CS 207: Discrete Structures

Lecture 19 – Counting and Combinatorics
Pigeon-Hole Principle

Sept 01 2016
Topics in Combinatorics

Basic counting techniques and applications

1. Basic counting techniques, double counting
2. Binomial theorem, permutations and combinations, Estimating $n!$
3. Recurrence relations and generating functions
4. Principle of Inclusion-Exclusion (PIE) and its applications.
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2. Binomial theorem, permutations and combinations, Estimating $n!$
3. Recurrence relations and generating functions
4. Principle of Inclusion-Exclusion (PIE) and its applications.
    ▶ Counting the number of surjections on $[n]$.
    ▶ Combinatorial proof of PIE
    ▶ Number of derangements $\sim \frac{1}{e}$. 

Topics in Combinatorics
Theorem: Principle of Inclusion-Exclusion (PIE)

Let $A_1, A_2, \ldots, A_n$ be finite sets. Then,

$$|A_1 \cup \ldots \cup A_n| = \sum_{1 \leq i \leq n} |A_i| - \sum_{1 \leq i < j \leq n} |A_i \cap A_j| + \sum_{1 \leq i < j < k \leq n} |A_i \cap A_j \cap A_k| - \ldots + (-1)^{n+1}|A_1 \cap \ldots \cap A_n|$$
This lecture

Pigeon-Hole Principle (PHP) and its applications

A simple formulation

Let \( k \in \mathbb{N} \). If \( k + 1 \) (or more) objects are to be placed in \( k \) boxes, then at least one box will have 2 (or more) objects.

How do you prove it?

A simple corollary

▶ Can a function from a set of \( k + 1 \) or more elements to a set with \( k \) elements be injective?
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Another simple application

For every \( n \in \mathbb{Z}^+ \), there exists a multiple of \( n \) whose decimal expansion only has 0's and 1's.
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- Consider $n + 1$ integers $k_1 = 1$, $k_2 = 11$, $k_3 = 111$, $\ldots$, $k_{n+1} = 1\ldots1$ (with $n+1$ 1's).
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- So among the $n + 1$ integers, by PHP, at least 2 must have the same remainder.
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- When any integer is divided by \( n \), the remainder can be either 0, 1, \ldots, \( n - 1 \), i.e., \( n \) choices.
- So among the \( n + 1 \) integers, by PHP, at least 2 must have the same remainder.
- That is, \( \exists i, j \), \( k_i = pn + d, k_j = qn + d \).
- But then \( |k_i - k_j| \) is a multiple of \( n \) and its decimal expansion only has 0′s and 1′s.
A (slightly) more general PHP

Pigeon-Hole Principle (Variant 1)

If $N$ objects are placed into $k$ boxes then there is at least one box with at least $\lceil N/k \rceil$ objects.

Suppose not. Then each box has strictly less than $\lfloor N/k \rfloor$ objects. Therefore, totally there can be strictly less than $N$ objects, which is a contradiction.

A simple example

How many cards must be selected from a pack of 52 cards so that at least three cards of the same suit are chosen?

- Four boxes, one for each suit; each selected card is put in one.
- If $N$ cards are selected then at least 1 box has $\lceil N/4 \rceil$ cards.
- To have $\geq 3$ cards from same suit, suffices $\lceil N/4 \rceil \geq 3$.
- Thus, $N = 9$.

But can we do better?

No.
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Another application of PHP

**Question**: Give a sequence of 10 real numbers with no subsequence of length 4 which is increasing or decreasing.
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**Theorem**

Every sequence of $n^2 + 1$ distinct real numbers contains a subsequence of length $n + 1$ which is either increasing or decreasing.
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1. Let $a_1, \ldots, a_{n^2+1}$ be a sequence of distinct real numbers.
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1. Let $a_1, \ldots, a_{n^2 + 1}$ be a sequence of distinct real numbers.
2. For each $k \in \{1 \ldots n^2 + 1\}$, let $(i_k, d_k)$ denote a pair:
   - $i_k =$ length of longest increasing subsequence starting from $a_k$
   - $d_k =$ length of longest decreasing subsequence starting from $a_k$
3. Suppose, there are no increasing/decreasing subsequences of length $n + 1$. Then $\forall k$, $i_k \leq n$ and $d_k \leq n$.
4. $\therefore$ by PHP, $\exists \ell, m, 1 \leq \ell < m \leq n^2 + 1$ s.t. $(i_\ell, d_\ell) = (i_m, d_m)$
5. We will show that this is not possible:
   - Case 1: $a_\ell < a_m$. Then $i_m \geq i_\ell + 1$, a contradiction.
   - Case 2: $a_\ell > a_m$. Then $d_\ell \geq d_m + 1$, a contradiction.
6. All $a_i$'s are distinct so this completes the proof.
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Another variant of PHP

**Theorem (PHP Variant 2)**

Suppose there are \( n \geq 1 + r(\ell - 1) \) objects which are colored with \( r \) different colors. Then there exist \( \ell \) objects all with the same color.
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Proof: (H.W)
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Suppose there are $n \geq 1 + r(\ell - 1)$ objects which are colored with $r$ different colors. Then there exist $\ell$ objects all with the same color.

Proof: (H.W)

- Is this coloring optimal?
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Proof: (H.W)

- Is this coloring optimal?
- That is, if fewer than \( 1 + r(\ell - 1) \) objects are given, is there a way of coloring them such that no \( \ell \) have the same color?
Back to the coloring game

The coloring game

- There are six points on board and two colored chalks.
The coloring game

- There are six points on board and two colored chalks.
- Divide class into 2 groups. Group 1 draws white lines and Group 2 draws blue lines between points.

Lemma: Any 2-coloring of edges of a graph on 6 nodes has a monochromatic triangle.

2-coloring of edges: coloring all edges of the graph using atmost 2 colors.

monochromatic (triangle): all edges have the same color.
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We will now show that this is impossible. That is, Lemma Any 2-coloring of edges of a graph on 6 nodes has a monochromatic triangle.

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Proof:
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Lemma

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Proof:

- Let 1, \ldots, 6 be the points, and red/blue the colors.
- Consider the edges 16, 26, 36, 46, 56.
- By PHP at least 3 of them must be same color, say 16, 26, 36 are red.
- Now there are two possibilities:
  - Either one of 12, 23, 31 is red (then we have a red triangle).
  - Else none of them are red, implies 123 is a blue triangle.  

\qed
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- What if there were 5 or lesser nodes?
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- Mathematical proofs and structures
- Counting and Combinatorics
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  - Propositions, proof techniques: contradiction, contrapositive
  - Induction: strong induction, well-ordering principle
  - Sets: finite and infinite sets, countable and uncountable sets
  - Functions: bijections (from e.g., \( \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N} \)), injections and surjections, Cantor’s diagonalization technique
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  - Principle of Inclusion-Exclusion (PIE) and its applications.
  - Pigeon-Hole Principle (PHP) and its applications.