CS228 : Logic for Computer Science

Module 2: The Problem of Satisfiability

Lecture 8: (Propositional) Resolution

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Topics Covered till date (Module 1: Propositional Logic)

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   - Questions of interest: Satisfiability, Validity

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   - Natural Deduction
   - Soundness and Completeness

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Next... (Module 2: The Problem of Satisfiability)

Resolution and SAT-solvers

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Coming up: Resolution, and the algorithms powering modern SAT solvers.

Clauses as sets and CNF formulas as set of sets

Recall from Lecture 7

- A clause $\ell_1 \vee \cdots \vee \ell_n$ is written as the set $\{\ell_1, \ldots, \ell_n\}$.
- A CNF formula $C_1 \wedge \cdots \wedge C_n$ as the set of clauses $\{C_1, \ldots, C_n\}$.
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Notation

We extend the definition of clauses to the empty set of literals: $\perp$ is the empty clause.

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- If F is in CNF, then we can interpret F as a set of clauses. Does this help?

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$$\textsc{Resolution} \quad \frac{F \vee G \qquad \neg F \vee H}{G \vee H}$$

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E.g.: I drink coffee, or I fall asleep in class. I do not drink coffee, or I understand the lecture. Therefore, I fall asleep in class, or I understand the lecture.

Resolution

Main idea to check (un)satisfiability of formula F

- Convert F to CNF form as set of clauses (Use Tseitin Encoding!).
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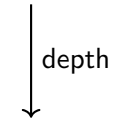

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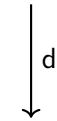

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4. Is it complete? That is, does it suffice to show unsatisfiability? i.e., when it terminates, if we didn't see $\perp$ does this imply F is satisfiable?

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Exercise 8.2: Prove this by induction on the length of the resolution sequence.

Proof of Exercise 8.2: Soundness of the Resolution Procedure

Suppose $C_1, \ldots, C_n$ is a resolution sequence from F ending in $C_n = \bot$: each C_i is either

- a clause of F, or
- the resolvent of two earlier clauses C_j, C_k $(j, k < i)$ in the sequence.

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2. **Inductive step:** C_i is the resolvent of C_j, C_k with $j, k < i$.
   - By the induction hypothesis (applied to $j, k < i$): $\pi \models C_j$ and $\pi \models C_k$.

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Suppose $C_1, \ldots, C_n$ is a resolution sequence from F ending in $C_n = \bot$: each C_i is either

- a clause of F, or
- the resolvent of two earlier clauses C_j, C_k $(j, k < i)$ in the sequence.

Claim (strong induction on i)

For every assignment π with $\pi \models F$: $\pi \models C_i$, for all $i = 1, \ldots, n$.

1. **Base case:** $C_i \in F$.
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- Taking $i = n$: if $\pi \models F$ then $\pi \models C_n = \bot$.
- But no assignment satisfies $\bot$ (the empty clause)!
- So no π satisfies F, i.e., $F \models \bot$, i.e., F is unsatisfiable. □

2. Termination of Resolution

Given formula F, we can write the resolution algo as:

Resolution Algorithm

1. Use Tseitin encoding to convert to F' in CNF form and write it as set of clauses.
2. Repeat till fixed point the following:
   - If possible, apply resolution and obtain new clause C i.e., $C \notin F'$. Set $F' := F' \cup \{C\}$.
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- So after these many steps we must stop, as there are no new clauses for sure!

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Before going to completeness, is this algorithm deterministic?

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- Notice: both resolvents here are tautologies (each contains a variable and its negation) – always true, hence useless for deriving $\bot$.
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- Thus, the algorithm is a non-deterministic proof search!
- Notice: both resolvents here are tautologies (each contains a variable and its negation) – always true, hence useless for deriving $\bot$.
- In practice, we focus on trying to ensure that we do not do unnecessary resolutions!
- However, note that we only have a single rule (resolution) instead of all of ND.
- Thus, we may be more effective in finding the proof strategy. This is what SAT solvers do!

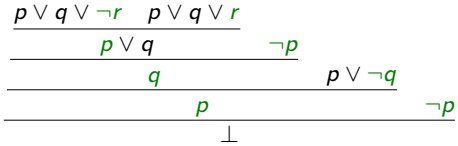
Exercises

Exercise 8.5. Show the following formulas using proof by resolution.

1. Show that $(p \vee q \vee \neg r) \wedge \neg p \wedge (p \vee q \vee r) \wedge (p \vee \neg q)$ is unsatisfiable.
2. Prove that $(p \to q) \wedge (q \to r) \vdash (p \to r)$ is valid.

Solutions: Exercise 8.5

(1) $F = (p \vee q \vee \neg r) \wedge \neg p \wedge (p \vee q \vee r) \wedge (p \vee \neg q)$



Hence F is unsatisfiable (depth 4).

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$$\dfrac{\dfrac{\dfrac{\dfrac{p \vee q \vee \neg r \quad p \vee q \vee r}{p \vee q} \quad \neg p}{q} \quad p \vee \neg q}{p} \quad \neg p}{\bot}$$

Hence F is unsatisfiable (depth 4).

(2) $(p \to q) \wedge (q \to r) \vdash (p \to r)$?

By refutation: negate the goal.

$F' = (p \to q) \wedge (q \to r) \wedge \neg(p \to r)$

But $\neg(p \to r) \equiv p \wedge \neg r$, giving clauses $\{(\neg p \vee q),\ (\neg q \vee r),\ p,\ \neg r\}$

$$\dfrac{\dfrac{\dfrac{\neg p \vee q \quad p}{q} \quad \neg q \vee r}{r} \quad \neg r}{\bot}$$

Hence unsatisfiable, so $(p \to q) \wedge (q \to r) \models (p \to r)$, i.e., $(p \to q) \wedge (q \to r) \vdash (p \to r)$. □

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4. Completeness

- Is resolution proof system, i.e., proof system where the only rule is Resolution, complete?
- In other words, if F is unsatisfiable, are we guaranteed to derive $F \vdash \bot$ via resolution?
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Clauses derivable with proofs of depth at most n

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Example

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Finiteness matters: for infinite sets of clauses this needs compactness, which we will not cover here.

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In the latter case, applying the resolution rule with $C = D = \emptyset$, we have the following resolution derivation of $\bot$.

$$\frac{p \qquad \neg p}{\bot}$$

...