CS228 : Logic for Computer Science

Module 1: Propositional Logic

Lecture 6: Completeness of Natural Deduction

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Recap: The Story so far

Module 1 Propositional Logic (Lectures 1–5)

- L1: Syntax
- L2: Parse Trees, Semantics & Problems of Interest
- L3: Formal Proofs & Natural Deduction
- L4: Natural Deduction Rules
- L5: Soundness & Completeness

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- L5: Soundness & Completeness

Soundness: Anything that is provable is (semantically) correct

If $F_1, \ldots, F_n \vdash G$ is valid, then $F_1, \ldots, F_n \models G$ holds

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Completeness: Anything that is correct is provable in ND!

If $F_1, \ldots, F_n \models G$ holds, then $F_1, \ldots, F_n \vdash G$ is valid

Recap: Completeness of Natural Deduction for propositional logic

Theorem (Anything that is correct is provable)

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- Step 1: Show that $\models F_1 \to (F_2 \to (\ldots(F_n \to G)\ldots))$ holds.
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Proof: Let $F_1, \ldots, F_n \models G$. Proof is in three steps

- Step 1: Show that $\models F_1 \rightarrow (F_2 \rightarrow (\ldots(F_n \rightarrow G)\ldots))$ holds. This is easy!
  - By contradiction, suppose $\not\models F_1 \rightarrow (F_2 \rightarrow (\ldots(F_n \rightarrow G)\ldots))$.
  - This nested implication is false only if there exists an assignment s.t. G evaluates to 0 and $F_1, \ldots F_n$ evaluate to 1 Check this by drawing its parse tree!.
- Step 2: Show that $\vdash F_1 \rightarrow (F_2 \rightarrow (\ldots(F_n \rightarrow G)\ldots))$ is valid
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  - Which contradicts $F_1, \ldots, F_n \models G$.
- Step 2: Show that $\vdash F_1 \rightarrow (F_2 \rightarrow (\ldots (F_n \rightarrow G) \ldots))$ is valid
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  - Take the proof from Step 2.
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  - Then apply MP n times to arrive at G.

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So, the only thing left is Step 2. This is the hard part!

Topic 6.1

Completing the completeness proof

Completeness of Natural Deduction for propositional logic

Step 2 follows from the following theorem:

Theorem

For any propositional formula G, if $\models G$ holds, then $\vdash G$ is valid. That is, if G is a tautology, then it is a theorem!

Proof

- Let $p_1, \ldots, p_m$ be the propositional variables in G; all 2^m valuations make G true.

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  - Step 2.2: Construct proofs for each of 2^m sequents.
  - Step 2.3: Assemble them to obtain proof of G

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- From just this, can derive a proof of G?
  - Step 2.1 Key Idea: Encode each line of truth table as a sequent!
    - If in line ℓ of truth table, p_i has value 1, then set $\hat{p_i} = p_i$ otherwise set $\hat{p_i} = \neg p_i$
    - Sequent for line ℓ: $\hat{p_1} \ldots \hat{p_m} \vdash G$
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    - Sequent for line ℓ: $\hat{p_1} \ldots \hat{p_m} \vdash G$
  - Step 2.2: Construct proofs for each of 2^m sequents.
    - For each line ℓ show that the sequent $\hat{p_1} \ldots \hat{p_m} \vdash G$ is valid.
  - Step 2.3: Assemble them to obtain proof of G

Completeness Proof contd

Lemma

Let F be any propositional formula with variables $p_1, \ldots, p_m$. Let ℓ be a line number in Truth Table of F. Let $\hat{p}_i = p_i$ if p_i has value 1 in line ℓ, and $= \neg p_i$ if p_i has value 0 in line ℓ. Then

1. $\hat{p}_1, \ldots, \hat{p}_m \vdash F$ is valid if F evaluates to 1 in line ℓ
2. $\hat{p}_1, \ldots, \hat{p}_m \vdash \neg F$ is valid if F evaluates to 0 in line ℓ

Completeness Proof contd

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Proof by Structural Induction on structure of F,

Completeness Proof contd

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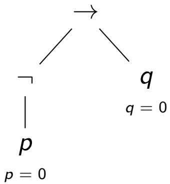
Proof by Structural Induction on structure of F, i.e., induction on height of parse tree for F.

An Example

Let $F = \neg p \rightarrow q$, i.e., $F_1 = \neg p$ and $F_2 = q$. We derive the proof for line ℓ below.

p	q	$\neg p$	F	
0	0	1	0	$\leftarrow \ell$
0	1	1	1	
1	0	0	1	
1	1	0	1	



Goal: $\hat{p}, \hat{q} \vdash \neg F$.

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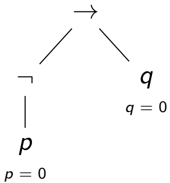

Goal: $\hat{p}, \hat{q} \vdash \neg F$.

- **Leaves:** $p = 0 \Rightarrow \hat{p} = \neg p$, so $\neg p \vdash \neg p$ (base case).

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Let $F = \neg p \to q$, i.e., $F_1 = \neg p$ and $F_2 = q$. We derive the proof for line ℓ below.

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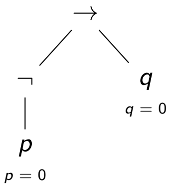

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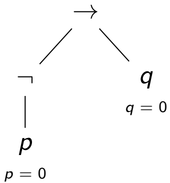

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- $\neg$ node ($F_1 = \neg p$): F_1 evaluates to 1 $\Rightarrow$ using the leaf,
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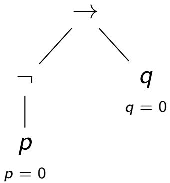

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- $\rightarrow$ node (F, root): F evaluates to 0 at this line. We have $\neg p$ (from the $\neg$ node) and $\neg q$ (from the leaf) as premises — using these, we derive $\neg F$ on the right:

1.	$\neg p$	premise
2.		

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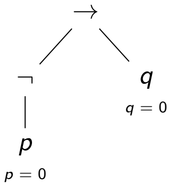

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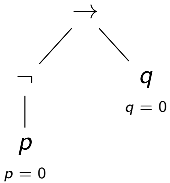

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3.	$\neg p \rightarrow q$	assumption
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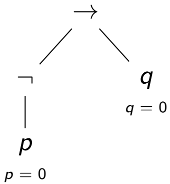

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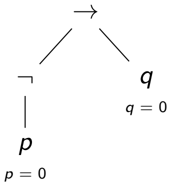

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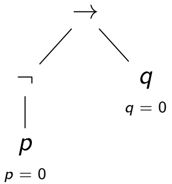
2.	$\neg q$	premise
3.	$\neg p \rightarrow q$	assumption
4.	q	MP 3,1
5.	$\perp$	$\neg e$ 4,2
6.		

An Example

Let $F = \neg p \to q$, i.e., $F_1 = \neg p$ and $F_2 = q$. We derive the proof for line ℓ below.

p	q	$\neg p$	F	
0	0	1	0	$\leftarrow \ell$
0	1	1	1	
1	0	0	1	
1	1	0	1	



Goal: $\hat{p}, \hat{q} \vdash \neg F$.

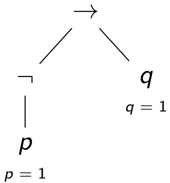
- Leaves: $p = 0 \Rightarrow \hat{p} = \neg p$, so $\neg p \vdash \neg p$ (base case). $q = 0 \Rightarrow \hat{q} = \neg q$, so $\neg q \vdash \neg q$ (base case).
- $\neg$ node ($F_1 = \neg p$): F_1 evaluates to 1 $\Rightarrow$ using the leaf, $\neg p \vdash \neg p$, i.e., $\neg p \vdash F_1$.
- $\to$ node (F, root): F evaluates to 0 at this line. We have $\neg p$ (from the $\neg$ node) and $\neg q$ (from the leaf) as premises — using these, we derive $\neg F$ on the right:

1.	$\neg p$	premise
2.	$\neg q$	premise
3.	$\neg p \to q$	assumption
4.	q	MP 3,1
5.	$\bot$	$\neg e$ 4,2
6.	$\neg(\neg p \to q)$	$\neg i$ 3-5

An Example (contd.)

Same $F = \neg p \to q$ ($F_1 = \neg p$, $F_2 = q$). Now let's try line ℓ: $p = 1, q = 1$.

p	q	$\neg p$	F	
0	0	1	0	
0	1	1	1	
1	0	0	1	
1	1	0	1	$\leftarrow \ell$

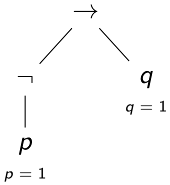


Goal: $\hat{p}, \hat{q} \vdash F$.

An Example (contd.)

Same $F = \neg p \to q$ ($F_1 = \neg p$, $F_2 = q$). Now let's try line ℓ: $p = 1, q = 1$.

p	q	$\neg p$	F	
0	0	1	0	
0	1	1	1	
1	0	0	1	
1	1	0	1	$\leftarrow \ell$



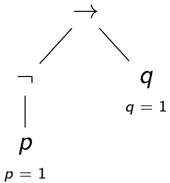
Goal: $\hat{p}, \hat{q} \vdash F$.

- Leaves: $p = 1 \Rightarrow \hat{p} = p$, so $p \vdash p$ (base case); $q = 1 \Rightarrow \hat{q} = q$, so $q \vdash q$ (base case) — both leaves now positive.

An Example (contd.)

Same $F = \neg p \rightarrow q$ ($F_1 = \neg p$, $F_2 = q$). Now let's try line ℓ: $p = 1, q = 1$.

p	q	$\neg p$	F	
0	0	1	0	
0	1	1	1	
1	0	0	1	
1	1	0	1	$\leftarrow \ell$



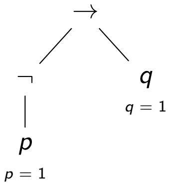
Goal: $\hat{p}, \hat{q} \vdash F$.

- Leaves: $p = 1 \Rightarrow \hat{p} = p$, so $p \vdash p$ (base case); $q = 1 \Rightarrow \hat{q} = q$, so $q \vdash q$ (base case) — both leaves now positive.
- $\neg$ node ($F_1 = \neg p$): F_1 now evaluates to 0 (since $p = 1$)! By IH, $p \vdash p$; use $\neg\neg i$ to get $p \vdash \neg\neg p$, i.e., $p \vdash \neg F_1$.

An Example (contd.)

Same $F = \neg p \to q$ ($F_1 = \neg p$, $F_2 = q$). Now let's try line ℓ: $p = 1, q = 1$.

p	q	$\neg p$	F	
0	0	1	0	
0	1	1	1	
1	0	0	1	
1	1	0	1	$\leftarrow \ell$



Goal: $\hat{p}, \hat{q} \vdash F$.

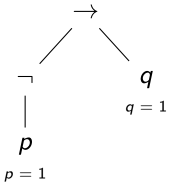
- Leaves: $p = 1 \Rightarrow \hat{p} = p$, so $p \vdash p$ (base case); $q = 1 \Rightarrow \hat{q} = q$, so $q \vdash q$ (base case) — both leaves now positive.
- $\neg$ node ($F_1 = \neg p$): F_1 now evaluates to 0 (since $p = 1$)! By IH, $p \vdash p$; use $\neg\neg i$ to get $p \vdash \neg\neg p$, i.e., $p \vdash \neg F_1$.
- $\to$ node (F, root): F_1 evaluates to 0, F_2 evaluates to 1 — a third sub-subcase! We have $p \vdash \neg F_1$ and $q \vdash F_2$ as premises — using these, we derive F on the right:

1.	p	premise
2.		

An Example (contd.)

Same $F = \neg p \to q$ ($F_1 = \neg p$, $F_2 = q$). Now let's try line ℓ: $p = 1, q = 1$.

p	q	$\neg p$	F	
0	0	1	0	
0	1	1	1	
1	0	0	1	
1	1	0	1	$\leftarrow \ell$



Goal: $\hat{p}, \hat{q} \vdash F$.

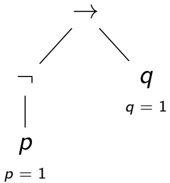
- Leaves: $p = 1 \Rightarrow \hat{p} = p$, so $p \vdash p$ (base case); $q = 1 \Rightarrow \hat{q} = q$, so $q \vdash q$ (base case) — both leaves now positive.
- $\neg$ node ($F_1 = \neg p$): F_1 now evaluates to 0 (since $p = 1$)! By IH, $p \vdash p$; use $\neg\neg i$ to get $p \vdash \neg\neg p$, i.e., $p \vdash \neg F_1$.
- $\to$ node (F, root): F_1 evaluates to 0, F_2 evaluates to 1 — a third sub-subcase! We have $p \vdash \neg F_1$ and $q \vdash F_2$ as premises — using these, we derive F on the right:

1.	p	premise
2.	q	premise
3.		

An Example (contd.)

Same $F = \neg p \to q$ ($F_1 = \neg p$, $F_2 = q$). Now let's try line ℓ: $p = 1, q = 1$.

p	q	$\neg p$	F	
0	0	1	0	
0	1	1	1	
1	0	0	1	
1	1	0	1	$\leftarrow \ell$



Goal: $\hat{p}, \hat{q} \vdash F$.

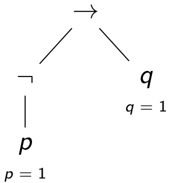
- Leaves: $p = 1 \Rightarrow \hat{p} = p$, so $p \vdash p$ (base case); $q = 1 \Rightarrow \hat{q} = q$, so $q \vdash q$ (base case) — both leaves now positive.
- $\neg$ node ($F_1 = \neg p$): F_1 now evaluates to 0 (since $p = 1$)! By IH, $p \vdash p$; use $\neg\neg i$ to get $p \vdash \neg\neg p$, i.e., $p \vdash \neg F_1$.
- $\to$ node (F, root): F_1 evaluates to 0, F_2 evaluates to 1 — a third sub-subcase! We have $p \vdash \neg F_1$ and $q \vdash F_2$ as premises — using these, we derive F on the right:

1.	p	premise
2.	q	premise
3.	$\neg\neg p$	$\neg\neg i$ 1
4.		

An Example (contd.)

Same $F = \neg p \to q$ ($F_1 = \neg p$, $F_2 = q$). Now let's try line ℓ: $p = 1, q = 1$.

p	q	$\neg p$	F	
0	0	1	0	
0	1	1	1	
1	0	0	1	
1	1	0	1	$\leftarrow \ell$



Goal: $\hat{p}, \hat{q} \vdash F$.

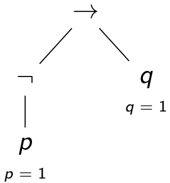
- Leaves: $p = 1 \Rightarrow \hat{p} = p$, so $p \vdash p$ (base case); $q = 1 \Rightarrow \hat{q} = q$, so $q \vdash q$ (base case) — both leaves now positive.
- $\neg$ node ($F_1 = \neg p$): F_1 now evaluates to 0 (since $p = 1$)! By IH, $p \vdash p$; use $\neg\neg i$ to get $p \vdash \neg\neg p$, i.e., $p \vdash \neg F_1$.
- $\to$ node (F, root): F_1 evaluates to 0, F_2 evaluates to 1 — a third sub-subcase! We have $p \vdash \neg F_1$ and $q \vdash F_2$ as premises — using these, we derive F on the right:

1.	p	premise
2.	q	premise
3.	$\neg\neg p$	$\neg\neg i$ 1
4.	$\neg p$	assumption
5.		

An Example (contd.)

Same $F = \neg p \to q$ ($F_1 = \neg p$, $F_2 = q$). Now let's try line ℓ: $p = 1, q = 1$.

p	q	$\neg p$	F	
0	0	1	0	
0	1	1	1	
1	0	0	1	
1	1	0	1	$\leftarrow \ell$



Goal: $\hat{p}, \hat{q} \vdash F$.

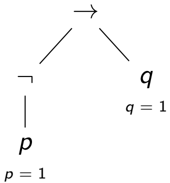
- Leaves: $p = 1 \Rightarrow \hat{p} = p$, so $p \vdash p$ (base case); $q = 1 \Rightarrow \hat{q} = q$, so $q \vdash q$ (base case) — both leaves now positive.
- $\neg$ node ($F_1 = \neg p$): F_1 now evaluates to 0 (since $p = 1$)! By IH, $p \vdash p$; use $\neg\neg i$ to get $p \vdash \neg\neg p$, i.e., $p \vdash \neg F_1$.
- $\to$ node (F, root): F_1 evaluates to 0, F_2 evaluates to 1 — a third sub-subcase! We have $p \vdash \neg F_1$ and $q \vdash F_2$ as premises — using these, we derive F on the right:

1.	p	premise
2.	q	premise
3.	$\neg\neg p$	$\neg\neg i$ 1
4.	$\neg p$	assumption
5.	$\perp$	$\neg e$ 4,3
6.		

An Example (contd.)

Same $F = \neg p \to q$ ($F_1 = \neg p$, $F_2 = q$). Now let's try line ℓ: $p = 1, q = 1$.

p	q	$\neg p$	F	
0	0	1	0	
0	1	1	1	
1	0	0	1	
1	1	0	1	$\leftarrow \ell$



Goal: $\hat{p}, \hat{q} \vdash F$.

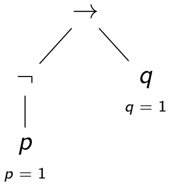
- Leaves: $p = 1 \Rightarrow \hat{p} = p$, so $p \vdash p$ (base case); $q = 1 \Rightarrow \hat{q} = q$, so $q \vdash q$ (base case) — both leaves now positive.
- $\neg$ node ($F_1 = \neg p$): F_1 now evaluates to 0 (since $p = 1$)! By IH, $p \vdash p$; use $\neg\neg i$ to get $p \vdash \neg\neg p$, i.e., $p \vdash \neg F_1$.
- $\to$ node (F, root): F_1 evaluates to 0, F_2 evaluates to 1 — a third sub-subcase! We have $p \vdash \neg F_1$ and $q \vdash F_2$ as premises — using these, we derive F on the right:

1.	p	premise
2.	q	premise
3.	$\neg\neg p$	$\neg\neg i$ 1
4.	$\neg p$	assumption
5.	$\bot$	$\neg e$ 4,3
6.	q	$\bot e$ 5
7.		

An Example (contd.)

Same $F = \neg p \to q$ ($F_1 = \neg p$, $F_2 = q$). Now let's try line ℓ: $p = 1, q = 1$.

p	q	$\neg p$	F	
0	0	1	0	
0	1	1	1	
1	0	0	1	
1	1	0	1	$\leftarrow \ell$



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- Leaves: $p = 1 \Rightarrow \hat{p} = p$, so $p \vdash p$ (base case); $q = 1 \Rightarrow \hat{q} = q$, so $q \vdash q$ (base case) — both leaves now positive.
- $\neg$ node ($F_1 = \neg p$): F_1 now evaluates to 0 (since $p = 1$)! By IH, $p \vdash p$; use $\neg\neg i$ to get $p \vdash \neg\neg p$, i.e., $p \vdash \neg F_1$.
- $\to$ node (F, root): F_1 evaluates to 0, F_2 evaluates to 1 — a third sub-subcase! We have $p \vdash \neg F_1$ and $q \vdash F_2$ as premises — using these, we derive F on the right:

1.	p	premise
2.	q	premise
3.	$\neg\neg p$	$\neg\neg i$ 1
4.	$\neg p$	assumption
5.	$\bot$	$\neg e$ 4,3
6.	q	$\bot e$ 5
7.	$\neg p \to q$	$\to i$ 4-6

Completeness Proof contd

Lemma

Let F be any propositional formula with variables $p_1, \ldots, p_m$. Let ℓ be a line number in Truth Table of F. Let $\hat{p}_i = p_i$ if p_i has value 1 in line ℓ, and $= \neg p_i$ if p_i has value 0 in line ℓ. Then

1. $\hat{p}_1, \ldots, \hat{p}_m \vdash F$ is valid if F evaluates to 1 in line ℓ
2. $\hat{p}_1, \ldots, \hat{p}_m \vdash \neg F$ is valid if F evaluates to 0 in line ℓ

Completeness Proof contd

Lemma

Let F be any propositional formula with variables $p_1, \ldots, p_m$. Let ℓ be a line number in Truth Table of F. Let $\hat{p}_i = p_i$ if p_i has value 1 in line ℓ, and $= \neg p_i$ if p_i has value 0 in line ℓ. Then

1. $\hat{p}_1, \ldots, \hat{p}_m \vdash F$ is valid if F evaluates to 1 in line ℓ
2. $\hat{p}_1, \ldots, \hat{p}_m \vdash \neg F$ is valid if F evaluates to 0 in line ℓ

Proof by Structural Induction on structure of F,

Completeness Proof contd

Lemma

Let F be any propositional formula with variables $p_1, \ldots, p_m$. Let ℓ be a line number in Truth Table of F. Let $\hat{p}_i = p_i$ if p_i has value 1 in line ℓ, and $= \neg p_i$ if p_i has value 0 in line ℓ. Then

1. $\hat{p}_1, \ldots, \hat{p}_m \vdash F$ is valid if F evaluates to 1 in line ℓ
2. $\hat{p}_1, \ldots, \hat{p}_m \vdash \neg F$ is valid if F evaluates to 0 in line ℓ

Proof by Structural Induction on structure of F, i.e., induction on height of parse tree for F.

- Base Case : If $F = p$, a proposition, then $p \vdash p$ and $\neg p \vdash \neg p$.

Completeness Proof contd

Lemma

Let F be any propositional formula with variables $p_1, \ldots, p_m$. Let ℓ be a line number in Truth Table of F. Let $\hat{p}_i = p_i$ if p_i has value 1 in line ℓ, and $= \neg p_i$ if p_i has value 0 in line ℓ. Then

1. $\hat{p}_1, \ldots, \hat{p}_m \vdash F$ is valid if F evaluates to 1 in line ℓ
2. $\hat{p}_1, \ldots, \hat{p}_m \vdash \neg F$ is valid if F evaluates to 0 in line ℓ

Proof by Structural Induction on structure of F, i.e., induction on height of parse tree for F.

- Base Case : If $F = p$, a proposition, then $p \vdash p$ and $\neg p \vdash \neg p$.
- Induction Hyp: Assume Lemma for formulae with parse tree height $\leq k - 1$

Completeness Proof contd

Lemma

Let F be any propositional formula with variables $p_1, \ldots, p_m$. Let ℓ be a line number in Truth Table of F. Let $\hat{p}_i = p_i$ if p_i has value 1 in line ℓ, and $= \neg p_i$ if p_i has value 0 in line ℓ. Then

1. $\hat{p}_1, \ldots, \hat{p}_m \vdash F$ is valid if F evaluates to 1 in line ℓ
2. $\hat{p}_1, \ldots, \hat{p}_m \vdash \neg F$ is valid if F evaluates to 0 in line ℓ

Proof by Structural Induction on structure of F, i.e., induction on height of parse tree for F.

- Base Case : If $F = p$, a proposition, then $p \vdash p$ and $\neg p \vdash \neg p$.
- Induction Hyp: Assume Lemma for formulae with parse tree height $\leq k - 1$
- Case Negation : $F = \neg F_1$

Completeness Proof contd

Lemma

Let F be any propositional formula with variables $p_1, \ldots, p_m$. Let ℓ be a line number in Truth Table of F. Let $\hat{p}_i = p_i$ if p_i has value 1 in line ℓ, and $= \neg p_i$ if p_i has value 0 in line ℓ. Then

1. $\hat{p}_1, \ldots, \hat{p}_m \vdash F$ is valid if F evaluates to 1 in line ℓ
2. $\hat{p}_1, \ldots, \hat{p}_m \vdash \neg F$ is valid if F evaluates to 0 in line ℓ

Proof by Structural Induction on structure of F, i.e., induction on height of parse tree for F.

- Base Case : If $F = p$, a proposition, then $p \vdash p$ and $\neg p \vdash \neg p$.
- Induction Hyp: Assume Lemma for formulae with parse tree height $\leq k - 1$
- Case Negation : $F = \neg F_1$
  - If F evaluates 1 at line ℓ, then F_1 evaluates to 0. By inductive hypothesis, $\hat{p}_1, \ldots, \hat{p}_m \vdash \neg F_1$, i.e., $\hat{p}_1, \ldots, \hat{p}_m \vdash F$.

Completeness Proof contd

Lemma

Let F be any propositional formula with variables $p_1, \ldots, p_m$. Let ℓ be a line number in Truth Table of F. Let $\hat{p}_i = p_i$ if p_i has value 1 in line ℓ, and $= \neg p_i$ if p_i has value 0 in line ℓ. Then

1. $\hat{p}_1, \ldots, \hat{p}_m \vdash F$ is valid if F evaluates to 1 in line ℓ
2. $\hat{p}_1, \ldots, \hat{p}_m \vdash \neg F$ is valid if F evaluates to 0 in line ℓ

Proof by Structural Induction on structure of F, i.e., induction on height of parse tree for F.

- Base Case : If $F = p$, a proposition, then $p \vdash p$ and $\neg p \vdash \neg p$.
- Induction Hyp: Assume Lemma for formulae with parse tree height $\leq k - 1$
- Case Negation : $F = \neg F_1$
  - If F evaluates 1 at line ℓ, then F_1 evaluates to 0. By inductive hypothesis, $\hat{p}_1, \ldots, \hat{p}_m \vdash \neg F_1$, i.e., $\hat{p}_1, \ldots, \hat{p}_m \vdash F$.
  - If F evaluates 0 at line ℓ, then F_1 evaluates to 1.

Completeness Proof contd

Lemma

Let F be any propositional formula with variables $p_1, \ldots, p_m$. Let ℓ be a line number in Truth Table of F. Let $\hat{p}_i = p_i$ if p_i has value 1 in line ℓ, and $= \neg p_i$ if p_i has value 0 in line ℓ. Then

1. $\hat{p}_1, \ldots, \hat{p}_m \vdash F$ is valid if F evaluates to 1 in line ℓ
2. $\hat{p}_1, \ldots, \hat{p}_m \vdash \neg F$ is valid if F evaluates to 0 in line ℓ

Proof by Structural Induction on structure of F, i.e., induction on height of parse tree for F.

- Base Case : If $F = p$, a proposition, then $p \vdash p$ and $\neg p \vdash \neg p$.
- Induction Hyp: Assume Lemma for formulae with parse tree height $\leq k-1$
- Case Negation : $F = \neg F_1$
  - If F evaluates 1 at line ℓ, then F_1 evaluates to 0. By inductive hypothesis, $\hat{p}_1, \ldots, \hat{p}_m \vdash \neg F_1$, i.e., $\hat{p}_1, \ldots, \hat{p}_m \vdash F$.
  - If F evaluates 0 at line ℓ, then F_1 evaluates to 1. By inductive hypothesis, $\hat{p}_1, \ldots, \hat{p}_m \vdash F_1$.

Completeness Proof contd

Lemma

Let F be any propositional formula with variables $p_1, \ldots, p_m$. Let ℓ be a line number in Truth Table of F. Let $\hat{p}_i = p_i$ if p_i has value 1 in line ℓ, and $= \neg p_i$ if p_i has value 0 in line ℓ. Then

1. $\hat{p}_1, \ldots, \hat{p}_m \vdash F$ is valid if F evaluates to 1 in line ℓ
2. $\hat{p}_1, \ldots, \hat{p}_m \vdash \neg F$ is valid if F evaluates to 0 in line ℓ

Proof by Structural Induction on structure of F, i.e., induction on height of parse tree for F.

- Base Case : If $F = p$, a proposition, then $p \vdash p$ and $\neg p \vdash \neg p$.
- Induction Hyp: Assume Lemma for formulae with parse tree height $\leq k - 1$
- Case Negation : $F = \neg F_1$
  - If F evaluates 1 at line ℓ, then F_1 evaluates to 0. By inductive hypothesis, $\hat{p}_1, \ldots, \hat{p}_m \vdash \neg F_1$, i.e., $\hat{p}_1, \ldots, \hat{p}_m \vdash F$.
  - If F evaluates 0 at line ℓ, then F_1 evaluates to 1. By inductive hypothesis, $\hat{p}_1, \ldots, \hat{p}_m \vdash F_1$. Use $\neg\neg i$ rule to obtain proof of $\hat{p}_1, \ldots, \hat{p}_m \vdash \neg\neg F_1$, i.e., $\hat{p}_1, \ldots, \hat{p}_m \vdash \neg F$.

Completeness Proof contd

Case $\rightarrow$: $F = F_1 \rightarrow F_2$

Completeness Proof contd

Case $\rightarrow$: $F = F_1 \rightarrow F_2$

Two subcases:

- Subcase 1: F evaluates to 0 in line ℓ, (exactly our first example!)
- Subcase 2: F evaluates to 1 in line ℓ,

Completeness Proof contd

Case $\to$: $F = F_1 \to F_2$

Two subcases:

- Subcase 1: F evaluates to 0 in line ℓ, (exactly our first example!)
  - then F_1 evaluates to 1 and F_2 to 0.
- Subcase 2: F evaluates to 1 in line ℓ,

Completeness Proof contd

Case $\rightarrow$: $F = F_1 \rightarrow F_2$

Two subcases:

- Subcase 1: F evaluates to 0 in line ℓ, (exactly our first example!)
  - then F_1 evaluates to 1 and F_2 to 0.
  - Let $\{q_1, \ldots, q_k\}$ be the variables of F_1 and let $\{r_1, \ldots, r_j\}$ be the variables in F_2; $\{q_1, \ldots, q_k\} \cup \{r_1, \ldots, r_j\} = \{p_1, \ldots, p_m\}$.
- Subcase 2: F evaluates to 1 in line ℓ,

Completeness Proof contd

Case $\rightarrow$: $F = F_1 \rightarrow F_2$

Two subcases:

- Subcase 1: F evaluates to 0 in line ℓ, (exactly our first example!)
  - then F_1 evaluates to 1 and F_2 to 0.
  - Let $\{q_1, \ldots, q_k\}$ be the variables of F_1 and let $\{r_1, \ldots, r_j\}$ be the variables in F_2; $\{q_1, \ldots, q_k\} \cup \{r_1, \ldots, r_j\} = \{p_1, \ldots, p_m\}$.
  - By inductive hypothesis, $\hat{q}_1, \ldots, \hat{q}_k \vdash F_1$ and $\hat{r}_1, \ldots, \hat{r}_j \vdash \neg F_2$.
- Subcase 2: F evaluates to 1 in line ℓ,

Completeness Proof contd

Case $\rightarrow$: $F = F_1 \rightarrow F_2$

Two subcases:

- Subcase 1: F evaluates to 0 in line ℓ, (exactly our first example!)
  - then F_1 evaluates to 1 and F_2 to 0.
  - Let $\{q_1, \ldots, q_k\}$ be the variables of F_1 and let $\{r_1, \ldots, r_j\}$ be the variables in F_2; $\{q_1, \ldots, q_k\} \cup \{r_1, \ldots, r_j\} = \{p_1, \ldots, p_m\}$.
  - By inductive hypothesis, $\hat{q}_1, \ldots, \hat{q}_k \vdash F_1$ and $\hat{r}_1, \ldots, \hat{r}_j \vdash \neg F_2$.
  - Then, $\hat{p}_1, \ldots, \hat{p}_m \vdash F_1 \wedge \neg F_2$.
- Subcase 2: F evaluates to 1 in line ℓ,

Completeness Proof contd

Case $\to$: $F = F_1 \to F_2$

Two subcases:

- Subcase 1: F evaluates to 0 in line ℓ, (exactly our first example!)
  - then F_1 evaluates to 1 and F_2 to 0.
  - Let $\{q_1, \ldots, q_k\}$ be the variables of F_1 and let $\{r_1, \ldots, r_j\}$ be the variables in F_2; $\{q_1, \ldots, q_k\} \cup \{r_1, \ldots, r_j\} = \{p_1, \ldots, p_m\}$.
  - By inductive hypothesis, $\hat{q}_1, \ldots, \hat{q}_k \vdash F_1$ and $\hat{r}_1, \ldots, \hat{r}_j \vdash \neg F_2$.
  - Then, $\hat{p}_1, \ldots, \hat{p}_m \vdash F_1 \land \neg F_2$.
  - **Exercise 6.1:** Prove that $F_1 \land \neg F_2 \vdash \neg(F_1 \to F_2)$ and conclude this case!
- Subcase 2: F evaluates to 1 in line ℓ,

Completeness Proof contd

Case $\rightarrow$: $F = F_1 \rightarrow F_2$

Two subcases:

- Subcase 1: F evaluates to 0 in line ℓ, (exactly our first example!)
  - then F_1 evaluates to 1 and F_2 to 0.
  - Let $\{q_1, \ldots, q_k\}$ be the variables of F_1 and let $\{r_1, \ldots, r_j\}$ be the variables in F_2; $\{q_1, \ldots, q_k\} \cup \{r_1, \ldots, r_j\} = \{p_1, \ldots, p_m\}$.
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- Subcase 2: F evaluates to 1 in line ℓ, there are 3 subsubcases.
  - If both F_1, F_2 evaluate to 1,
  - If both F_1, F_2 evaluate to 0,
  - If F_1 evaluates to 0 and F_2 evaluates to 1, (our second example!)

Completeness Proof contd

Case $\to$: $F = F_1 \to F_2$

Two subcases:

- Subcase 1: F evaluates to 0 in line ℓ, (exactly our first example!)
  - then F_1 evaluates to 1 and F_2 to 0.
  - Let $\{q_1, \ldots, q_k\}$ be the variables of F_1 and let $\{r_1, \ldots, r_j\}$ be the variables in F_2; $\{q_1, \ldots, q_k\} \cup \{r_1, \ldots, r_j\} = \{p_1, \ldots, p_m\}$.
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  - If both F_1, F_2 evaluate to 1, then $\hat{p}_1, \ldots, \hat{p}_m \vdash F_1 \wedge F_2$.
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Completeness Proof contd

Case $\rightarrow$: $F = F_1 \rightarrow F_2$

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    Exercise 6.2(i): Prove $F_1 \wedge F_2 \vdash F_1 \rightarrow F_2$ and conclude.
  - If both F_1, F_2 evaluate to 0,
  - If F_1 evaluates to 0 and F_2 evaluates to 1, (our second example!)

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    Exercise 6.2(i): Prove $F_1 \wedge F_2 \vdash F_1 \rightarrow F_2$ and conclude.
  - If both F_1, F_2 evaluate to 0, then $\hat{p}_1, \ldots, \hat{p}_m \vdash \neg F_1 \wedge \neg F_2$.
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  - **Exercise 6.1:** Prove that $F_1 \wedge \neg F_2 \vdash \neg(F_1 \rightarrow F_2)$ and conclude this case!
- Subcase 2: F evaluates to 1 in line ℓ, there are 3 subsubcases.
  - If both F_1, F_2 evaluate to 1, then $\hat{p}_1, \ldots, \hat{p}_m \vdash F_1 \wedge F_2$.
    Exercise 6.2(i): Prove $F_1 \wedge F_2 \vdash F_1 \rightarrow F_2$ and conclude.
  - If both F_1, F_2 evaluate to 0, then $\hat{p}_1, \ldots, \hat{p}_m \vdash \neg F_1 \wedge \neg F_2$.
    Exercise 6.2(ii): Prove $\neg F_1 \wedge \neg F_2 \vdash F_1 \rightarrow F_2$ and conclude.
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Completeness Proof contd

Case $\rightarrow$: $F = F_1 \rightarrow F_2$

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- Subcase 1: F evaluates to 0 in line ℓ, (exactly our first example!)
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  - If both F_1, F_2 evaluate to 1, then $\hat{p}_1, \ldots, \hat{p_m} \vdash F_1 \wedge F_2$.
    Exercise 6.2(i): Prove $F_1 \wedge F_2 \vdash F_1 \rightarrow F_2$ and conclude.
  - If both F_1, F_2 evaluate to 0, then $\hat{p}_1, \ldots, \hat{p_m} \vdash \neg F_1 \wedge \neg F_2$.
    Exercise 6.2(ii): Prove $\neg F_1 \wedge \neg F_2 \vdash F_1 \rightarrow F_2$ and conclude.
  - If F_1 evaluates to 0 and F_2 evaluates to 1, (our second example!) then $\hat{p}_1, \ldots, \hat{p_m} \vdash \neg F_1 \wedge F_2$.
    Exercise 6.2(iii): Prove $\neg F_1 \wedge F_2 \vdash F_1 \rightarrow F_2$, and complete this case.

Completeness proof (Contd.)

Exercise 6.3

- Complete similar proofs for the cases $\wedge, \vee$.

Completeness proof (Contd.)

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Or at least read it from Huth & Ryan's book!

Completeness proof (Contd.)

Exercise 6.3

- Complete similar proofs for the cases $\wedge, \vee$.

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This completes the proof of the Lemma, finishing Step 2.2 of the overall proof.

Completeness of Natural Deduction for propositional logic

Back to Step 2: If $\models G$ holds, then $\vdash G$ is valid; if G is a tautology, then it is a theorem!

Proof

- Let $p_1, \ldots, p_m$ be the propositional variables in G; all 2^m valuations make G true.

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- Let $p_1, \ldots, p_m$ be the propositional variables in G; all 2^m valuations make G true.
- From this, we derive a proof of G
  - Step 2.1 Key Idea: Encode each line of truth table as a sequent!
    - If in line ℓ of truth table, p_i has value 1, then set $\hat{p}_i = p_i$ otherwise set $\hat{p}_i = \neg p_i$
    - Sequent for line ℓ: $\hat{p}_1 \ldots \hat{p_m} \vdash G$
  - Step 2.2: Use Previous Lemma to Construct proofs for each of 2^m sequents.
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    - Sequent for line ℓ: $\hat{p_1} \ldots \hat{p_m} \vdash G$
  - Step 2.2: Use Previous Lemma to Construct proofs for each of 2^m sequents.
    - For each line ℓ, by Previous Lemma, we have $\hat{p_1} \ldots \hat{p_m} \vdash G$ is valid.
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Completeness of Natural Deduction for propositional logic

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Proof

- Let $p_1, \ldots, p_m$ be the propositional variables in G; all 2^m valuations make G true.
- From this, we derive a proof of G
  - Step 2.1 Key Idea: Encode each line of truth table as a sequent!
    - If in line ℓ of truth table, p_i has value 1, then set $\hat{p_i} = p_i$ otherwise set $\hat{p_i} = \neg p_i$
    - Sequent for line ℓ: $\hat{p_1} \ldots \hat{p_m} \vdash G$
  - Step 2.2: Use Previous Lemma to Construct proofs for each of 2^m sequents.
    - For each line ℓ, by Previous Lemma, we have $\hat{p_1} \ldots \hat{p_m} \vdash G$ is valid.
  - Step 2.3: Assemble the sequents to obtain proof of G
    - Use LEM repeatedly on all the propositional variables of G and $\vee$-e to obtain $\vdash G$.

An example

Consider the tautology $F = p \wedge q \to p$.

Exercise 6.4 Show that it is a theorem by applying all steps of proof above.

- Has two propositions p, q. So 4 valuations
- Then applying Lemma we have (derive them!):
  - $p, q \vdash p \wedge q \to p$
  - $\neg p, q \vdash p \wedge q \to p$
  - $p, \neg q \vdash p \wedge q \to p$
  - $\neg p, \neg q \vdash p \wedge q \to p$
- Using LEM to show $\vdash F$...

An example: Assembling the Proof

Assemble the four sequents via LEM and $\vee e$, applied three times — exactly as in Huth & Ryan:

1.	$p \vee \neg p$	LEM
2.		

An example: Assembling the Proof

Assemble the four sequents via LEM and $\vee e$, applied three times — exactly as in Huth & Ryan:

1.	$p \vee \neg p$	LEM
2.	p	assumption
3.	$q \vee \neg q$	LEM
4.		

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1.	$p \vee \neg p$	LEM
2.	p	assumption
3.	$q \vee \neg q$	LEM
4.	q	assumption
5.	$p \wedge q \to p$	Use Lemma
6.		

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1.	$p \vee \neg p$	LEM
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6.	$\neg q$	assumption
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8.		

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1.	$p \vee \neg p$	LEM
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5.	$p \wedge q \rightarrow p$	Use Lemma
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8.	$p \wedge q \rightarrow p$	$\vee e$ 3,4-5,6-7
9.		

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9.	$\neg p$	assumption
10.	$q \vee \neg q$	LEM
11.		

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10.	$q \vee \neg q$	LEM
11.	q	assumption
12.	$p \wedge q \to p$	Use Lemma
13.		

An example: Assembling the Proof

Assemble the four sequents via LEM and $\vee e$, applied three times — exactly as in Huth & Ryan:

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11.	q	assumption
12.	$p \wedge q \to p$	Use Lemma
13.	$\neg q$	assumption
14.	$p \wedge q \to p$	Use Lemma
15.	$p \wedge q \to p$	$\vee e$ 10,11-12,13-14
16.		

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1.	$p \vee \neg p$	LEM
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16.	$p \wedge q \rightarrow p$	$\vee e$ 1,2-8,9-15

Conclusion!

Propositional Logic is sound and complete.

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This means we can freely switch between semantic ($\models$) and syntactic ($\vdash$) reasoning — pick whichever is easier for the problem at hand!

Are there other proof systems

- What we have looked at is called Gentzen-style Natural Deduction.
- Was introduced in 1934!

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- If you want to prove F, you should prove F, not disprove $\neg F$!
- No proof by contradiction also!

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- No proof by contradiction also!

Exercise 6.5 Why are Intuitionistic logics interesting? What was Brouwer-Hilbert controversy?

Topic 6.2

Remarks and Extensions

A notable point

Showing invalidity of a sequent

- To show a sequent $F_1, \ldots F_n \vdash G$ is valid, you just derive a proof.

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Use soundness!

1. Just find an assignment such that $F_1 = 1, \ldots F_n = 1$ and $G = 0$.

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Showing invalidity of a sequent

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- How do you show that it is NOT valid?

Use soundness!

1. Just find an assignment such that $F_1 = 1, \ldots F_n = 1$ and $G = 0$.
2. This means $F_1 \ldots F_n \not\models G$.

A notable point

Showing invalidity of a sequent

- To show a sequent $F_1, \dots F_n \vdash G$ is valid, you just derive a proof.
- How do you show that it is NOT valid?

Use soundness!

1. Just find an assignment such that $F_1 = 1, \dots F_n = 1$ and $G = 0$.
2. This means $F_1 \dots F_n \not\models G$.
3. Now soundness implies: if $F_1 \dots F_n \vdash G$ is valid, then $F_1 \dots F_n \models G$ holds; the contrapositive gives: if $F_1 \dots F_n \not\models G$, then $F_1 \dots F_n \nvdash G$.

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Exercise 6.6: Prove or disprove: $p \vdash p \wedge q$ valid.

Another note regarding reusing what has been derived

Consider the theorem

$\vdash p \rightarrow (q \rightarrow p)$

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So, when can you copy? From above ... unless they depend on temporary assumptions whose box has been closed!