CS228 : Logic for Computer Science

Module 1: Propositional Logic

Lecture 5: Soundness and Completeness of Natural Deduction

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Recap: Rules of Natural Deduction

Conjunction:

$$\frac{F \qquad G}{F \wedge G}\wedge i \qquad \frac{F \wedge G}{F}\wedge e_1 \qquad \frac{F \wedge G}{G}\wedge e_2$$

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Double Negation:

$$\frac{\neg\neg F}{F} \neg\neg e \qquad \frac{G}{\neg\neg G} \neg\neg i$$

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Implication (MP, MT & Intro):

$$\frac{F \qquad F \rightarrow G}{G} \text{MP} \qquad \frac{F \rightarrow G \qquad \neg G}{\neg F} \text{MT} \qquad \frac{\boxed{\begin{array}{c} F \\ \vdots \\ G \end{array}}}{F \rightarrow G} \rightarrow i$$

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Disjunction:

$$\frac{F}{F \vee G}\vee i_1 \qquad \frac{G}{F \vee G}\vee i_2 \qquad \frac{F \vee G \qquad \boxed{\begin{array}{c} F \\ \vdots \\ H \end{array}} \qquad \boxed{\begin{array}{c} G \\ \vdots \\ H \end{array}}}{H}\vee e$$

Single Negation Rules

- We have seen $\neg\neg e$ and $\neg\neg i$, the elimination and introduction of double negation.

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- What about introducing and eliminating single negations?

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$$\frac{\perp}{F}\ \perp e$$

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$$\frac{\perp}{F}\ \perp e \qquad\qquad \frac{F \qquad \neg F}{\perp}\ \neg e$$

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$$\frac{\bot}{F}\ \bot e \qquad\qquad \frac{F \qquad \neg F}{\bot}\ \neg e$$

Fun fact: $\bot e$ has a name of its own — Ex Falso Quodlibet, Latin for "from a falsehood, anything follows" — also called the Principle of Explosion!

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What about introducing negations?

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Exercise 5.1: Derive $p \rightarrow \neg p \vdash \neg p$.

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1.	$p \rightarrow \neg p$	premise
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1.	$p \to \neg p$	premise
2.	p	assumption
3.		

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$$\frac{\bot}{F}\ \bot e \qquad\qquad \frac{F \qquad \neg F}{\bot}\ \neg e$$

$$\frac{\boxed{\begin{matrix} F \\ \vdots \\ \bot \end{matrix}}}{\neg F}\ \neg i$$

Exercise 5.1: Derive $p \to \neg p \vdash \neg p$.

1.	$p \to \neg p$	premise
2.	p	assumption
3.	$\neg p$	MP 1,2
4.		

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$$\frac{\boxed{\begin{array}{c} F \\ \vdots \\ \perp \end{array}}}{\neg F}\ \neg i$$

Exercise 5.1: Derive $p \to \neg p \vdash \neg p$.

1.	$p \to \neg p$	premise
2.	p	assumption
3.	$\neg p$	MP 1,2
4.	$\perp$	$\neg e$ 2,3
5.	$\neg p$	$\neg i$ 2-4

Some Derived Rules

Popquiz!

(i) Prove that Modus Tollens rule can be derived from the other rules above! Are there any other such rules that can be derived? Hint: what about $\neg\neg i$?

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(ii) Prove "proof by contradiction", i.e., show that if from $\neg F$ we can derive $\perp$, then we can deduce F. Hint: use only $\rightarrow$ intro/elim rules and negation intro/double negation elimination.

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(iii) Prove the theorem called Law of Excluded Middle (also called LEM): $\vdash F \vee \neg F$ is valid!

Some Derived Rules

Exercise 5.2

(i) Prove that Modus Tollens rule can be derived from the other rules above! Are there any other such rules that can be derived? Hint: what about $\neg\neg i$?

(ii) Prove "proof by contradiction", i.e., show that if from $\neg F$ we can derive $\bot$, then we can deduce F. Hint: use only $\rightarrow$ intro/elim rules and negation intro/double negation elimination.

(iii) Prove the theorem called Law of Excluded Middle (also called LEM): $\vdash F \vee \neg F$ is valid!

Exercise 5.2(i)-(ii): MT and Proof by Contradiction

(i) $F \to G, \neg G \vdash \neg F$

1.	$F \to G$	premise
2.	$\neg G$	premise
3.	F	assumption
4.	G	MP 1,3
5.	$\perp$	$\neg e$ 4,2
6.	$\neg F$	$\neg i$ 3-5

(ii) $\neg F \to \perp \vdash F$

1.	$\neg F \to \perp$	given (via $\to i$)
2.	$\neg F$	assumption
3.	$\perp$	MP 1,2
4.	$\neg\neg F$	$\neg i$ 2-3
5.	F	$\neg\neg e$ 4

Exercise 5.2(iii): Law of Excluded Middle (LEM)

$$\vdash F \vee \neg F$$

Exercise 5.2(iii): Law of Excluded Middle (LEM)

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1.	$\neg(F \vee \neg F)$	assumption
2.		

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Exercise 5.2(iii): Law of Excluded Middle (LEM)

$$\vdash F \vee \neg F$$

1.	$\neg(F \vee \neg F)$	assumption
2.	F	assumption
3.	$F \vee \neg F$	$\vee i_1$ 2
4.		

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$$\vdash F \vee \neg F$$

1.	$\neg(F \vee \neg F)$	assumption
2.	F	assumption
3.	$F \vee \neg F$	$\vee i_1$ 2
4.	$\perp$	$\neg e$ 3,1
5.		

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$$\vdash F \vee \neg F$$

1.	$\neg(F \vee \neg F)$	assumption
2.	F	assumption
3.	$F \vee \neg F$	$\vee i_1$ 2
4.	$\bot$	$\neg e$ 3,1
5.	$\neg F$	$\neg i$ 2-4
6.		

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1.	$\neg(F \vee \neg F)$	assumption
2.	F	assumption
3.	$F \vee \neg F$	$\vee i_1$ 2
4.	$\perp$	$\neg e$ 3,1
5.	$\neg F$	$\neg i$ 2-4
6.	$F \vee \neg F$	$\vee i_2$ 5
7.		

Exercise 5.2(iii): Law of Excluded Middle (LEM)

$$\vdash F \vee \neg F$$

1.	$\neg(F \vee \neg F)$	assumption
2.	F	assumption
3.	$F \vee \neg F$	$\vee i_1$ 2
4.	$\perp$	$\neg e$ 3,1
5.	$\neg F$	$\neg i$ 2-4
6.	$F \vee \neg F$	$\vee i_2$ 5
7.	$\perp$	$\neg e$ 6,1
8.		

Exercise 5.2(iii): Law of Excluded Middle (LEM)

$$\vdash F \lor \neg F$$

1.	$\neg(F \lor \neg F)$	assumption
2.	F	assumption
3.	$F \lor \neg F$	$\lor i_1$ 2
4.	$\bot$	$\neg e$ 3,1
5.	$\neg F$	$\neg i$ 2-4
6.	$F \lor \neg F$	$\lor i_2$ 5
7.	$\bot$	$\neg e$ 6,1
8.	$\neg\neg(F \lor \neg F)$	$\neg i$ 1-7
9.		

Exercise 5.2(iii): Law of Excluded Middle (LEM)

$$\vdash F \vee \neg F$$

1.	$\neg(F \vee \neg F)$	assumption
2.	F	assumption
3.	$F \vee \neg F$	$\vee i_1$ 2
4.	$\perp$	$\neg e$ 3,1
5.	$\neg F$	$\neg i$ 2-4
6.	$F \vee \neg F$	$\vee i_2$ 5
7.	$\perp$	$\neg e$ 6,1
8.	$\neg\neg(F \vee \neg F)$	$\neg i$ 1-7
9.	$F \vee \neg F$	$\neg\neg e$ 8

Summary: All Rules of Natural Deduction

Conjunction:

$$\frac{F \qquad G}{F \wedge G} \wedge i \qquad \frac{F \wedge G}{F} \wedge e_1 \qquad \frac{F \wedge G}{G} \wedge e_2$$

Disjunction:

$$\frac{F}{F \vee G} \vee i_1 \qquad \frac{G}{F \vee G} \vee i_2 \qquad \frac{F \vee G \qquad \boxed{\begin{array}{c} F \\ \vdots \\ H \end{array}} \qquad \boxed{\begin{array}{c} G \\ \vdots \\ H \end{array}}}{H} \vee e$$

Double Negation:

$$\frac{\neg\neg F}{F} \neg\neg e$$

Implication:

$$\frac{F \qquad F \rightarrow G}{G} \text{MP} \qquad \frac{\boxed{\begin{array}{c} F \\ \vdots \\ G \end{array}}}{F \rightarrow G} \rightarrow i$$

Negation & Bot:

$$\frac{\boxed{\begin{array}{c} F \\ \vdots \\ \bot \end{array}}}{\neg F} \neg i \qquad \frac{F \qquad \neg F}{\bot} \neg e \qquad \frac{\bot}{F} \bot e$$

Derived Rules:

$$\frac{F \rightarrow G \qquad \neg G}{\neg F} \text{MT} \qquad \frac{\boxed{\begin{array}{c} \neg F \\ \vdots \\ \bot \end{array}}}{F} \text{PBC}$$

$$\frac{G}{\neg\neg G} \neg\neg i$$

LEM: $\vdash F \vee \neg F$

Topic 5.1

Soundness of Natural Deduction for Propositional Logic

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Theorem (What is derivable is true!)

If $F_1, \ldots, F_n \vdash G$ is valid, then $F_1, \ldots, F_n \models G$ holds

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If $F_1, \ldots, F_n \vdash G$ is valid, then $F_1, \ldots, F_n \models G$ holds

i.e., G being provable from $F_1, \ldots, F_n$ means: for every valuation, whenever $F_1, \ldots, F_n$ evaluate to 1, G also evaluates to 1.

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Proof: By Induction (on what?!)

- Assume $F_1, \ldots, F_n \vdash G$.

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Proof: By Induction (on what?!)

- Assume $F_1, \ldots, F_n \vdash G$.
- Thus there is some proof that yields G. Induction is on number of lines, k, of this proof!

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- Thus there is some proof that yields G. Induction is on number of lines, k, of this proof!
- When $k = 1$, there is only one line in the proof, which is the premise, say F. This gives the sequent $F \vdash F$. But in this case $F \models F$!

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- Induction Hypothesis: Whenever $F_1, \ldots, F_n \vdash G$ is derived using a proof of length $\leq k - 1$, we have $F_1, \ldots, F_n \models G$.

Soundness of Natural Deduction for Propositional Logic

Theorem (What is derivable is true!)

If $F_1, \ldots, F_n \vdash G$ is valid, then $F_1, \ldots, F_n \models G$ holds

i.e., G being provable from $F_1, \ldots, F_n$ means: for every valuation, whenever $F_1, \ldots, F_n$ evaluate to 1, G also evaluates to 1.

Proof: By Induction (on what?!)

- Assume $F_1, \ldots, F_n \vdash G$.
- Thus there is some proof that yields G. Induction is on number of lines, k, of this proof!
- When $k = 1$, there is only one line in the proof, which is the premise, say F. This gives the sequent $F \vdash F$. But in this case $F \models F$!
- Induction Hypothesis: Whenever $F_1, \ldots, F_n \vdash G$ is derived using a proof of length $\leq k - 1$, we have $F_1, \ldots, F_n \models G$.
  Note! We are using "Strong Induction"

Soundness of Natural Deduction for Propositional Logic (Contd.)

- Consider a proof with k lines.
- How did we arrive at G? Which proof rule gave G as the last line?

Soundness of Natural Deduction for Propositional Logic (Contd.)

- Consider a proof with k lines.
- How did we arrive at G? Which proof rule gave G as the last line? Case Analysis!
  - Case 1: Assume G was obtained using $\wedge i$ rule.
    - Then G is of the form $G_1 \wedge G_2$.

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    - i.e., we have shorter proofs $F_1, \ldots, F_n \vdash G_1$ and $F_1, \ldots, F_n \vdash G_2$
    - By inductive hypothesis, $F_1, \ldots, F_n \models G_1$ and $F_1, \ldots, F_n \models G_2$. This implies, $F_1, \ldots, F_n \models G_1 \wedge G_2$ (by definition!).

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  - Case 2: Assume G was obtained using $\rightarrow i$, i.e., G is of the form $G_1 \rightarrow G_2$.
    - A box starting with G_1 was opened at some line $k_1 < k$.

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    - A box starting with G_1 was opened at some line $k_1 < k$.
    - Last line in box is G_2 and line just after the box is G.

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    - A box starting with G_1 was opened at some line $k_1 < k$.
    - Last line in box is G_2 and line just after the box is G.
    - Add G_1 as premise, along with $F_1, \ldots, F_n$, i.e., get proof $F_1, \ldots, F_n, G_1 \vdash G_2$, of length $k-1$. By inductive hypothesis, $F_1, \ldots, F_n, G_1 \models G_2$. But, this is same as $F_1, \ldots, F_n \models G_1 \rightarrow G_2$

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    - Add G_1 as premise, along with $F_1, \ldots, F_n$, i.e., get proof $F_1, \ldots, F_n, G_1 \vdash G_2$, of length $k-1$. By inductive hypothesis, $F_1, \ldots, F_n, G_1 \models G_2$. But, this is same as $F_1, \ldots, F_n \models G_1 \rightarrow G_2$
  - All other cases are by considering each rule separately.

Soundness of Natural Deduction for Propositional Logic (Contd.)

- Consider a proof with k lines.
- How did we arrive at G? Which proof rule gave G as the last line? Case Analysis!
  - Case 1: Assume G was obtained using $\wedge i$ rule.
    - Then G is of the form $G_1 \wedge G_2$.
    - G_1 and G_2 were proved earlier, say at lines $k_1, k_2 < k$.
    - i.e., we have shorter proofs $F_1, \ldots, F_n \vdash G_1$ and $F_1, \ldots, F_n \vdash G_2$
    - By inductive hypothesis, $F_1, \ldots, F_n \models G_1$ and $F_1, \ldots, F_n \models G_2$. This implies, $F_1, \ldots, F_n \models G_1 \wedge G_2$ (by definition!).
  - Case 2: Assume G was obtained using $\rightarrow i$, i.e., G is of the form $G_1 \rightarrow G_2$.
    - A box starting with G_1 was opened at some line $k_1 < k$.
    - Last line in box is G_2 and line just after the box is G.
    - Add G_1 as premise, along with $F_1, \ldots, F_n$, i.e., get proof $F_1, \ldots, F_n, G_1 \vdash G_2$, of length $k-1$. By inductive hypothesis, $F_1, \ldots, F_n, G_1 \models G_2$. But, this is same as $F_1, \ldots, F_n \models G_1 \rightarrow G_2$
  - All other cases are by considering each rule separately.
  - **Exercise 5.4: Check all other 10 cases and complete the proof!**

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  - **Exercise 5.4: Check all other 10 cases and complete the proof!** $\square$

Topic 5.2

Completeness of Natural Deduction for Propositional Logic

Completeness

What we saw till now

Soundness: Anything that is provable is (semantically) correct

If $F_1, \ldots, F_n \vdash G$ is valid, then $F_1, \ldots, F_n \models G$ holds

Completeness

What we saw till now

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Next is the more interesting part!

Completeness

What we saw till now

Soundness: Anything that is provable is (semantically) correct

If $F_1, \ldots, F_n \vdash G$ is valid, then $F_1, \ldots, F_n \models G$ holds

Next is the more interesting part!

Completeness: Anything that is correct is provable in ND!

If $F_1, \ldots, F_n \models G$ holds, then $F_1, \ldots, F_n \vdash G$ is valid

Completeness of Natural Deduction for propositional logic

Theorem (Anything that is correct is provable)

If $F_1, \ldots, F_n \models G$ holds, then $F_1, \ldots, F_n \vdash G$ is valid

Completeness of Natural Deduction for propositional logic

Theorem (Anything that is correct is provable)

If $F_1, \ldots, F_n \models G$ holds, then $F_1, \ldots, F_n \vdash G$ is valid

Whenever $F_1, \ldots, F_n$ semantically entail G, then G can be derived from $F_1, \ldots, F_n$ using the ND rules.

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- Step 1: Show that $\models F_1 \rightarrow (F_2 \rightarrow (\ldots (F_n \rightarrow G) \ldots))$ holds.
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- Step 1: Show that $\models F_1 \to (F_2 \to (\ldots(F_n \to G)\ldots))$ holds. This is easy!
  - By contradiction, suppose $\not\models F_1 \to (F_2 \to (\ldots(F_n \to G)\ldots))$.
  - This is a nested implication, is false only if there exists an assignment s.t. G evaluates to 0 and $F_1, \ldots F_n$ evaluate to 1 Check this by drawing its parse tree!.
- Step 2: Show that $\vdash F_1 \to (F_2 \to (\ldots(F_n \to G)\ldots))$ is valid
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  - Which contradicts $F_1, \ldots, F_n \models G$.
- Step 2: Show that $\vdash F_1 \rightarrow (F_2 \rightarrow (\ldots(F_n \rightarrow G)\ldots))$ is valid
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    - Take the proof from Step 2.

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    - Take the proof from Step 2.
    - Add $F_1, \ldots F_n$ as premises as done in soundness proof.
    - Then apply MP n times to arrive at G.

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So, the only thing left is step 2 from step 1. This is the hard part!

Completeness of Natural Deduction for propositional logic

Step 2 follows from the following theorem:

Theorem

For any propositional formula G, if $\models G$ holds, then $\vdash G$ is valid. That is, if G is a tautology, then it is a theorem!

Proof

- Let $p_1, \ldots, p_m$ be the propositional variables in G; all 2^m valuations make G true.

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    - If in line ℓ of truth table, p_i has value 1, then set $\hat{p}_i = p_i$ otherwise set $\hat{p}_i = \neg p_i$
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    - For each line ℓ show that the sequent $\hat{p_1} \ldots \hat{p_m} \vdash G$ is valid.
  - Step 2.3: Assemble them to obtain proof of G

Completeness Proof contd

Lemma

Let F be any propositional formula with variables $p_1, \ldots, p_m$. Let ℓ be a line number in Truth Table of F. Let $\hat{p}_i = p_i$ if p_i has value 1 in line ℓ, and $= \neg p_i$ if p_i has value 0 in line ℓ. Then

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Proof by Structural Induction on structure of F, i.e., induction on height of parse tree for F.

- Base Case : If $F = p$, a proposition, then $p \vdash p$ and $\neg p \vdash \neg p$.

Completeness Proof contd

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Let F be any propositional formula with variables $p_1, \ldots, p_m$. Let ℓ be a line number in Truth Table of F. Let $\hat{p}_i = p_i$ if p_i has value 1 in line ℓ, and $= \neg p_i$ if p_i has value 0 in line ℓ. Then

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  - If F evaluates 1 at line ℓ, then F_1 evaluates to 0. By inductive hypothesis, $\hat{p}_1, \ldots, \hat{p}_m \vdash \neg F_1$, i.e., $\hat{p}_1, \ldots, \hat{p}_m \vdash F$.

Completeness Proof contd

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  - If F evaluates 0 at line ℓ, then F_1 evaluates to 1.

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  - If F evaluates 0 at line ℓ, then F_1 evaluates to 1. By inductive hypothesis, $\hat{p}_1, \ldots, \hat{p}_m \vdash F_1$.

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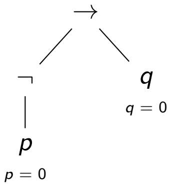
  - If F evaluates 0 at line ℓ, then F_1 evaluates to 1. By inductive hypothesis, $\hat{p}_1, \ldots, \hat{p}_m \vdash F_1$. Use $\neg\neg i$ rule to obtain proof of $\hat{p}_1, \ldots, \hat{p}_m \vdash \neg\neg F_1$, i.e., $\hat{p}_1, \ldots, \hat{p}_m \vdash \neg F$.

Seeing the Induction: An Example

Let $F = \neg p \to q$, i.e., $F_1 = \neg p$ and $F_2 = q$. We derive the proof for line ℓ below.

p	q	$\neg p$	F	
0	0	1	0	$\leftarrow \ell$
0	1	1	1	
1	0	0	1	
1	1	0	1	



Goal: $\hat{p}, \hat{q} \vdash \neg F$.

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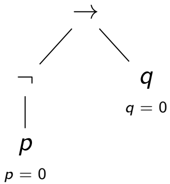

Goal: $\hat{p}, \hat{q} \vdash \neg F$.

- **Leaves:** $p = 0 \Rightarrow \hat{p} = \neg p$, so $\neg p \vdash \neg p$ (base case).

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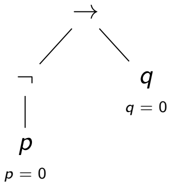
1	0	0	1	
1	1	0	1	



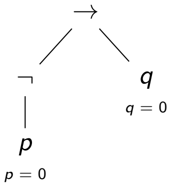
Goal: $\hat{p}, \hat{q} \vdash \neg F$.

- **Leaves:** $p = 0 \Rightarrow \hat{p} = \neg p$, so $\neg p \vdash \neg p$ (base case).
  $q = 0 \Rightarrow \hat{q} = \neg q$, so $\neg q \vdash \neg q$ (base case).

Seeing the Induction: An Example

Let $F = \neg p \to q$, i.e., $F_1 = \neg p$ and $F_2 = q$. We derive the proof for line ℓ below.

p	q	$\neg p$	F	
0	0	1	0	$\leftarrow \ell$
0	1	1	1	
1	0	0	1	
1	1	0	1	



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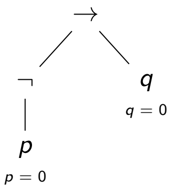
- $\neg$ node ($F_1 = \neg p$): F_1 evaluates to 1 $\Rightarrow$ using the leaf,
  $\neg p \vdash \neg p$, i.e., $\neg p \vdash F_1$.

Seeing the Induction: An Example

Let $F = \neg p \to q$, i.e., $F_1 = \neg p$ and $F_2 = q$. We derive the proof for line ℓ below.

p	q	$\neg p$	F	
0	0	1	0	$\leftarrow \ell$
0	1	1	1	
1	0	0	1	
1	1	0	1	



Goal: $\hat{p}, \hat{q} \vdash \neg F$.

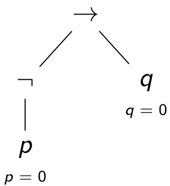
- **Leaves:** $p = 0 \Rightarrow \hat{p} = \neg p$, so $\neg p \vdash \neg p$ (base case).
  $q = 0 \Rightarrow \hat{q} = \neg q$, so $\neg q \vdash \neg q$ (base case).
- $\neg$ **node** $(F_1 = \neg p)$: F_1 evaluates to 1 $\Rightarrow$ using the leaf, $\neg p \vdash \neg p$, i.e., $\neg p \vdash F_1$.
- $\to$ **node** (F, root): F evaluates to 0 at this line. We have $\neg p$ (from the $\neg$ node) and $\neg q$ (from the leaf) as premises — using these, we derive $\neg F$ on the right:

1.	$\neg p$	premise
2.		

Seeing the Induction: An Example

Let $F = \neg p \rightarrow q$, i.e., $F_1 = \neg p$ and $F_2 = q$. We derive the proof for line ℓ below.

p	q	$\neg p$	F	
0	0	1	0	$\leftarrow \ell$
0	1	1	1	
1	0	0	1	
1	1	0	1	



Goal: $\hat{p}, \hat{q} \vdash \neg F$.

- Leaves: $p = 0 \Rightarrow \hat{p} = \neg p$, so $\neg p \vdash \neg p$ (base case).
  $q = 0 \Rightarrow \hat{q} = \neg q$, so $\neg q \vdash \neg q$ (base case).
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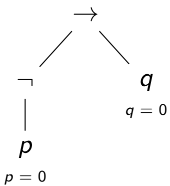
- $\rightarrow$ node (F, root): F evaluates to 0 at this line. We have $\neg p$ (from the $\neg$ node) and $\neg q$ (from the leaf) as premises — using these, we derive $\neg F$ on the right:

1.	$\neg p$	premise
2.	$\neg q$	premise
3.		

Seeing the Induction: An Example

Let $F = \neg p \to q$, i.e., $F_1 = \neg p$ and $F_2 = q$. We derive the proof for line ℓ below.

p	q	$\neg p$	F	
0	0	1	0	$\leftarrow \ell$
0	1	1	1	
1	0	0	1	
1	1	0	1	



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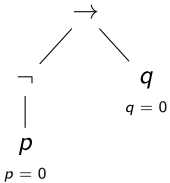
1.	$\neg p$	premise
2.	$\neg q$	premise
3.	$\neg p \to q$	assumption
4.		

Seeing the Induction: An Example

Let $F = \neg p \to q$, i.e., $F_1 = \neg p$ and $F_2 = q$. We derive the proof for line ℓ below.

p	q	$\neg p$	F	
0	0	1	0	$\leftarrow \ell$
0	1	1	1	
1	0	0	1	
1	1	0	1	



Goal: $\hat{p}, \hat{q} \vdash \neg F$.

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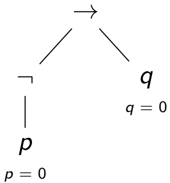
1.	$\neg p$	premise
2.	$\neg q$	premise
3.	$\neg p \to q$	assumption
4.	q	MP 3,1
5.		

Seeing the Induction: An Example

Let $F = \neg p \to q$, i.e., $F_1 = \neg p$ and $F_2 = q$. We derive the proof for line ℓ below.

p	q	$\neg p$	F	
0	0	1	0	$\leftarrow \ell$
0	1	1	1	
1	0	0	1	
1	1	0	1	



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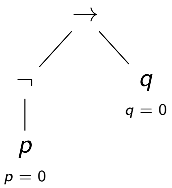
1.	$\neg p$	premise
2.	$\neg q$	premise
3.	$\neg p \to q$	assumption
4.	q	MP 3,1
5.	$\bot$	$\neg e$ 4,2
6.		

Seeing the Induction: An Example

Let $F = \neg p \rightarrow q$, i.e., $F_1 = \neg p$ and $F_2 = q$. We derive the proof for line ℓ below.

p	q	$\neg p$	F	
0	0	1	0	$\leftarrow \ell$
0	1	1	1	
1	0	0	1	
1	1	0	1	



Goal: $\hat{p}, \hat{q} \vdash \neg F$.

- Leaves: $p = 0 \Rightarrow \hat{p} = \neg p$, so $\neg p \vdash \neg p$ (base case).
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1.	$\neg p$	premise
2.	$\neg q$	premise
3.	$\neg p \rightarrow q$	assumption
4.	q	MP 3,1
5.	$\perp$	$\neg e$ 4,2
6.	$\neg(\neg p \rightarrow q)$	$\neg i$ 3-5