CS 207: Discrete Structures

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Lecture 06 – Countable and uncountable sets
Chapter 2: Basic mathematical structures

Sets and Functions

- Finite and infinite sets, Russell’s paradox, axioms of ZFC.
- Functions, their properties, associativity, inverse.
- Types of functions: surjective, injective and bijective.
- Two sets have the same “size” or cardinality if there is a bijection between them.
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- Functions, their properties, associativity, inverse.
- Types of functions: surjective, injective and bijective.
- Two sets have the same “size” or cardinality if there is a bijection between them.

Today’s class

- Countable and countably infinite sets.
Recap: properties of functions on finite and infinite sets

Some important properties (H.W.: Prove them!)

- ∃ bij from $A$ to $B$ and $B$ to $C$, implies ∃ bij from $A$ to $C$.
- ∃ bij from $A$ to $B$, implies ∃ bij from $B$ to $A$.
- ∃ inj from $A$ to $B$, implies ∃ surj from $B$ to $A$ (and vice-versa)
- Schröder-Bernstein Theorem: ∃ surj from $A$ to $B$ and ∃ surj $B$ to $A$, implies ∃ bij from $A$ to $B$. 

Theorem

Let $A$ be a set and $b \not\in A$. Then $A$ is infinite iff there is a bijection from $A$ to $A \cup \{b\}$.

Corollary

For any infinite set $A$, there is a surjection from $A$ to $\mathbb{N}$.

Is there also an injection? Are all (infinite) sets bijective to $\mathbb{N}$?
Recap: properties of functions on finite and infinite sets

Some important properties (H.W.: Prove them!)

- \( \exists \text{bij} \) from \( A \) to \( B \) and \( B \) to \( C \), implies \( \exists \text{bij} \) from \( A \) to \( C \).
- \( \exists \text{bij} \) from \( A \) to \( B \), implies \( \exists \text{bij} \) from \( B \) to \( A \).
- \( \exists \text{inj} \) from \( A \) to \( B \), implies \( \exists \text{surj} \) from \( B \) to \( A \) (& vice-versa)
- **Schröder-Bernstein Theorem:** \( \exists \text{surj} \) from \( A \) to \( B \) and \( \exists \text{surj} \) \( B \) to \( A \), implies \( \exists \text{bij} \) from \( A \) to \( B \).

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Proof: Hilbert’s hotel argument. But how do you formalize it?
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Proof: Hilbert’s hotel argument. But how do you formalize it?

$$f(x) = \begin{cases} 
-2x & \text{if } x \leq 0 \\
2x - 1 & \text{else}
\end{cases}$$
Comparing infinite sets using functions

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Some questions...
- Is there a bijection between $\mathbb{N} \times \mathbb{N}$ to $\mathbb{N}$?
- Is there a bijection between $\mathbb{N} \times \mathbb{N} \times \mathbb{N}$ to $\mathbb{N}$?
- Is there a bijection from $\mathbb{Q}$ to $\mathbb{N}$?
- Is there a bijection from the set of all subsets of $\mathbb{N}$ to $\mathbb{N}$?
- Is there a bijection from $\mathbb{R}$ to $\mathbb{N}$?
Countable and countably infinite sets

<table>
<thead>
<tr>
<th>Definition</th>
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<tbody>
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Examples: even numbers, number of horses,...

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Countable and countably infinite sets

**Definition**

- For a given set $C$, if there is a bijection from $C$ to $\mathbb{N}$, then $C$ is called **countably infinite**.
- A set is **countable** if it is finite or countably infinite.

Examples: even numbers, number of horses, ...

By previous corollary ($\exists$ surj from any infinite set to $\mathbb{N}$)

Countably infinite sets are the “smallest” infinite sets.

What are the other properties of countable sets?
Are the following sets countable?

That is, is there a bijection from these sets to $\mathbb{N}$?

- the set of all integers $\mathbb{Z}$
- $\mathbb{N} \times \mathbb{N}$
- $\mathbb{N} \times \mathbb{N} \times \mathbb{N}$
- the set of rationals $\mathbb{Q}$
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To show these it suffices to show that

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To show these it suffices to show that

- there is an injection from these sets to $\mathbb{N}$
- or there is a surjection from $\mathbb{N}$ (or any countable set) to these sets.
Union of countable sets is countable

Let $A = \{a_0, \ldots, \}$ be a countably infinite set and $B$ be a set. Then, is $A \cup B$ countable, under the following conditions?

1. $B = \{b_0\}$ is a singleton
2. $B = \{b_0, \ldots, b_n\}$ is a finite set
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➤ Rather, choose \( \{a_0, b_0, a_1, b_1, \ldots\} \), then \( b_i \) is at \( (2i + 1)^{th} \) position.
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► Formally, define a bijection \( f : (A \cup B) \rightarrow \mathbb{N} \) by \( f(a_i) = 2i \) and \( f(b_i) = 2i + 1 \)
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Formally, define a bijection $f : (A \cup B) \rightarrow \mathbb{N}$ by $f(a_i) = 2i$ and $f(b_i) = 2i + 1$

Are we done? What is missing?
Theorem: The cartesian product of two countably infinite sets is countably infinite

Proof: Let $A, B$ be countably infinite. Find a way to “number” the elements in $A \times B = \{(a, b) \mid a \in A, b \in B\}$.

That is, define a bijection from $A \times B$ to $\mathbb{N}$. 

Corollaries

$\mathbb{N} \times \mathbb{N}$, $\mathbb{N} \times \mathbb{N} \times \mathbb{N}$, $\mathbb{N} \times \mathbb{Z} \times \mathbb{N}$ are countable.

The set of (positive) rationals is countable. Hint: Show that $f(a, b) = a/b$ is a surjection. How does the result follow?
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Countable sets and functions

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Comparing $\mathbb{N}$ and set of all subsets of $\mathbb{N}$

Theorem (Cantor, 1891)
There is no bijection between $\mathbb{N}$ and the set of all subsets of $\mathbb{N}$.