CS 207: Discrete Structures

Instructor : S. Akshay

Aug 2, 2016
Lecture 08 – Basic mathematical structures
Equivalence relations and partitions
Recap: Relations

Definition: Relation

- A relation $R$ from $A$ to $B$ is a subset of $A \times B$. If $(a, b) \in R$, we also write this as $a R b$.

We write $R(A, B)$ for a relation from $A$ to $B$ and just $R(A)$ if $A = B$. Also if $A$ is clear from context, we just write $R$.

- All functions are relations.
- Relational databases are practical examples.
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Representations of a relation from $A$ to $B$.

- As a set of ordered pairs of elements, i.e., subset of $A \times B$, as a directed graph, as a (database) table.
Some special types of relations

Let us fix a set $S$.

- A relation $R(S)$ is called reflexive if for all $a \in S$, $aRa$.
  Example: relation $R_2$. 
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An relation which satisfies all these three properties is called an **equivalence relation**.

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Examples

- **Reflexive:** $\forall a \in S, aRa$.
- **Symmetric:** $\forall a, b \in S, aRb$ implies $bRa$.
- **Transitive:** $\forall a, b, c \in S, aRb, bRc$ implies $aRc$.
- **Equivalence:** Reflexive, Symmetric and Transitive.
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Partitions of a set – grouping “like” elements

**Definition**

A partition of a set \( S \) is a set \( P \) of its subsets such that

- if \( S' \in P \), then \( S' \neq \emptyset \).
- \( \bigcup_{S' \in P} S' = S \): its union covers entire set \( S \).
- If \( S_1, S_2 \in P \), then \( S_1 \cap S_2 = \emptyset \): sets are disjoint.

Example: natural numbers partitioned into even and odd...

**Theorem**

Every partition of set \( S \) gives rise to a canonical equivalence relation \( R \) on \( S \), namely,

- \( aRb \) if \( a \) and \( b \) belong to the same set in the partition of \( S \).
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Is the converse true? Can we generate a partition from every equivalence relation?
Equivalence classes

**Definition**

- Let $R$ be an equivalence relation on set $S$, and let $a \in S$.
- Then the **equivalence class** of $a$, denoted $[a]$, is the set of all elements related to it, i.e., $[a] = \{ b \in S \mid (a, b) \in R \}$.
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Let $R$ be an equivalence relation on $S$. Let $a, b \in S$. Then, the following statements are equivalent:

1. $aRb$
2. $[a] = [b]$
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Proof Sketch: (1) to (2) symm and trans, (2) to (3) refl, (3) to (1) symm and trans. (H.W.: Do the proof formally.)
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2. Conversely, given a partition $P$ of $S$, there is an equivalence relation $R$ whose equivalence classes are exactly the sets of $P$. 

Proof sketch of (1): Union, non-emptiness follows from reflexivity. The rest (pairwise disjointness) follows from the previous lemma.

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More “applications” of equivalence relations

Defining new objects using equivalence relations

Consider

\[ R = \{ ((a, b), (c, d)) \mid (a, b), (c, d) \in \mathbb{Z} \times (\mathbb{Z} \setminus \{0\}), (ad = bc) \}. \]

- Then the equivalence classes of \( R \) define the rational numbers.
- e.g., \([\frac{1}{2}] = [\frac{2}{4}] \) are two names for the same rational number.
- Indeed, when we write \( \frac{p}{q} \) we implicitly mean \( \left[ \frac{p}{q} \right] \).
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Can we define integers and real numbers starting from naturals by using equivalence classes?