CS 207: Discrete Structures

Lecture 21 – Counting, Combinatorics and Graph Theory
A glimpse of Ramsey Theory

Sept 19 2016
Non-trivial applications of the Pigeon-Hole Principle (PHP)
Edge coloring problems

Non-trivial applications of the Pigeon-Hole Principle (PHP)

Results we saw last class

1. Any 2-coloring of a graph on 6 nodes has either a red triangle or a blue triangle.
   ▶ 6 is the optimal such number. Thus, \( R(3, 3) = 6 \).
Edge coloring problems

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2. Any 2-coloring of a graph on 10 nodes has either a red triangle or a blue complete graph on 4 nodes.
### Edge coloring problems

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### Edge coloring problems

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| 1. Any 2-coloring of a graph on 6 nodes has either a **red triangle** or a **blue triangle**.  
  ▶ 6 is the optimal such number. Thus, \( R(3, 3) = 6 \).
| 2. Any 2-coloring of a graph on 10 nodes has either a **red triangle** or a **blue complete graph** on 4 nodes.
| 3. Any 2-coloring of a graph on 9 nodes has either a **red triangle** or a **blue complete graph** on 4 nodes.  
  ▶ Is 9 the optimal such number? \( R(3, 4) \leq 9 \).  
  ▶ (H.W?) Prove that \( R(3, 4) = 9! \)
| 4. Any 2-coloring of a graph on 18 nodes has a **monochromatic complete graph** on 4 nodes. |
Mixing counting and combinatorics with graph theory

Today’s class

Generalizing the coloring game
An introduction to Ramsey theory.
Ramsey’s theorem

Recall:

Definition

For $k, \ell \in \mathbb{N}$, $R(k, \ell)$ denotes the minimum number of nodes such that any 2-coloring of a (complete) graph on $R(k, \ell)$ nodes has

- either, a complete graph on $k$-nodes with all red edges
- or, a complete graph on $\ell$-nodes with all blue edges
Ramsey’s theorem

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Ramsey’s theorem (simplified version)

For all \( k, \ell \in \mathbb{N} \), \( R(k, \ell) \) exists, i.e., it is finite.
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**Ramsey’s theorem (simplified version)**

For all \( k, \ell \in \mathbb{N} \), \( R(k, \ell) \) exists, i.e., it is finite. In fact,

\[
R(k, \ell) \leq \binom{k + \ell - 2}{k - 1}
\]
Ramsey theory: A search for order in disorder!

Every structure no matter how disordered must contain some regular sub-part!

E.g., any 2-coloring on a complete graph of 10 nodes contains either a complete graph of 3 nodes of one color or a complete graph of 4 nodes of the other color.
Ramsey theory: A search for order in disorder!

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▶ Suppose in a group of people any two are friends or enemies.
Ramsey theory: A search for order in disorder!

Every structure no matter how disordered must contain some regular sub-part!

E.g., any 2-coloring on a complete graph of 10 nodes contains either a complete graph of 3 nodes of one color or a complete graph of 4 nodes of the other color.

- Suppose in a group of people any two are friends or enemies.
- In any set of 10 people there must be either 3 mutual friends or 4 mutual enemies.
Proof of Ramsey’s theorem

- What is $R(n, 2) = R(2, n)$?
Proof of Ramsey’s theorem

- What is $R(n, 2) = R(2, n)$?
- What is $R(1, 1)$? $R(n, 1) = R(1, n)$?
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For all integers $k, \ell \geq 2$, $R(k, \ell)$ is finite.
Proof of Ramsey’s theorem

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For all integers $k, \ell \geq 2$, $R(k, \ell)$ is finite.

Proof:
- By strong induction on $k + \ell$. 
Proof of Ramsey’s theorem

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Proof:
- By strong induction on $k + \ell$.
- Base case: $R(2, 2) = 2$. 

Proof of Ramsey’s theorem

- What is $R(n, 2) = R(2, n)$?
- What is $R(1, 1)$? $R(n, 1) = R(1, n)$?

For all integers $k, \ell \geq 2$, $R(k, \ell)$ is finite.

Proof:

- By strong induction on $k + \ell$.
- Base case: $R(2, 2) = 2$.
- Suppose it is true for all $k, \ell$ such that $k + \ell < N$. We will show that $R(k, \ell)$ is finite by showing

\[ R(k, \ell) \leq R(k - 1, \ell) + R(k, \ell - 1) \]

where $R(k - 1, \ell)$ and $R(k, \ell - 1)$ exist by induction hypothesis since $k + \ell - 1 < N$. 
Proof of Ramsey’s theorem contd.

By ind hyp assume that $R(k-1, \ell)$ and $R(k, \ell - 1)$ exist. Then,

Claim: $R(k, \ell) \leq R(k-1, \ell) + R(k, \ell - 1)$
Proof of Ramsey’s theorem contd.

By ind hyp assume that $R(k - 1, \ell)$ and $R(k, \ell - 1)$ exist. Then,

Claim: $R(k, \ell) \leq R(k - 1, \ell) + R(k, \ell - 1)$

i.e., given a 2-colored complete graph with $R(k - 1, \ell) + R(k, \ell - 1)$ nodes, it has either a complete red graph with $k$ nodes or a complete blue graph with $\ell$ nodes.
Proof of Ramsey’s theorem contd.

By ind hyp assume that $R(k - 1, \ell)$ and $R(k, \ell - 1)$ exist. Then,

Claim: $R(k, \ell) \leq R(k - 1, \ell) + R(k, \ell - 1)$

Consider complete graph with $R(k - 1, \ell) + R(k, \ell - 1)$ nodes.
Proof of Ramsey’s theorem contd.

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Consider complete graph with $R(k-1, \ell) + R(k, \ell-1)$ nodes.

Clearly $M + N = R(k-1, \ell) + R(k, \ell-1) - 1$. 
Proof of Ramsey’s theorem contd.

By ind hyp assume that \( R(k - 1, \ell) \) and \( R(k, \ell - 1) \) exist. Then,

**Claim:** \( R(k, \ell) \leq R(k - 1, \ell) + R(k, \ell - 1) \)

Consider complete graph with \( R(k - 1, \ell) + R(k, \ell - 1) \) nodes.

\[
\begin{align*}
R(k - 1, \ell) + R(k, \ell - 1) - 1
\end{align*}
\]

- Clearly \( M + N = R(k - 1, \ell) + R(k, \ell - 1) - 1 \).
- By PHP, either \( M \geq R(k - 1, \ell) \) or \( N \geq R(k, \ell - 1) \).
Proof of Ramsey’s theorem contd.

By ind hyp assume that $R(k-1, \ell)$ and $R(k, \ell-1)$ exist. Then,

**Claim:** $R(k, \ell) \leq R(k-1, \ell) + R(k, \ell-1)$

Consider complete graph with $R(k-1, \ell) + R(k, \ell-1)$ nodes.

- **Case 1:** $M \geq R(k-1, \ell)$. 
Proof of Ramsey’s theorem contd.

By ind hyp assume that $R(k - 1, \ell)$ and $R(k, \ell - 1)$ exist. Then,

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Consider complete graph with $R(k - 1, \ell) + R(k, \ell - 1)$ nodes.

- Case 1: $M \geq R(k - 1, \ell)$. Either complete blue graph on $\ell$ nodes
Proof of Ramsey’s theorem contd.

By ind hyp assume that $R(k-1, \ell)$ and $R(k, \ell-1)$ exist. Then,

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Consider complete graph with $R(k-1, \ell) + R(k, \ell-1)$ nodes.

▶ Case 1: $M \geq R(k-1, \ell)$. Either complete blue graph on $\ell$ nodes or complete red graph on $k-1$ nodes + $x$
Proof of Ramsey’s theorem contd.

By ind hyp assume that $R(k-1, \ell)$ and $R(k, \ell - 1)$ exist. Then,

**Claim:** $R(k, \ell) \leq R(k-1, \ell) + R(k, \ell - 1)$

Consider complete graph with $R(k-1, \ell) + R(k, \ell - 1)$ nodes.

- **Case 1:** $M \geq R(k-1, \ell)$. ✓
- **Case 2:** $N \geq R(k, \ell - 1)$ leads to same argument. (Do it!) ✓
Proof of Ramsey’s theorem contd.

By ind hyp assume that $R(k - 1, \ell)$ and $R(k, \ell - 1)$ exist. Then, 

Claim: $R(k, \ell) \leq R(k - 1, \ell) + R(k, \ell - 1)$

Consider complete graph with $R(k - 1, \ell) + R(k, \ell - 1)$ nodes.

Thus in all cases, we have $R(k, \ell) \leq R(k - 1, \ell) + R(k, \ell - 1)$. \qed
Proof of Ramsey’s theorem

Ramsey’s theorem (simplified version)

For all $k, \ell \geq 2$, $R(k, \ell)$ exists, i.e., it is finite. Further,

$$R(k, \ell) \leq \binom{k + \ell - 2}{k - 1}$$

Proof:
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Proof: Now, this should be trivial!
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- By induction on $k + \ell$ as before.
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- Base case for \( k = \ell = 2 \) is done.
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Proof:

- By induction on \( k + \ell \) as before.
- Base case for \( k = \ell = 2 \) is done.
- By what we just showed and induction hypothesis we have:

\[
R(k, \ell) \leq R(k - 1, \ell) + R(k, \ell - 1) \\
\leq \binom{k + \ell - 3}{k - 2} + \binom{k + \ell - 3}{k - 1} = \binom{k + \ell - 2}{k - 1}
\]
Ramsey theory

Some interesting facts

- The general Ramsey theorem extends this to any finite number of colors (not just 2).
- Several applications, vast research area!
- Exact values are known only for 6 or so entries: \( R(3, 3) = 6, \ R(3, 4) = 9, \ R(4, 4) = 18, \ldots \ R(3, 8) = 28 \) or 29...
- Only bounds are known for rest. (see wiki on this...)
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▶ What about lower bounds?
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So how hard is it? Paul Erdős is supposed to have said:

*Suppose an evil alien would tell mankind “Either you tell me the value of $R(5, 5)$ or I will exterminate the human race.” … It would be best to try to compute it, both by mathematics and with a computer. If he would ask for the value of $R(6, 6)$, the best thing would be to destroy him before he destroys us, because we couldn’t.*