CS228 : Logic for Computer Science

Module 3: Predicate Logic

Lecture 13: Semantics of Predicate Logic

Instructor: S. Akshay

IIT Bombay, India

Logistics

Syllabus for Mid-Semester Exam

- Module 1: Propositional Logic
- Module 2: The Problem of Satisfiability
- All lectures till and including Lecture 11

Mid-Semester Exam

- Venue: LA 201, LA 002; PwD students at KR 125
- Seating arrangement will be shared on Piazza
- 1 A4 size paper allowed as cheatsheet, but MUST be handwritten with blue ink – no photocopy, etc.

Topics Covered till date

Module 1: Propositional Logic

1. Syntax and Semantics ause
2. Formal proof systems
3. Normal Forms

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Module 3: Predicate Logic

1. Syntax of predicate logic (Lecture 12)
2. Today: semantics of predicate logic

Quick Recap: Syntax of Predicate Logic

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Today: what do these formulas actually *mean*?

Topic 13.1

Semantics of Predicate Logic

Semantics of FOL

Consider an example:

Example 13.1 $\psi_1 := \forall x \forall y (P(x,y) \rightarrow \exists z (\neg(z = x) \wedge \neg(z = y) \wedge P(x,z) \wedge P(z,y)))$

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Truth value of any formula φ depends on (i) underlying set from which variables take values and (ii) interpretation of constants, functions and predicates, i.e., Vocabulary!

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- Vocabulary $\mathcal{V}$. E.g., $\mathcal{V} = \{a, f, =, P\}$
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   - We will also use $\alpha[x \mapsto u]$ to denote the env which maps x to u (other vars are mapped according to α).

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    - With $x = 0$ it is not satisfied.
    - But, $x = 1$ works! And that is all we need.

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  - Finally, is there a value of x from the universe $\mathbb{N}$ that makes their conjunction true?
    - With $x = 0$ it is not satisfied.
    - But, $x = 1$ works! And that is all we need.
    - In other words, the env α, where we set $x = 1$ (ignoring the earlier value) satisfies the formula, since $(1 < 2 \wedge 1 \neq 0)$ is true!
  - Thus, we can conclude $M, \alpha \models \psi_2$!

Semantics of FOL: An Example First

- **Example 13.2** Consider again $\psi_2 := \exists x(P(x, f(y, a)) \land \neg(x = a))$ over Vocabulary $\mathcal{V} = \{a, f, =, P\}$
- Also, let $M = (\mathbb{N}, (0, +, <))$ be a $\mathcal{V}$-structure, and $\alpha(x) = 0, \alpha(y) = 2$ be the env.
- Does M, α satisfy ψ_2?
  - First compute $f(y, a) = y + a = 2 + 0 = 2$ (using $\alpha(y) = 2, a^M = 0$).
  - Is $P(x, f(y, a))$ true, i.e. is $x < 2$?
  - Is $\neg(x = a)$ true, i.e. is $x \neq 0$?
  - Is their conjunction true, i.e. is $x < 2$ and $x \neq 0$?
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Now, let's make this definition precise and generic!

Semantics of FOL: Formal Definition

Given $\mathcal{V}$-structure/model M and env α, we define $M, \alpha \models \varphi$ (satisfaction relation)

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We omit $\vee, \to, \Leftrightarrow$ here since they reduce to $\wedge, \neg$ in the usual way (e.g. $\varphi_1 \to \varphi_2 \equiv \neg\varphi_1 \vee \varphi_2$) – same treatment as $\wedge$ above.

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Satisfaction only depends on the “free” environment

Exercise 13.2

Show that if α, α' are identical on the set of free variables, then for any formula φ and any $\mathcal{V}$-structure M, $M, \alpha \models \varphi$ iff $M, \alpha' \models \varphi$.

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hint: use structural induction!

- Thus, if φ has no free variables, i.e., it is a sentence, then α does not matter!

For a sentence φ, we just write $M \models \varphi$.

Recall: Truth Depends on the Structure

In Example 13.1 $\psi_1 := \forall x \forall y (P(x,y) \rightarrow \exists z (\neg(z = x) \wedge \neg(z = y) \wedge P(x,z) \wedge P(z,y)))$

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- For structure $M_1 = (\mathbb{R}, <)$: $M_1 \models \psi_1$ (density!)

What is the vocab here?
Why is there no α?

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What happens when $M' = (\mathbb{N}, (1, \times, <))$? Is there an env which makes the formula true/false?

Revisiting substitution

Exercise 13.4 Let $\psi_3(x) := \exists y\,(y > x)$ over $\mathcal{V} = \{>\}$, with $M = (\mathbb{N}, >)$.

1. What is *free*(ψ_3)? For any α, does $M, \alpha \models \psi_3$ hold?
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Recall Substitution in FOL: $t = y$ is not free for x in ψ_3: (y occurs in t, and the free x is under the scope of $\exists y$)

- Substituting anyway, step by step:

$$\exists y\,(y > x) \quad \longrightarrow \quad \exists y\,(y > y)$$

- So $\psi_3[y/x] = \exists y\,(y > y)$
- Is *this* true?

Revisiting substitution

Exercise 13.4 Let $\psi_3(x) := \exists y\,(y > x)$ over $\mathcal{V} = \{>\}$, with $M = (\mathbb{N}, >)$.

1. What is $\mathit{free}(\psi_3)$? For any α, does $M, \alpha \models \psi_3$ hold?
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The truth value flipped. This is exactly why substitution is restricted to terms that are free for x.

Semantic entailment

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- ψ is **valid** iff $M, \alpha \models \psi$ holds for all M and α.
- Γ is **consistent** iff there is at least one M and α such that $M, \alpha \models \varphi$ for all $\varphi \in \Gamma$.

Note: M ranges over *all* universes and *all* interpretations of $\mathcal{V}$ on them – not a fixed universe!

Exercise 13.5 Give 2 examples of valid and 2 of satisfiable (but not valid) formulas with the vocab given in Example 13.2.

Exercise 13.5 Solution

Vocab: constant a, unary function f, binary predicate P (plus always-present $=$).

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2. $\exists x\, P(x, a)$ – true if $U = \mathbb{Z}$, P interpreted as "$<$", $a = 0$ (witness $x = -1$); false in $M_4 = (\mathbb{N}, (0, +, <))$ since no natural number is < 0. Satisfiable, not valid.

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Exercise 13.6 Prove the following: Quantifier equivalences

- $\forall\ x\varphi \equiv \neg\exists\ x\neg\varphi$
- $\forall x \forall y \varphi \equiv \forall y \forall x \varphi$
- $\forall x(\varphi_1 \land \varphi_2) \equiv (\forall x \varphi_1) \land (\forall x \varphi_2)$
- $\exists x(\varphi_1 \lor \varphi_2) \equiv (\exists x \varphi_1) \lor (\exists x \varphi_2)$

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What about satisfiability? Also undecidable!