\[
D_n = n! \sum_{i=0}^{n} \frac{(-1)^i}{i!}
\]

- Now it is easy to see that \( \lim_{n \to \infty} \frac{D_n}{n!} = \frac{1}{e} \).
- In other words, \( \forall \delta > 0, \exists N_\delta \in \mathbb{N} \), such that for all \( n > N_\delta \),
  \[ |\frac{D(n)}{n!} - \frac{1}{e}| \leq \delta. \]
- (H.W.) Prove this by using the Taylor’s expansion for \( e^{-1} \).