

CS228 : Logic for Computer Science

Module 1: Propositional Logic

Lecture 3: Formal Proofs and Natural Deduction

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Recap

Propositional Logic

- ▶ Syntax
 - ▶ Atomic propositions
 - ▶ Propositional formula
 - ▶ Parse trees

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- ▶ Satisfiability and validity
- ▶ Semantic entailment and equivalence

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Problems of interest

- ▶ **Satisfiability** and **validity**
 - ▶ Does there exist a model/satisfying assignment?
- ▶ **Semantic entailment** and **equivalence**

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 - ▶ If two formulas have the same truth table over the union of their propositional variables, then they are equivalent!
 - ▶ What about the converse? Are there two equivalent formulas that have different truth tables?

Semantic equivalence can be used to simplify formulas

- ▶ $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$ (Associativity)

p	q	r		
0	0	0		
0	0	1		
0	1	0		
0	1	1		
1	0	0		
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0	0	0	0	0	0	0
0	0	1	0	0	0	0
0	1	0	0	0	0	0
0	1	1	0	0	1	0
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- ▶ $(p \wedge q) \vee r \equiv (p \vee r) \wedge (q \vee r)$ (Distributivity)

p	q	r	$p \wedge q$	$(p \wedge q) \wedge r$	$q \wedge r$	$p \wedge (q \wedge r)$
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0	0	1	0	0	0	0
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Pop Quiz!

Question 1 (1 mark)

Let p = “you attend the lectures”, q = “you fall asleep”, r = “you will pass the course”.

Write a propositional formula for: “If you attend the lectures, and you don't fall asleep, then you will pass the course.”

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Show, using truth tables, that $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$ (Distributivity).

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Question 3 (3 marks)

Consider the formula $\varphi = (p \rightarrow q) \wedge (q \rightarrow r) \rightarrow (p \rightarrow r)$.

- (i) Construct its truth table.
- (ii) Is φ valid, i.e., does it evaluate to true under all assignments?
- (iii) Is $\neg\varphi$ satisfiable and/or valid?

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1. Give an example of a universal connective. Are there several universal connectives?

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2. * Show $\oplus$ is not a universal connective!

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Suppose some $\oplus$ -only formula $\varphi(p, q_1, \dots, q_n)$ satisfied $\varphi \equiv \neg p$. Set p and all q_i to false: by Lemma, φ evaluates to 0, but $\neg p$ evaluates to 1 when $p = 0$. **Contradiction!** □

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So what's the big problem?

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Solution:

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Solution:

1. Formal proof systems
2. Knowledge compilation

Topic 3.1

A Formal Proof system

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- ▶ Is $((p \vee q) \rightarrow (q \wedge r)) \wedge (\neg((p \vee q) \rightarrow (q \wedge r)))$ **satisfiable**?
- ▶ Does $p \wedge q \models p$ **hold**?
- ▶ If q is true then $p \vee q$ is **also true**.
- ▶ If I have my umbrella open, and I only open it when it's raining, then **it is raining**.

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We use **rules** between propositions and connectives!

Can we formalize them?

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$$F_1, \dots, F_n \vdash G$$

to express the *claim* that, starting from $F_1, \dots, F_n$, by applying the “rules” repeatedly, one can derive G .

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- ▶ Another formula G : call it **conclusion**
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$$F_1, \dots, F_n \vdash G$$

to express the *claim* that, starting from $F_1, \dots, F_n$, by applying the “rules” repeatedly, one can derive G .

- ▶ This expression is called a **sequent**.
- ▶ The sequent is said to be **valid** if such a derivation actually exists, i.e., you can actually derive G from $F_1, \dots, F_n$ using the rules.

A proof system

What are our guiding principles to define rules?

E.g., we want to obtain $F, F \rightarrow G \vdash G$, but not $p, q \vdash p \wedge \neg q$.

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How do we define correctness? **Semantic entailment**, i.e., via **Truth tables**!

(General) Semantic Entailment

$$F_1, \dots, F_n \models G$$

if for all valuations in which $F_1, \dots, F_n$ evaluate to 1, G also evaluates to 1.

(Recall: a **valuation** is just an assignment of a truth value to each atomic proposition.)

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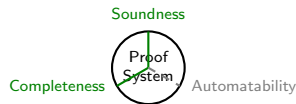
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 - ▶ If $F_1, \dots, F_n \models G$ holds, then $F_1 \dots F_n \vdash G$ is valid.
- ▶ **Automatability**



(a logician might not mind losing this 3rd spoke!)

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A formal proof system: Natural deduction

Natural deduction rules (from Huth & Ryan's Book)

12 Axioms or rules:

1. Introducing and Eliminating $\wedge$
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3. Introducing and Eliminating $\vee$
4. Negation rules: Introducing, Eliminating and Double negation
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4 Derived/inferred rules to make life easier!



Gerhard Gentzen (1909-1945)

Conjunction Rules

If my cat is brown and it is fat then my cat is brown and fat!

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$$\frac{F \quad G}{F \wedge G} \wedge i$$

$$\frac{F \wedge G}{F} \wedge e_1$$

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► Rules to Introduce and Elimination $\wedge$

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2. r premise

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3. q $\wedge_{e_2} 1$

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When clear we omit subscripts e_1, e_2 and just write e .