CS 207: Discrete Structures

Graph theory
Stable matchings

Lecture 32
Oct 18 2016
Topic 3: Graph theory

Topics in Graph theory

1. Basics concepts and definitions.
2. Eulerian graphs: Using degrees of vertices.
6. Perfect matchings in bipartite graphs: Using neighbour sets. – Hall’s theorem
7. Applications of Hall’s theorem: Minimum vertex covers – Konig-Egervary’s theorem
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2. Eulerian graphs: Using degrees of vertices.
6. Perfect matchings in bipartite graphs: Using neighbour sets. – Hall’s theorem
7. Applications of Hall’s theorem: Minimum vertex covers – Konig-Egervary’s theorem
8. Today: Stable matchings...
A min-max theorem

Theorem (Konig ’31, Egervary ’31)

If $G$ is a bipartite graph, then the size of the maximum matching of $G$ equals the size of the minimum vertex cover of $G$. 

Proof. (For details, see Douglas West, Chapter 3.1).

- Size of any vertex cover $\geq$ size of any matching.
- Thus, it suffices to show that we can achieve a matching which has size equal to min vertex cover.
- Take a minimum vertex cover $Q$, partition into $R = Q \cap X$ and $T = Q \cap Y$.
- Consider subgraphs $H, H'$ induced by $R \cup (Y \setminus T)$, $T \cup (X \setminus R)$.
- Show that $H$ has a matching that saturates $R$ into $Y \setminus T$; $H'$ has a matching saturating $T$ (Use minimality of $Q$).
- Together this forms desired matching ($\therefore$ $H, H'$ are disjoint).
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Next topic: Stable matchings
## Stable matchings

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<tr>
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<th>Girls</th>
</tr>
</thead>
<tbody>
<tr>
<td>• 1</td>
<td>• A</td>
</tr>
<tr>
<td>• 2</td>
<td>• B</td>
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<tr>
<td>• 3</td>
<td>• C</td>
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Stable matchings

Boys

\[ C > B > E > A > D \]

\[ ABEC \]

\[ DCBA \]

\[ ACDB \]

\[ ABDE \]

Girls

\[ A : 35214 \]

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Let us try a “greedy” marriage strategy for boys.
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Stable matchings

\[
C > B > E > A > D
\]

Boys
- \(ABECD\) \(\rightarrow\) 2
- \(DCBAE\) \(\rightarrow\) 3
- \(ACDBE\) \(\rightarrow\) 4
- \(ABDEC\) \(\rightarrow\) 5

Girls
- \(A : 35214\)
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Danger!
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Danger! 4 prefers C to B and C prefers 4 to 1. Divorce!
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Qn: Can you match everyone without such Rogue couples?!
More than just a funny puzzle

- Matching hospitals and residents.
- Matching dancing partners.
- Matching students with jobs.
Stable matchings

Definition

Given a matching $M$ in a graph with preference lists of nodes.

- **Unstable pair**: Two vertices $x, y$ such that $x$ prefers $y$ to its assigned vertex and vice versa.
- $x, y$ would be happier by eloping.
- Qn: Find a perfect matching with no unstable pairs. Such a matching is called a **Stable Matching**.
Roommates Problem

- $A : BCD$
- $B : CAD$
- $C : ABD$
- $D : ABC$

▶ What can you observe from this?
Roommates Problem

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- $B: CAD$
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What can you observe from this?

- Everybody hates $D$. 

Diagram:

A $\rightarrow$ B $\rightarrow$ C $\rightarrow$ A

D
Roommates Problem

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- $B : CAD$
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What can you observe from this?
- Stable matchings don’t always exist.
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- What can you observe from this?
- Stable matchings don’t always exist.
- So, do they exist for bipartite graphs and how can we prove this?
The proposal algorithm

Given: bipartite graph, preference list for $n$ men/women

- 8am: Every man goes to first woman on his list not yet crossed off, and proposes to her!
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The above loop is repeated every day until there are no more rejected suitors. On that day, the women says “yes” to her “maybe” guy!
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- Does this algorithm terminate?
- If yes, does it produce a stable matching when it terminates?
Termination and Correctness of the proposal algo

- Try out the algo on the example.
Termination and Correctness of the proposal algo

Lemmas

1. The algo terminates.
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Termination and Correctness of the proposal algo

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   - The algo terminates within $n^2$ days.
   - For each day (except last), at least one woman is crossed off some man’s list.
   - As there are $n$ men and each has list of size $n$, algo must terminate in $n^2$ days.
Termination and Correctness of the proposal algo

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1. The algo terminates.
2. If $W$ says maybe to $M$ on $k^{th}$ day, then on every subsequent day she says maybe to someone whom she likes at least as much as $M$. 
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- We will show that \( W' \) prefers some other \( M' \) and hence no unstable pair.
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  - \( W' \) rejected him only because she preferred some \( M'' \) to \( M \).
  - By Lemma 2, she likes her final partner at least as much as \( M'' \), so better than \( M \).
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- If $(M, W)$ is pair in current matching, s.t., $M$ prefers $W'$.
- We will show that $W'$ prefers some other $M'$ and hence no unstable pair.
- Thus no man can be part of an unstable pair, implies stable matching.
The proposal algorithm: who does better?

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Theorem (H.W.: Prove this!)
The male-proposal algorithm is male-optimal (and "woman-pessimal").

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Further reading

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- How many stable marriages are possible?
- Can you do better by lying? Boys - no!, Girls - yes!
- What if there are brother-sisters (who should not be matched)?
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