CS 207: Discrete Structures

Instructor: S. Akshay

July 25, 2016
Lecture 04 – Basic Mathematical Structures
**Logistics**

<table>
<thead>
<tr>
<th>Tutorial timings</th>
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<td><strong>Mondays 5.15pm</strong></td>
<td>Venue to be announced.</td>
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Recap of last three lectures

Chapter 1: Mathematical reasoning

- Propositions, predicates.
- Axioms, Theorems and Types of proofs: contradiction, contrapositive, etc.
- Principle of Mathematical Induction
- Well-ordering principle and Strong Induction
Recap of last three lectures

### Chapter 1: Mathematical reasoning
- Propositions, predicates.
- Axioms, Theorems and Types of proofs: contradiction, contrapositive, etc.
- Principle of Mathematical Induction
- Well-ordering principle and Strong Induction

### Today:- Chapter 2: Basic Mathematical Structures
- Finite and infinite sets, Functions
- Relations
What is a set?

- A set is an unordered collection of objects.
- The objects in a set are called its elements.
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More formally,

Let $P$ be a property. Any collection of objects that are defined by (or satisfy) $P$ is a set, i.e., $S = \{x \mid P(x)\}$.
Examples and properties

- We have already seen examples: \( \mathbb{Z}, \mathbb{N}, \mathbb{R} \), set of all horses,...
- Let \( A, B \) be two sets. Recall the usual definitions:
  - Equality \( A = B \), Subset \( A \subseteq B \),
  - Cartesian product \( A \times B = \{(a, b) \mid a \in A, b \in B\} \)
  - Union \( A \cup B = \{x \mid a \in A \text{ or } b \in B\} \)
  - Intersection \( A \cap B = \{x \mid a \in A \text{ and } b \in B\} \)
  - Empty set \( \emptyset \),
  - Power set of \( A = \mathcal{P}(A) = \text{set of all subsets of } A \).
  - If \( U \) is the universe, then the complement of \( A \), \( \overline{A} = A^c = \{x \in U \mid x \notin A\} \).
Some simple boring stuff about sets

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So, what is the difference between \( \{\emptyset\} \) and \( \emptyset \)?
Not so simple...

A barber is a man in town who only shaves those who don’t shave themselves.

**Barber’s paradox:** Does the barber shave himself?
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**Russell’s paradox**

Let $S = \{X \mid X \not\in X\}$

Then if $S \in S$, then $S \not\in S$ and if $S \not\in S$, then $S \in S$!
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How do you resolve this?

*Figure : Bertrand Russell (1872-1970)*
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### Axiomatic approach to set theory (ZFC!)

Start with a few objects defined. Then for a set $A$ and a property $P$, $S = \{ x \in A \mid P(x) \}$ is a set.
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Why does this definition get rid of Russell’s paradox?
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- if \( (S \in S) \): from the definition of \( S \), \( S \in A \) and \( S \not\in S \), which is a contradiction.
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- if \( (S \in S) \): from the definition of \( S \), \( S \in A \) and \( S \notin S \), which is a contradiction.
- if \( (S \notin S) \): from the definition, either \( S \notin A \) or \( S \in S \). But we have assumed that \( S \notin S \). Hence, \( S \notin A \). No contradiction!
Is the fun done? How do we compare sets?

- For two finite sets, this is easy, just count the number of elements and compare them!
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- Turns out we need functions... but first...
Hilbert’s hotel

- Suppose there is a hotel with infinitely many rooms.
- And suppose they are all full (like in IIT guest house).
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And suppose they are all full (like in IIT guest house).

1. Can you accommodate 1 or finitely many more guests, by shifting around the existing guests?
2. What if infinitely many more guests arrive?
3. What if infinitely many more trains with infinitely many more guests arrive? (H.W)
Functions

What you did above was to define functions...

**Definition**

Let $A$, $B$ be two sets. A **function** $f$ from $A$ to $B$ is an assignment of exactly one element of $B$ to each element of $A$. 
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i.e., $f : A \to B$ is a subset $R$ of $A \times B$ such that

(i) $\forall a \in A$, $\exists b \in B$ such that $(a, b) \in R$, and

(ii) if $(a, b) \in R$ and $(a, c) \in R$, then $b = c$. 

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- We write $f(a) = b$ and call $b$ the image of $a$.
- $\text{Range}(f) = \{ b \in B \mid \exists a \in A \text{ s.t. } f(a) = b \}$, $\text{Domain}(f) = A$
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**Composition of functions**

- If $g : A \to B$ and $f : B \to C$, then $f \circ g : A \to C$ is defined by $f \circ g(x) = f(g(x))$.

- Defined only if $\text{Range}(g) \subseteq \text{Domain}(f)$.

- Example: if $f(x) = x^2$, $g(x) = x - x^3$ with $f, g : \mathbb{R} \to \mathbb{R}$, what is $f \circ g(x), g \circ f(x)$?
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Composition of functions is associative

- If $h : A \rightarrow B$ and $g : B \rightarrow C$ and $f : C \rightarrow D$, then
  $f \circ (g \circ h) = (f \circ g) \circ h$.

Check it! (H.W.)
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**Inverse of a function**

- If $f : A \rightarrow B$ is a ??? function, then $f^{-1} : B \rightarrow A$ defined by $f^{-1}(b) = a$ if $f(a) = b$, is called its inverse.