CS 207: Discrete Structures

Graph theory
Bipartite graphs and their characterization, subgraphs

Lecture 24
Sept 22 2016
Topic 3: Graph theory

Topics covered in the last two lectures and this one:

▷ What is a Graph?
▷ Paths, cycles, walks and trails; connected graphs.
▷ Eulerian graphs and a characterization in terms of degrees of vertices.
▷ Bipartite graphs and a characterization.

Reference: Section 1.1, 1.2 of Chapter 1 from Douglas West.
Another application of Eulerian graphs

If we want to draw a given connected graph $G$ on paper, how many times must we stop and move the pen? No segment should be drawn twice.

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Theorem

For a connected graph with $|V| > 1$ and exactly $2k$ odd vertices, the minimum number of trails that decompose it is $\max\{k, 1\}$. 
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Proof idea: We will show that (i) at least these many trails are required and (ii) these many trails suffice.

- A trail touches each vertex an even no. of times, except if the trail is not closed, then the endpoints are touched odd no. of times
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- Each trail has only 2 ends implies we use at least $k$ trails to satisfy $2k$ odd vertices.
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- We need at least one trail since $G$ has an edge.
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- i.e., if we partition $G$ into trails, each odd vertex in $G$ must have a non-closed walk starting or ending at it.
- Each trail has only 2 ends implies we use at least $k$ trails to satisfy $2k$ odd vertices.
- We need at least one trail since $G$ has an edge.
- Thus, we have shown that at least $\max\{k, 1\}$ trails are required.
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- If $k = 0$, one trail suffices (i.e., an Eulerian walk by previous Thm)
Theorem

For a connected graph with $|V| > 1$ and exactly $2k$ odd vertices, the minimum number of trails that decompose it is $\max\{k, 1\}$.

Proof idea: We will show that (i) at least these many trails are required and (ii) these many trails suffice.

- If $k = 0$, one trail suffices (i.e., an Eulerian walk by previous Thm)
- If $k > 0$ we need to prove that $k$ trails suffice.
  - Pair up odd vertices in $G$ (in any order) and form $G'$ by adding an edge between them.
  - $G'$ is connected, by previous Thm has an Eulerian walk $C$.
  - Traverse $C$ in $G'$ and for each time we cross an edge of $G'$ not in $G$, start a new trail (lift pen!).
  - Thus, we get $k$ trails decomposing $G$. □
Some simple types of Graphs

- We have already seen some: connected graphs.
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- paths, cycles.
Some simple types of Graphs

- We have already seen some: connected graphs.
- paths, cycles.
- Are there other interesting classes of graphs?
Bipartite graphs

Definition

A graph is called bipartite, if the vertices of the graph can be partitioned into $V = X \cup Y$, $X \cap Y = \emptyset$ s.t., $\forall e = (u, v) \in E$,

- either $u \in X$ and $v \in Y$
- or $v \in X$ and $u \in Y$

Example: $m$ jobs and $n$ people, $k$ courses and $\ell$ students.

- How can we check if a graph is bipartite?
- Can we characterize bipartite graphs?
Characterizing bipartite graphs using cycles.

- Recall: A path or a cycle has length $n$ if the number of edges in it is $n$.
- A path (or cycle) is called odd (or even) if its length is odd (or even, respectively).

Lemma

Every closed odd walk contains an odd cycle.

Proof: By induction on the length of the given closed odd walk.

Exercise!
Characterizing bipartite graphs using cycles.

Lemma
Every closed odd walk contains an odd cycle.

Theorem, Konig, 1936
A graph is bipartite iff it has no odd cycle.

Proof:
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Proof:
- (\(\implies\)) direction is easy.
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Proof:
- $(\implies)$ direction is easy.
- Let $G$ be bipartite with $(V = X \cup Y)$. Then, every walk in $G$ alternates between $X, Y$. 
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**Lemma**
Every closed odd walk contains an odd cycle.

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**Proof:**
- (⇒) direction is easy.
- Let $G$ be bipartite with $(V = X \cup Y)$. Then, every walk in $G$ alternates between $X, Y$.

$\Rightarrow$ if we start from $X$, each return to $X$ can only happen after an even number of steps.

$\Rightarrow G$ has no odd cycles.
Lemma
Every closed odd walk contains an odd cycle.

Theorem, Konig, 1936
A graph is bipartite iff it has no odd cycle.

Proof:

\( \iff \)
Suppose \( G \) has no odd cycle, then let us construct the bipartition. Wlog assume \( G \) is connected.
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Lemma
Every closed odd walk contains an odd cycle.

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Proof:

- (⇐) Suppose $G$ has no odd cycle, then let us construct the bipartition. Wlog assume $G$ is connected.
- Let $u \in V$. Break $V$ into
  
  \begin{align*}
  X &= \{v \in V \mid \text{length of shortest path } P_{uv} \text{ from } u \text{ to } v \text{ is even}\}, \\
  Y &= \{v \in V \mid \text{length of shortest path } P_{uv} \text{ from } u \text{ to } v \text{ is odd}\},
  \end{align*}
Characterizing bipartite graphs using cycles.

**Lemma**
Every closed odd walk contains an odd cycle.

**Theorem, Konig, 1936**
A graph is bipartite iff it has no odd cycle.

Proof:

- \((\Leftarrow)\) Suppose \(G\) has no odd cycle, then let us construct the bipartition. Wlog assume \(G\) is connected.
- Let \(u \in V\). Break \(V\) into:
  \[X = \{v \in V \mid \text{length of shortest path } P_{uv} \text{ from } u \text{ to } v \text{ is even}\},\]
  \[Y = \{v \in V \mid \text{length of shortest path } P_{uv} \text{ from } u \text{ to } v \text{ is odd}\},\]
- If there is an edge \(vv'\) between two vertices of \(X\) or two vertices of \(Y\), this creates a closed odd walk: \(uP_{uv}vv'P_{v'u}u\).
Characterizing bipartite graphs using cycles.

**Lemma**

Every closed odd walk contains an odd cycle.

**Theorem, Konig, 1936**

A graph is bipartite iff it has no odd cycle.

**Proof:**

- \((\Longleftarrow)\) Suppose \(G\) has no odd cycle, then let us construct the bipartition. Wlog assume \(G\) is connected.
- Let \(u \in V\). Break \(V\) into
  \[X = \{v \in V \mid \text{length of shortest path } P_{uv} \text{ from } u \text{ to } v \text{ is even}\},\]
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- If there is an edge \(vv'\) between two vertices of \(X\) or two vertices of \(Y\), this creates a closed odd walk: \(uP_{uv}vv'P_{v'u}u\).
- By Lemma, it must contain an odd cycle: contradiction.
- This along with \(X \cap Y = \emptyset\) and \(X \cup Y = V\), implies \(X, Y\) is a bipartition.