CS228 : Logic for Computer Science

Module 1: Propositional Logic

Lecture 2: Parse Trees, Semantics and Problems of Interest

Instructor: S. Akshay

IIT Bombay, India

Logistics

Updates

- Safe should be up and running

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- Optional Help Session: For a few students, already contacted!

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We will go from simple to more complicated

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Recap: Propositional Logic

1. What are propositions?
2. Atomic propositions and how to combine them
3. Syntax of propositional logic: Propositional formulae

Recall: Propositional formulas

Is this a proposition?

It is raining.

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Topic 2.1

Parse Trees

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- A better solution: Tree representation!

Representation as Parse trees

Definition: A parse tree of a formula $F \in$ **Prop** is a tree s.t.,

- the root is F,
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Example

$$
\begin{array}{ccc}
 & (p_1 \rightarrow (\neg p_2 \leftrightarrow (p_1 \wedge p_3))) & \\
\swarrow & & \searrow \\
p_1 & & (\neg p_2 \leftrightarrow (p_1 \wedge p_3))
\end{array}
$$

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Example

$$\begin{tikzcd}
& (p_1 \to (\neg p_2 \leftrightarrow (p_1 \wedge p_3))) \arrow[dl] \arrow[dr] & & \\
p_1 & & (\neg p_2 \leftrightarrow (p_1 \wedge p_3)) \arrow[dl] \arrow[dr] & \\
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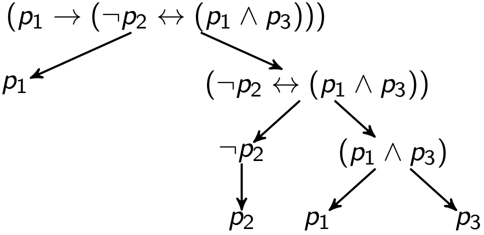

Theorem

$F \in$ **Prop** iff F has a parse tree

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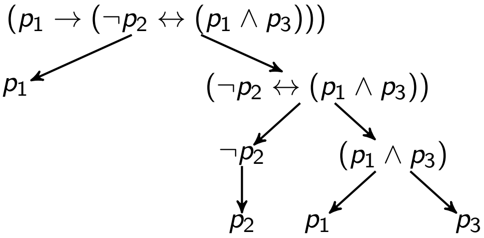

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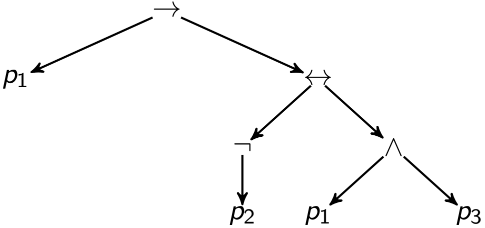

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- Formulas on internal nodes and leaves are called sub-formulas.
- Don't need brackets in the tree!

Representation as Parse trees / DAGs

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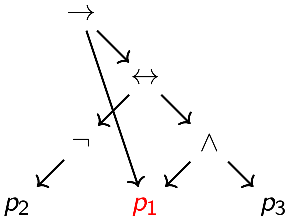

- Therefore, writing the tree allows to check that formula is well-formed.
- Formulas on internal nodes and leaves are called sub-formulas.
- Share leaves to get a directed-acyclic graph (DAG) instead of Tree!

Some exercises

Exercise 2.1

- Are these formulas in **Prop**? If so, draw the parse trees and dags for the formulas below:
  1. $(p \wedge (\neg q \rightarrow \neg p))$
  2. $\neg((\neg q \wedge (p \rightarrow r)) \wedge (r \rightarrow q))$
  3. $(s \vee ((\neg p) \rightarrow (\neg p)))$
  4. $(p \rightarrow (\neg q \wedge r \vee \neg p))$

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- So far: Parse Trees — how to represent a formula's structure (and share sub-formulas via DAGs).
- Next: Semantics — given a well-formed formula, what does it actually *mean*? How do we compute its truth value?

Topic 2.2

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Definition

- An *assignment* is a map from **AP** $\to \mathcal{B}$.
- A *Valuation or a model m* of a propositional formula $F \in$ **Prop** is an assignment of each atomic proposition in F to a truth value.

Summarizing the truth values: Truth Tables

- For each assignment of variables, i.e., propositions, we obtain a value of a formula F.

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For example, if $F = p \wedge q$ we have:

p	q	$p \wedge q$
1	1	1
1	0	0
0	1	0
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For example, for arbitrary formula G and H:

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- What is the truth table for $\neg p \vee q$ and what is its link with $\rightarrow$?

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- Note that Truth table for True is just $\begin{array}{|c|}\hline \top \\ \hline 1 \\ \hline \end{array}$ and False is $\begin{array}{|c|}\hline \bot \\ \hline 0 \\ \hline \end{array}$.

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How do we generalize this for an arbitrary formula F?

Evaluating a formula

Question

If $p_1 = 1$, $p_2 = 1$, $p_3 = 0$, what is $(p_1 \rightarrow (\neg p_2 \leftrightarrow (p_1 \wedge p_3)))$?

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$$(p_1 \rightarrow (\neg p_2 \leftrightarrow (p_1 \wedge p_3))) =?$$

$$p_1 = 1 \qquad (\neg p_2 \leftrightarrow (p_1 \wedge p_3)) =?$$

$$\neg p_2 =? \qquad (p_1 \wedge p_3) =?$$

$$p_2 = 1 \qquad p_1 = 1 \qquad p_3 = 0$$

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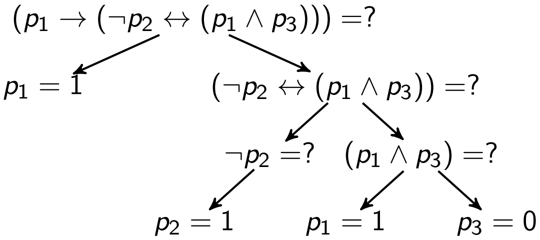

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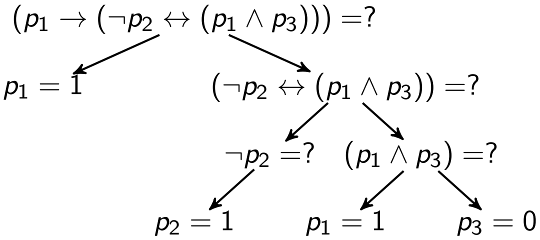

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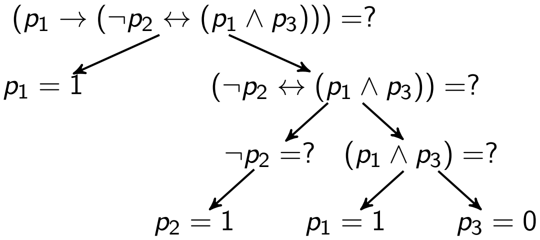

- For a given formula F and a given assignment, how long does it take to evaluate it? A. Linear time in size of the tree!
- But what is the size of the tree in worst case?

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0	0	0	0			0		0		0	
0	0	1	0			0		0		1	
0	1	0	0			1		0		0	
0	1	1	0			1		0		1	
1	0	0	1			0		1		0	
1	0	1	1			0		1		1	
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0	1	0	0		0	1		0	0	0
0	1	1	0		0	1		0	0	1
1	0	0	1		1	0		1	0	0
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0	0	1	0		1	0	0	0	0	1
0	1	0	0		0	1	1	0	0	0
0	1	1	0		0	1	1	0	0	1
1	0	0	1		1	0	0	1	0	0
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0	1	1	0	1	0	1	1	0	0	1
1	0	0	1	0	1	0	0	1	0	0
1	0	1	1	1	1	0	1	1	1	1
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0	0	1	0	1	1	0	0	0	0	1
0	1	0	0	1	0	1	1	0	0	0
0	1	1	0	1	0	1	1	0	0	1
1	0	0	1	0	1	0	0	1	0	0
1	0	1	1	1	1	0	1	1	1	1
1	1	0	1	1	0	1	1	1	0	0
1	1	1	1	0	0	1	0	1	1	1

- Can think of it as going up the Parse tree.
- So we are applying an inductive evaluation!

Semantics of Propositional formula as Truth Tables

Consider: $(p_1 \rightarrow (\neg p_2 \leftrightarrow (p_1 \wedge p_3)))$

p_1	p_2	p_3	$(p_1$	$\rightarrow$	$(\neg$	p_2	$\leftrightarrow ($	p_1	$\wedge$	p_3)))
0	0	0	0	1	1	0	0	0	0	0
0	0	1	0	1	1	0	0	0	0	1
0	1	0	0	1	0	1	1	0	0	0
0	1	1	0	1	0	1	1	0	0	1
1	0	0	1	0	1	0	0	1	0	0
1	0	1	1	1	1	0	1	1	1	1
1	1	0	1	1	0	1	1	1	0	0
1	1	1	1	0	0	1	0	1	1	1

- Can think of it as going up the Parse tree.
- So we are applying an inductive evaluation!
  - $[|F \wedge G|] = 1$ iff $[|F|] = 1$ and $[|G|] = 1$
  - $[|\neg G|] = 1$ iff $[|G|] = 0$
  - Complete the rest!

Back to the example beyond good and evil

- In a country, every person is either good or evil (i.e., NOT good).
- Good people always tell the truth and evil people always lie.
- You meet two citizens A and B. A says, "I am evil or B is good". What are A and B?

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- Step 1: Introduce atomic propositions: p_A= A is good, p_B= B is good.
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Exercise 2.3

Write out the Truth Table for F

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p_A	p_B	$\neg p_A$	$\neg p_A \vee p_B$	F
0	0	1	1	0
0	1	1	1	0
1	0	0	0	0
1	1	0	1	1

More examples

A prisoner must choose between three rooms. One room has a Lady and other two have tigers. If the prisoner chooses the room with the Lady, he wins! If he chooses a wrong room, he (probably) gets eaten by the tiger!
Each room has a sign in front of it. BUT Only one of the signs is true; the other two are false.

(**Room1**)	*The Lady is not in this room*
(**Room2**)	*The Lady is not in this room*
(**Room3**)	*The Lady is in Room 2*

Which room has the Lady?!

Exercise 2.4

1. Take the encoding of this problem you had written last lecture. Write the Truth Table for that formula.
2. Use the Truth Table to solve the problem.

Topic 2.3

Problems of Interest

Satisfiability

Definition (Satisfiability)

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How did we solve it?Check for 1 entry.

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Example: $(p \vee q)$ is not valid but $(p \vee \neg p)$ is valid.

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- Can we read it from the Truth Table?! Is $G = (p_1 \rightarrow (\neg p_2 \leftrightarrow (p_1 \wedge p_3)))$ valid?
- Just check if all entries in col for G are 1.

p_1	p_2	p_3	G
0	0	0	1
0	0	1	1
0	1	0	1
0	1	1	1
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In other words, if you have an algorithm to check for Satisfiability of propositional formulas, you can use it to check for validity of formulas and vice versa!

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Example: $(p \to q) \equiv (\neg p \vee q)$.