CS228 : Logic for Computer Science

Module 2: The Problem of Satisfiability

Lecture 10: Clause Learning and the CDCL Algorithm

Instructor: S. Akshay

IIT Bombay, India

Topics Covered till date

Module 1: Propositional Logic

1. Syntax and Semantics
   - Questions of interest: Satisfiability, Validity

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Module 2: The Problem of Satisfiability

1. Propositional Resolution - soundness, termination and completeness

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Module 2: The Problem of Satisfiability

1. Propositional Resolution - soundness, termination and completeness
2. The DPLL algorithm
3. Today: clause learning, the CDCL algorithm, and modern SAT solvers

A Quick Recap: DPLL

A DPLL run makes a sequence of

- decisions: choose an unassigned variable p and a bit b, set $\pi(p) = b$

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Today

We open up DPLL's backtracking step: instead of just undoing the last decision, learn from a conflict which decisions actually caused it and jump back further – this is clause learning, and DPLL + clause learning is CDCL.

Implementation and Optimizations

In fact, in DPLL in practice

- The "pure" literal check is not done (too expensive!)
- Branching literal is not chosen randomly!
- Algo is implemented iteratively; with backtrack stack maintained explicitly! (may help to backtrack more than one level!)

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- **clause learning**
- 2-watched literals
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We will look at one such optimization – clause learning – in detail, next.

Topic 10.1

Clause learning

Revisiting DPLL

Exercise 10.1: run DPLL on this set of clauses, deciding p_6, p_7, p_1 in that order: p_6=0, p_7=0, p_1=1

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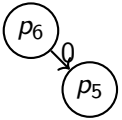

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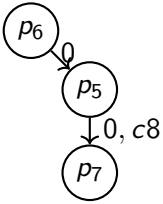

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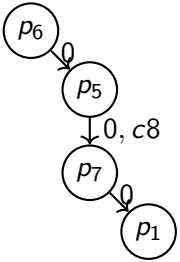

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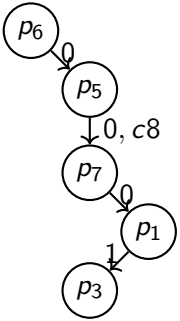

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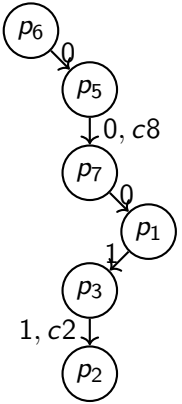

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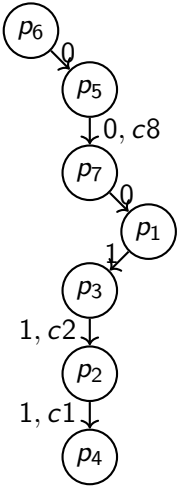

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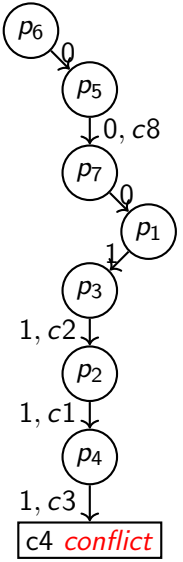

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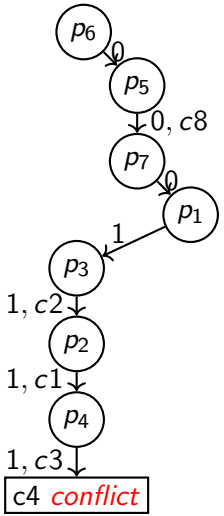

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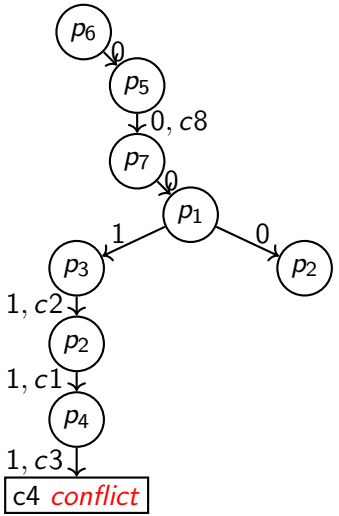

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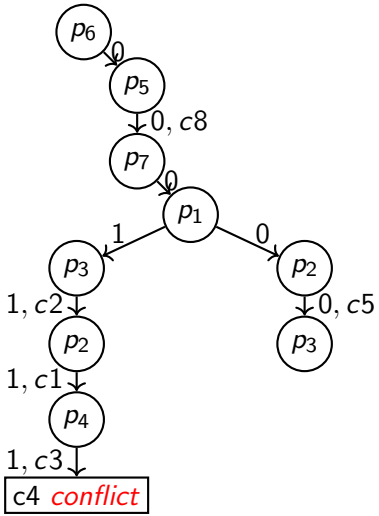

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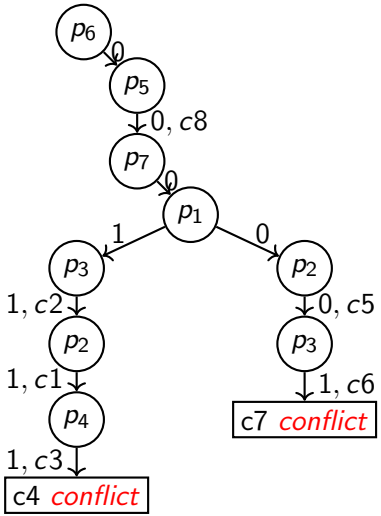

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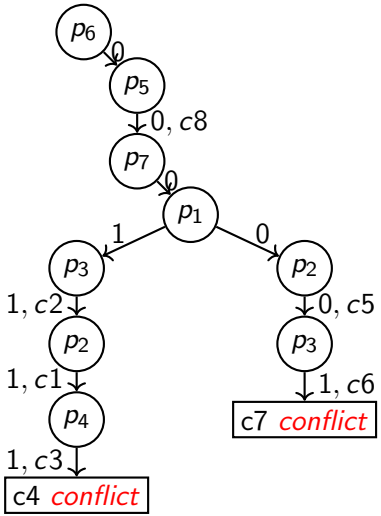

Both values of p_1 fail – backtrack further, to p_7

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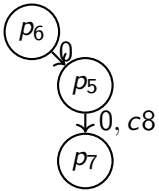

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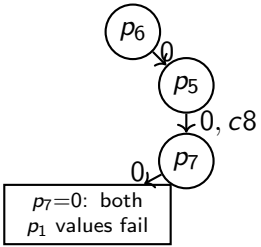

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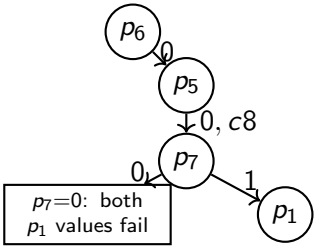

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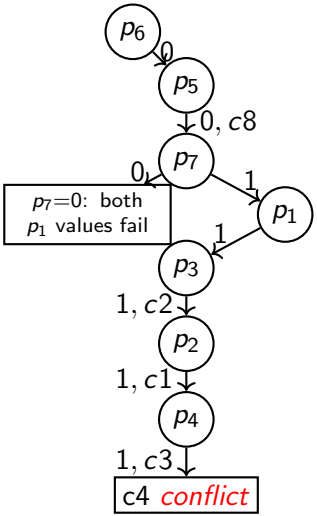

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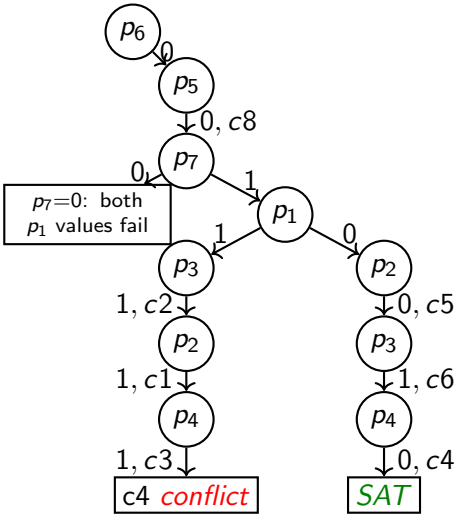

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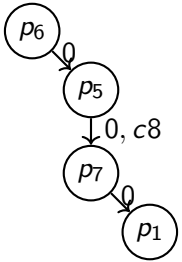

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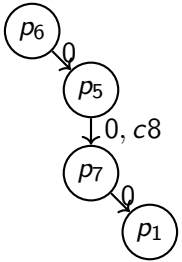
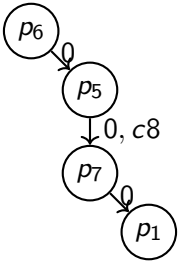


Definition

The decision level of a true literal is the number of decisions after which the literal was made true.

Decision level

Consider just the partial run so far, up to p_1.



Definition

The decision level of a true literal is the number of decisions after which the literal was made true.

Given the above run, we write $\neg p_6@1$ and $\neg p_5@1$ (p_5 was propagated while only one decision had been made), $\neg p_7@2$.

Example: implication graph

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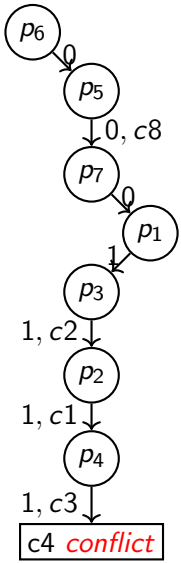

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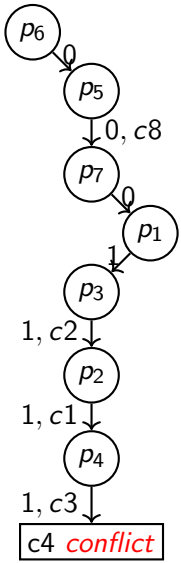

Implication graph

$\neg p_6$@1

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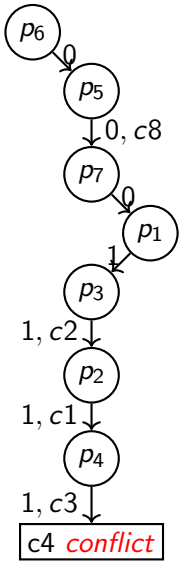

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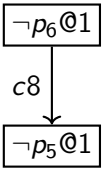

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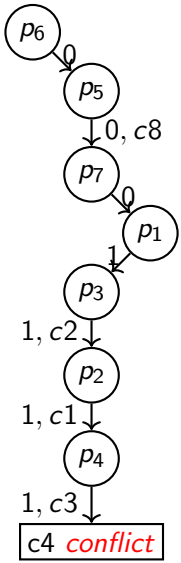

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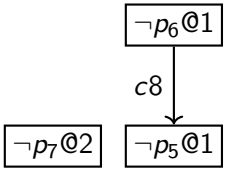

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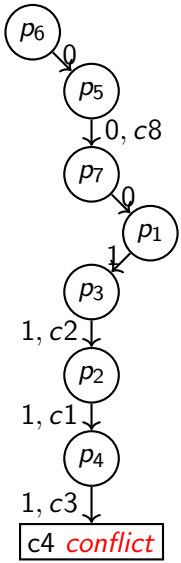

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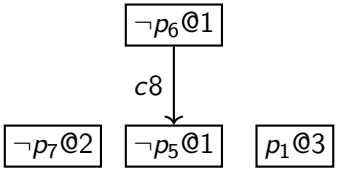

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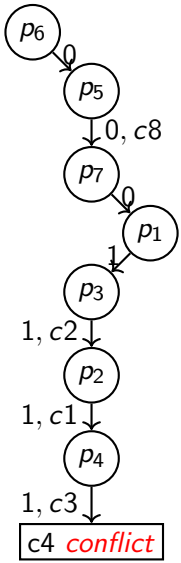

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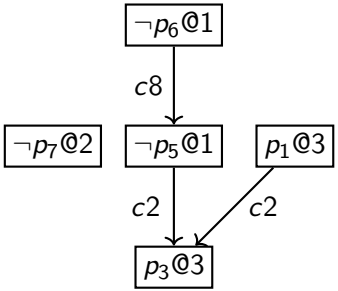

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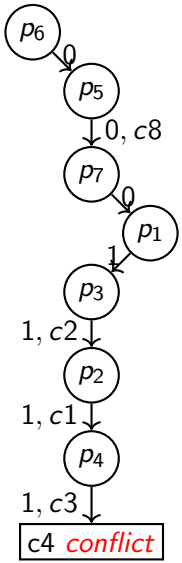

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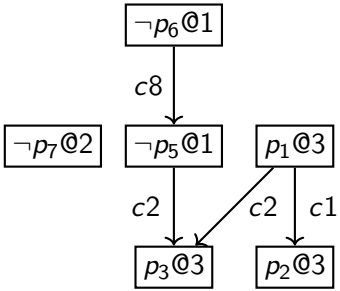

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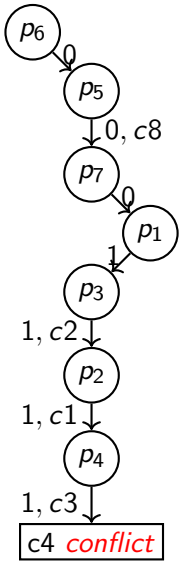

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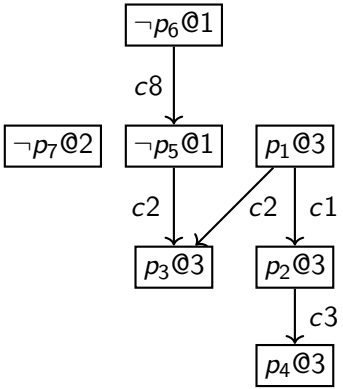

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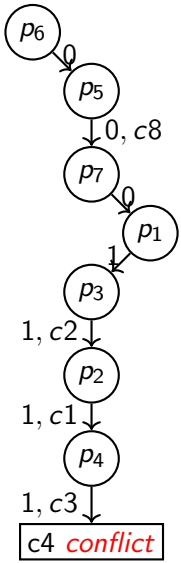

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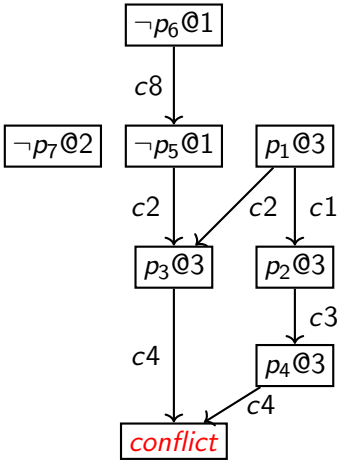

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So what is the use of drawing this implication graph?

Conflict clause

We traverse the implication graph backwards to find the set of decisions that caused the conflict.

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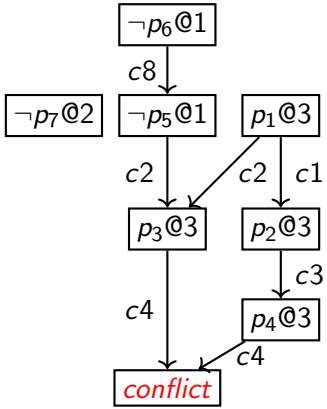

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So what do we do with this learned conflict clause?

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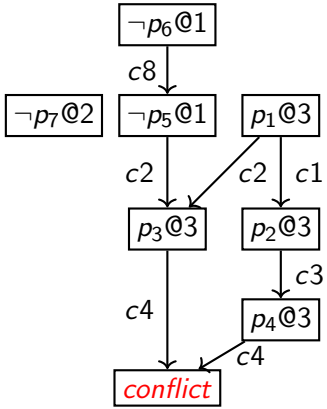

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So what do we do with this learned conflict clause?
Add it to our formula!

What happens when we add the learned clause?

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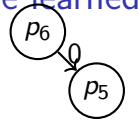

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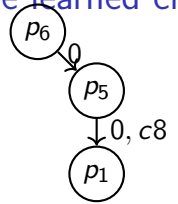

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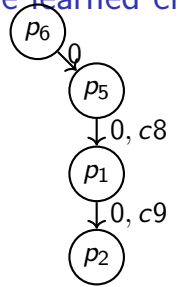

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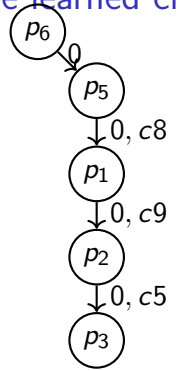

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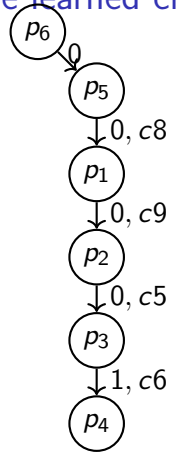

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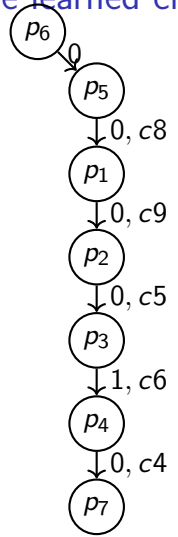

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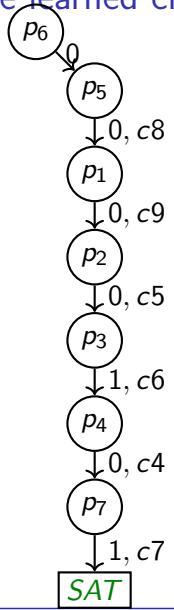

p_6 is the only decision – everything else is forced!

Clause learning

Clause learning heuristics

- add conflict clause in the input clauses and
- backtrack to just after the second last conflicting decision, and proceed like DPLL

Clause learning

Clause learning heuristics

- add conflict clause in the input clauses and
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Lemma

Let C be the conflict clause. Then

1. $F \models C$ (so adding C does not change the set of satisfying assignments)

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2. the conflicting partial assignment will never be tried again

Exercise 10.2

1. Prove the above Lemma. (Hint: for part 1, show $F \wedge \neg C$ is unsat.)
2. Can you prove $F \vdash C$ using resolution?

Implication graphs and resolution proofs!

$$c_1 = (\neg p_1 \vee p_2)$$
$$c_2 = (\neg p_1 \vee p_3 \vee p_5)$$
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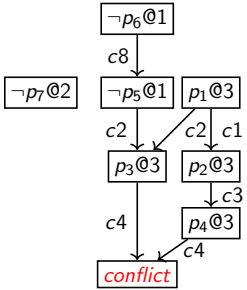

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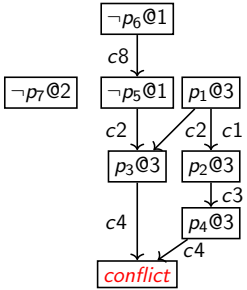

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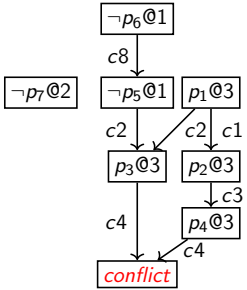

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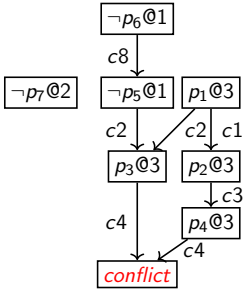

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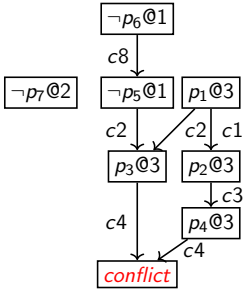

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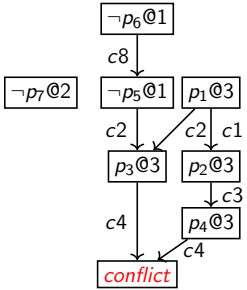

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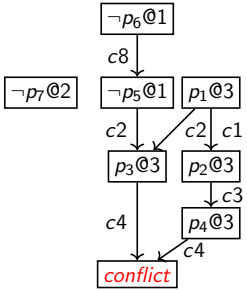

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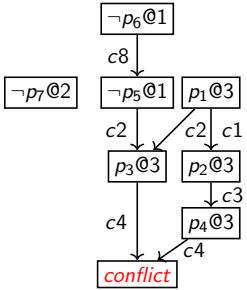

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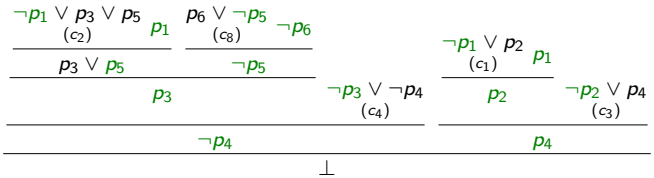

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2. Records past work of the SAT solver

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In the previous example, we made decisions : $m(p_6) = 0$, $m(p_7) = 0$, and $m(p_1) = 1$

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- There are many other clever choices for conflict clauses.
- DPLL+conflicts = CDCL (Conflict-driven clause learning!)

Algorithm 10.1: CDCL

Input: CNF F

$m := \emptyset$; $dl := 0$; $dstack := \lambda x.0$;

UnitPropagation(m, F);

do

while ;

- UnitPropagation(m, F) - applies unit propagation and extends m

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  if *m is partial* **then**

    $dstack(dl) := m.size()$;

    $dl := dl + 1$; Decide(m, F); UnitPropagation(m, F) ;

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  // backtracking
  while $m \not\models F$ **do**
    if $dl = 0$ **then return** *unsat*;
    $(C, dl) :=$ AnalyzeConflict(m, F);
    $m.resize(dstack(dl))$; $F := F \cup \{C\}$; m := UnitPropagation(m, F);
  // Boolean decision
  if m *is partial* **then**
    $dstack(dl) := m.size()$;
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- UnitPropagation(m, F) - applies unit propagation and extends m
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  if m *is partial* **then**

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while m *is partial or* $m \not\models F$;

return *sat*

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Other improvements

Decide

Use involved heuristics to decide which unassigned variable to branch from.

- VSIDS: score each variable by how often it appears in recent conflicts (score decays over time); branch on the highest-scoring variable.

Exercise 10.3: for Exercise 10.1's clauses, show that deciding p_1 first reaches SAT with no conflict at all – decision order matters!

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- Clause deletion: learned clauses pile up fast – periodically discard the less useful ones to bound memory and keep propagation fast.

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Resets

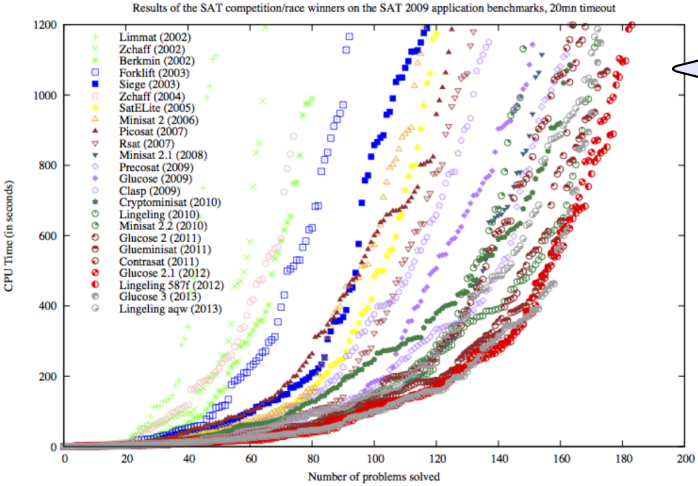
At some point give up and re-start!

Topic 10.2

SAT technology and its impact

Efficiency of SAT solvers over the years

(chart as of 2014 – solvers have improved substantially further since!)



Every solver here, from 2002 on, is a Chaff-lineage CDCL solver – this whole plot is watching CDCL refine itself for two decades!

Cactus plot:

X-axis: # problems solved

Y-axis: CPU time taken

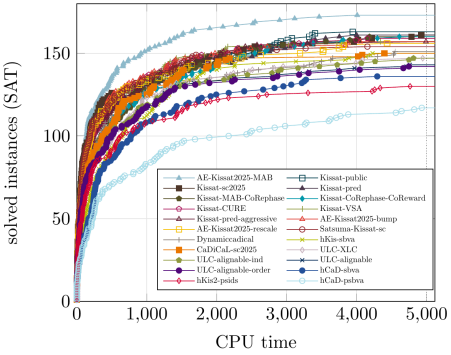
each curve = one solver

further right & lower = faster

Source: `http://satsmt2014.forsyte.at/files/2014/07/SAT-introduction.pdf`

SAT solvers today: Competition 2025

Main (Sequential) Track SAT Plot



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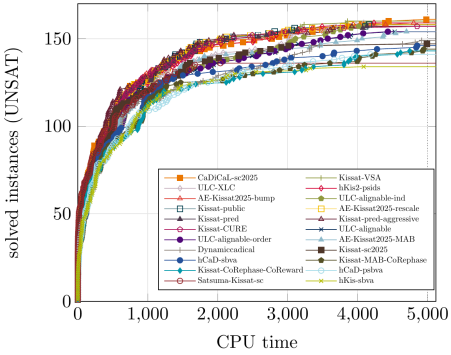

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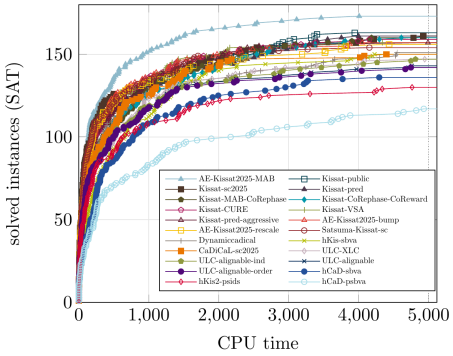

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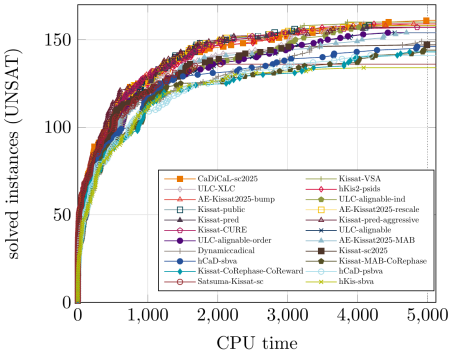

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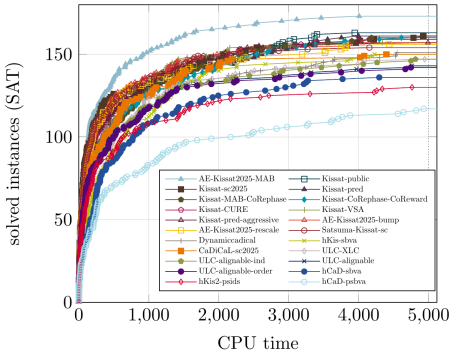

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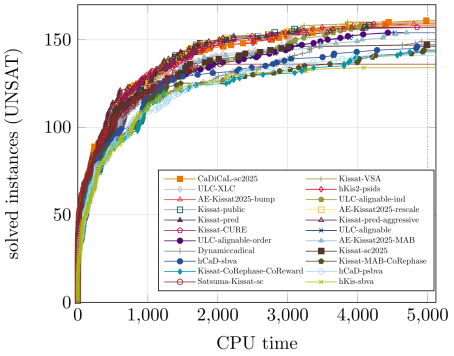

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Impact of SAT technology

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A few are listed below

- Hardware verification and design assistance
  Almost all hardware/EDA companies have their own SAT solver
- AI/Planning: many resource allocation problems are convertible to SAT
- Security: analysis of crypto algorithms
- Solving hard problems, *e. g.*, travelling salesman problem

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- Hardware verification and design assistance
  Almost all hardware/EDA companies have their own SAT solver
- AI/Planning: many resource allocation problems are convertible to SAT
- Security: analysis of crypto algorithms
- Solving hard problems, *e. g.*, travelling salesman problem

Next class

We will see a SAT solver in action! Come with your laptop, with a solver installed – instructions will be given on Piazza.