CS228 : Logic for Computer Science

Module 3: Predicate Logic

Lecture 12: Syntax of Predicate Logic

Instructor: S. Akshay

IIT Bombay, India

Topics Covered till date

Module 1: Propositional Logic

1. Syntax and Semantics
   - Questions of interest: Satisfiability, Validity

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   - Natural Deduction, Soundness and Completeness

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1. Propositional Resolution - soundness, termination and completeness

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  - Variables – to be able to say "$\forall\ x\ S(x)$" instead of writing each student out...
- e.g., $\forall x(S(x) \rightarrow \exists y(T(y) \wedge Y(x,y)))$ For all x, if x is a student, then there is some y which is a teacher such that x is younger than y.

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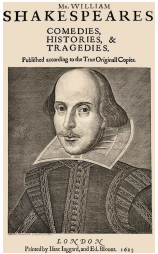

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Fun fact: this proverb is a common misquote of Shakespeare's The Merchant of Venice – the original line is "All that glisters is not gold."

Topic 12.1

Syntax of Predicate Logic

Another difference from Propositional Logic

Exercise 12.2 Encode into Predicate logic.

Asif and Lavanya have the same grandfather.

- Use "variables" a for "Asif", ℓ for "Lavanya"
- Binary predicate F for "being father", i.e., $F(x, y)$ if y is father of x.

Function Symbols: Suppose we had a function symbol f(x) to mean father of x.

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Important note: Equality $=$ is a special binary predicate symbol. Always assumed to be present!

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Fixed symbols

- All symbols from prop. logic: $\wedge, \vee, \neg, \rightarrow, (,)$
- New ones: $\exists, \forall$ (quantifiers)

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- $f(a)$ (father of Asif), $f(f(a))$ (grandfather of Asif) are terms too!
- Careful: S(Osim) (recall the earlier example!) is *not* a term!
  - f is a function symbol: maps objects to objects, so $f(a)$ *denotes an object* – a term.
  - S is a predicate symbol: maps objects to truth values, so S(Osim) is True or False – not a term!

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Why "First Order"?

Called First order logic because quantification is only over individual variables...

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Binding rules as expected!

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Exercise 12.3

1. Draw the Parse tree for $\forall x((P(x) \rightarrow Q(x)) \wedge S(x, y))$.
2. Encode into Predicate logic and draw its Parse tree: every son of my father is my brother.

Encoding "every son of my father is my brother"

Using Predicates Only:

$F(x, y)$: x father of y; $S(x, y)$: x son of y; $B(x, y)$: x brother of y; m: "me".

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$$\forall x \forall y (F(x, m) \wedge S(y, x) \rightarrow B(y, m))$$

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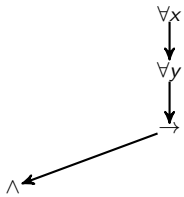

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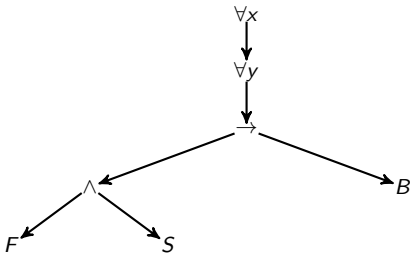

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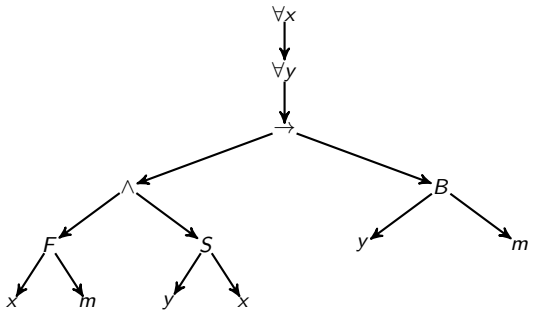

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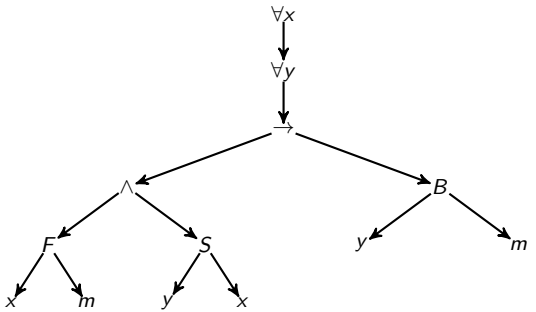

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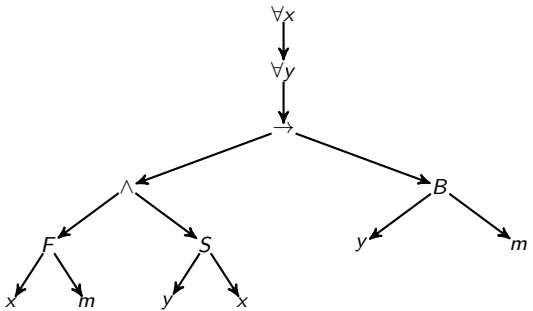

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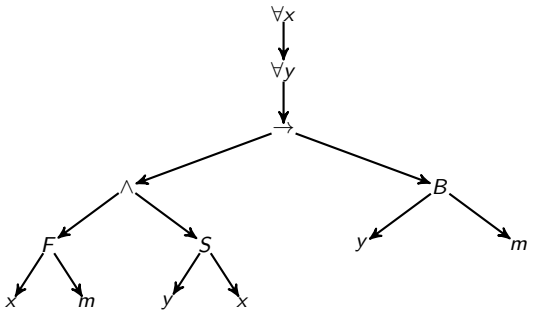

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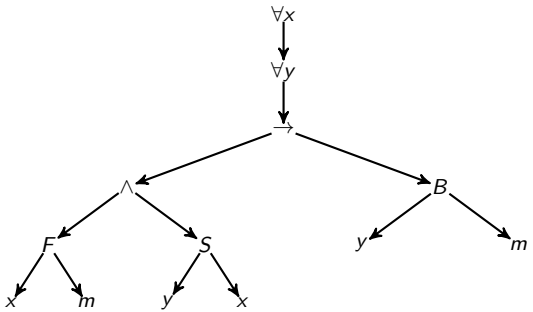

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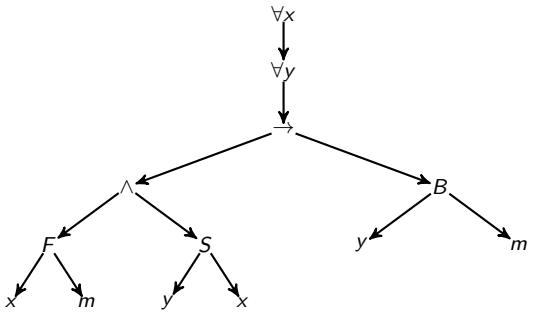

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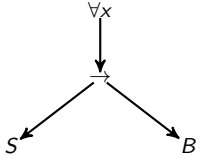

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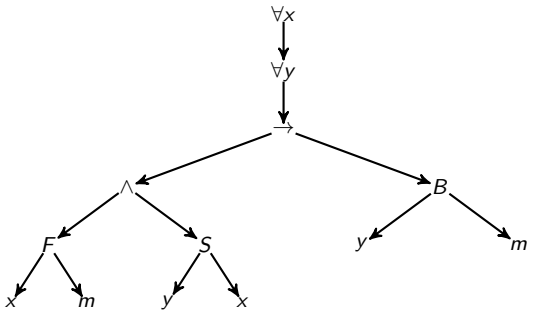

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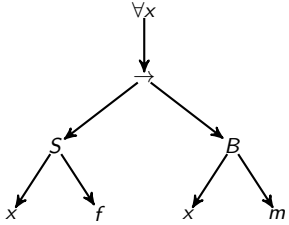

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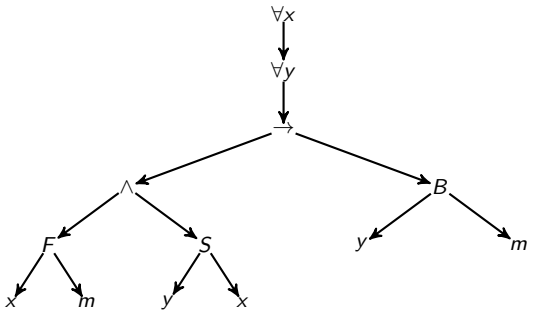

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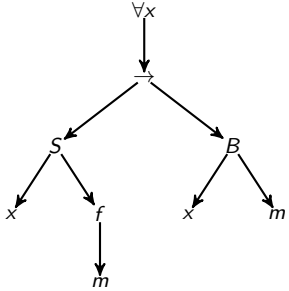

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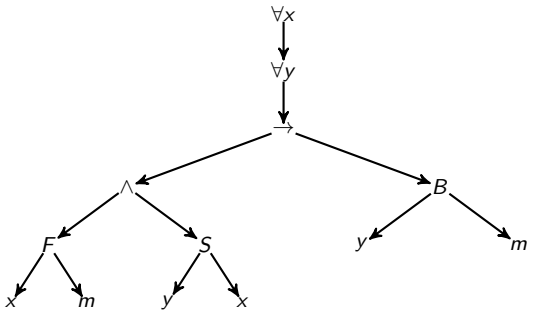

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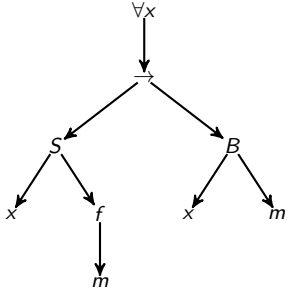

Much more compact than the predicates-only version – this is exactly why function symbols are useful!

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3. Are free and bound variables complements? Why or why not?

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Recursive definition of $bnd(\varphi)$, alongside $free(\varphi)$

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Definition

Term t is free for x in φ if no free occurrence of x in φ is in the "scope" of $\forall y$ or $\exists y$ for any var y occurring in t.

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- If we naively try substituting $t = f(y, x)$ for x in ψ, we get $\exists y R(f(y, x), y) \vee \forall x R(z, x)$
- Problem: the y from t is now captured by $\exists y$ – it's no longer free like it was in t!

Definition

Term t is free for x in φ if no free occurrence of x in φ is in the "scope" of $\forall y$ or $\exists y$ for any var y occurring in t. e.g., $f(z, x)$ is free for x in ψ, but $f(y, x)$ is not.

Substitution in FOL

Variables are placeholders; so we must be able to replace them with concrete terms: substitution.

- Suppose $x \in \mathit{free}(\varphi)$ and t is any term.
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$\varphi[t/x]$ is formula obtained by replacing each free occurrence of x in φ by t, if t is free for x in φ.

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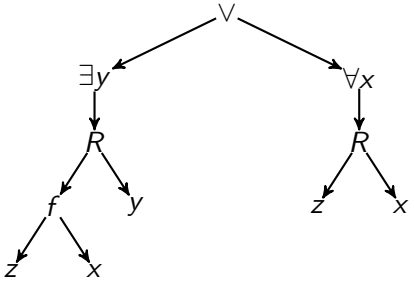
For ψ defined above: $\psi[f(z, x)/x] := \exists y R(f(z, x), y) \vee \forall x R(z, x)$

Substitution in FOL: Correct vs. Captured

Correct: $\psi[f(z,x)/x]$

$\exists y\, R(f(z,x),y) \vee \forall x\, R(z,x)$



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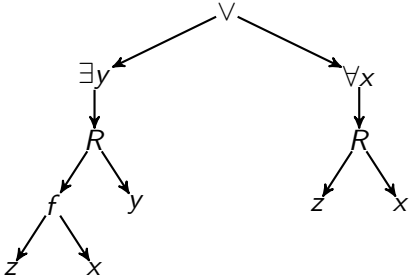

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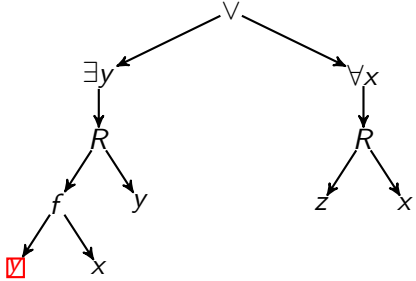

The y inside $f(y,x)$ (boxed in red) is now captured by $\exists y$ – it's a different leaf than the term's original free y, but shares its scope!

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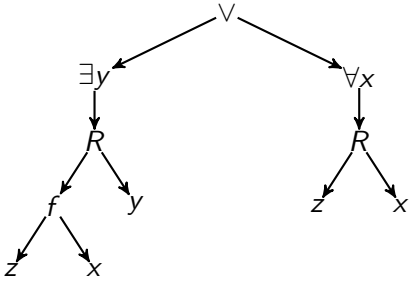

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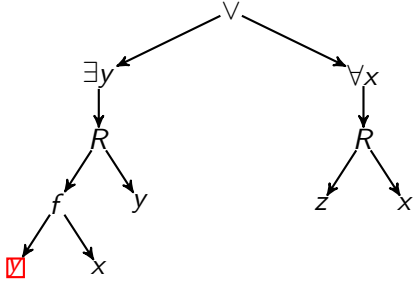

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- Fix: rename the bound y to a fresh y' first
- Rewrite ψ as $\exists y'\, R(x,y') \vee \forall x\, R(z,x)$ (same meaning, y is just a placeholder), *then* substitute.