CS228 : Logic for Computer Science

Module 1: Propositional Logic

Lecture 7: Normal Forms

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IIT Bombay, India

So far: Propositional Logic

Module 1: Propositional Logic - Encoding knowledge into simple formulas

Logic: a formal language & calculus for reasoning

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- Propositional Logic Semantics (via truth tables)

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Can we say when these problems are easy?

- Exploit their structure!

Topic 7.1

Normal Forms

Normal forms

Questions

1. Can some representation make the formula easier to analyze?
   - Example: is $(x \vee \neg x) \vee (\neg x \wedge y) \vee (x \wedge y \wedge w) \vee (\neg x \wedge \neg z \wedge \neg w)$ valid?
2. Are there "Normal forms" which allow for easy and uniform algorithmic analysis?

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     - Would you write the truth table with 16 rows?
     - Easier: would you try to derive it from no premises?
     - Even easier: just observe that it's a disjunction of many subformulas, and the first one is valid, so the entire thing is valid!
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2. Are there "Normal forms" which allow for easy and uniform algorithmic analysis?
   - Building algorithms and tools for handling arbitrary formulas is difficult
   - Normalization also results in standardization and interoperability of tools.

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- $(p \oplus q) \equiv (p \vee q) \wedge (\neg p \vee \neg q)$
- $(p \leftrightarrow q) \equiv \neg(p \oplus q)$

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So our propositional formulas are only in terms of $\neg, \vee, \wedge$.

Topic 7.2

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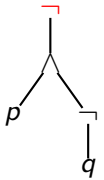
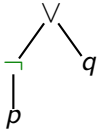


$\neg(p \wedge \neg q)$ ✗ Not in NNF



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Negation normal form (contd.)

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Convert the following formulas into NNF

- $\neg(p \to q)$
- $\neg(\neg((s \to \neg(p \leftrightarrow q))) \oplus (\neg q \vee r))$
- $\neg((p \to q) \to (p \to q))$
- $\neg\neg\neg p$

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Exercise 7.2 (Is the NNF "canonical"?)

Do there exist two distinct formulae that are both in NNF but are equivalent?

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Since $\vee$ is associative, commutative, and idempotent, a clause is sometimes referred to as a set of literals.
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idempotent:
$p \vee p \equiv p$
(repeats don't matter)

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Exercise 7.3 Which of the following are in CNF?

- $(p \vee q) \wedge (\neg p \wedge q)$
- $(p \wedge q) \vee (\neg p \vee q)$
- $p \rightarrow q$
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$CNF ::= Clause \mid Clause \wedge CNF$

$Clause ::= p \mid \neg p \mid Clause \vee Clause$

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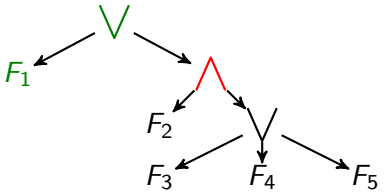
- Step 3: Use distributivity (see next slide!)

...

CNF conversion (contd.)

Proof(contd.)

Now the formulas have the following form with literals at leaves, e.g., $F_1 \vee (F_2 \wedge (F_3 \vee F_4 \vee F_5))$.



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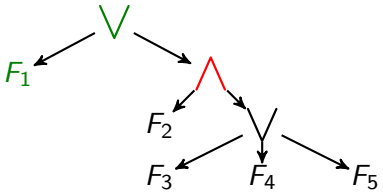

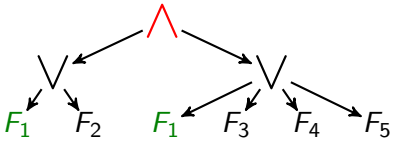

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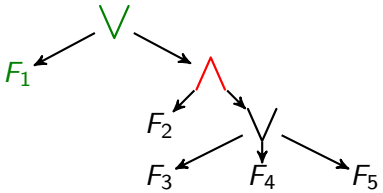

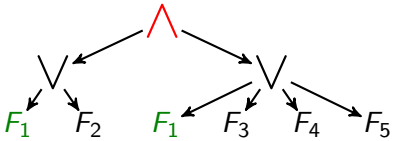

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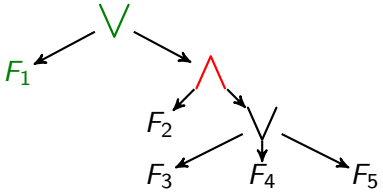

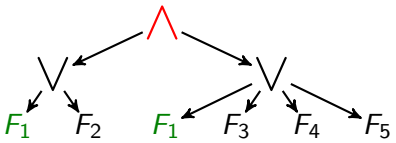

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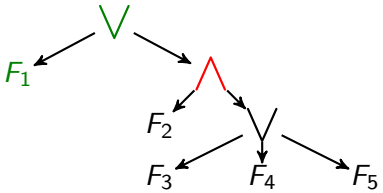

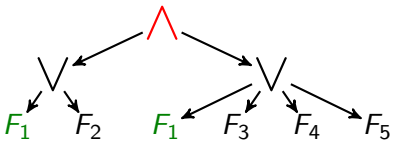

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Eventually, we will obtain a formula that is CNF. (Note how F_1 gets duplicated.) □

Exercise 7.5

Convert $(a \vee b) \wedge (c \vee (d \wedge e))$ to CNF using distributivity.

Solution: $(a \vee b) \wedge (c \vee (d \wedge e)) \equiv (a \vee b) \wedge (c \vee d) \wedge (c \vee e)$

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(NP-complete, informally: among the hardest problems whose solutions can be *verified* in polynomial time – if you could solve CNF-SAT quickly, you could solve *every* such problem quickly.)

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So given formula F, an important question is to convert it to CNF.

Topic 7.4

Tseitin's encoding

Conversion to CNF

We already saw a naive algorithm

- Step 1: Convert F to NNF.
- Step 2: Flatten the $\wedge$ and $\vee$ gates, e.g., $(x \wedge y) \wedge (z \wedge y) \wedge (w \wedge z) \equiv (x \wedge y \wedge z \wedge w)$
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Exercise 7.7 Give an example of a propositional formula with $2n$ variables whose conversion to CNF produces a formula with 2^n clauses.

Conversion to CNF (contd.)

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Tseitin's encoding

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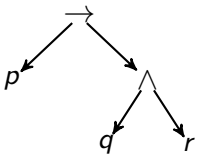

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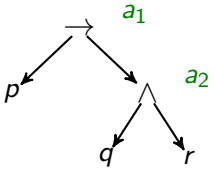

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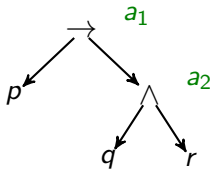

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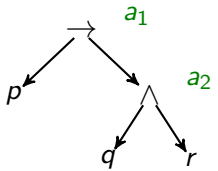

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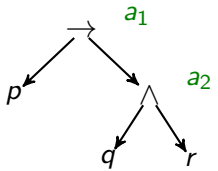

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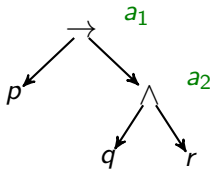

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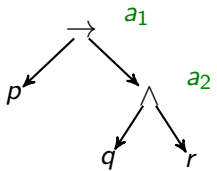

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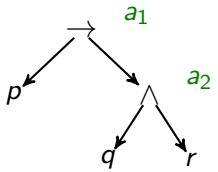

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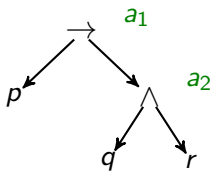

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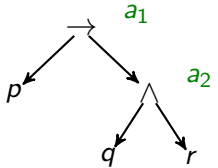

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Theorem: The Tseitin algo converts any formula into an equisatisfiable CNF formula.

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Proof: Exercise!

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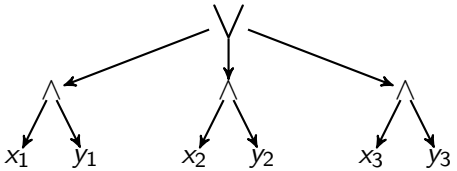

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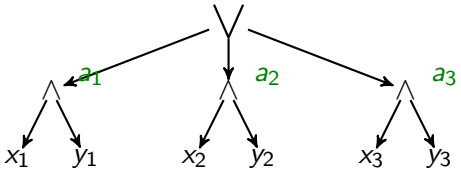

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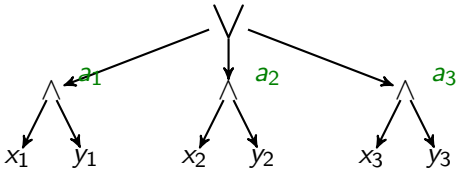

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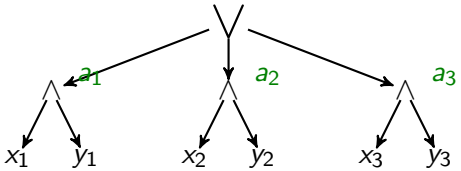

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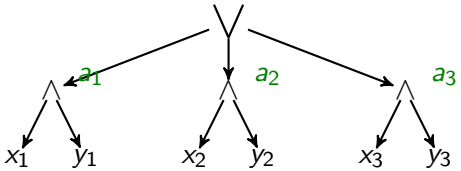
$\wedge(\neg a_2 \vee x_2) \wedge (\neg a_2 \vee y_2) \wedge (a_2 \vee \neg x_2 \vee \neg y_2)$
$\wedge(\neg a_3 \vee x_3) \wedge (\neg a_3 \vee y_3) \wedge (a_3 \vee \neg x_3 \vee \neg y_3)$
$\wedge(a_1 \vee a_2 \vee a_3)$

9 variables, 10 clauses – matches $3n + 1 = 10$ for $n = 3$.

Tseitin encoding of F_3

$F_3 = (x_1 \wedge y_1) \vee (x_2 \wedge y_2) \vee (x_3 \wedge y_3)$ – 6 variables.



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$$G_3 = (\neg a_1 \vee x_1) \wedge (\neg a_1 \vee y_1) \wedge (a_1 \vee \neg x_1 \vee \neg y_1)$$
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$$\wedge (a_1 \vee a_2 \vee a_3)$$

9 variables, 10 clauses – matches $3n + 1 = 10$ for $n = 3$. (If instead we added one extra variable a_4 for the root, we'd need $n + 1$ more clauses: 4 for $n = 3$, giving $14 = 4n + 2$.)

Topic 7.5

Disjunctive normal form

Disjunctive normal form (DNF)

Definition

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So, CNF makes validity easy, DNF makes satisfiability easy.