CS310 : Automata Theory 2019

Lecture 31: Rice's theorem and other undecidable problems

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Recap

Turing machines and computability

- 1. Definition of Turing machines: high level and low-level descriptions
- 2. Variants of Turing machines
- 3. Decidable and Turing recognizable languages
- 4. Church-Turing Hypothesis
- 5. Undecidability and a proof technique by diagonalization
 - A universal TM lang $L_{TM}^A = \{ \langle M, w \rangle \mid M \text{ is a TM and } M \text{ accepts } w \}$
- 6. Reductions: a powerful way to show undecidability.
- 7. Rice's theorem



Rice's Theorem



Rice's Theorem

Let *P* be a non-trivial property of r.e. languages. Then $\mathcal{L}_P = \{ \langle M \rangle \mid L(M) \in P \}$ is undecidable.

For the following, is Rice's theorem applicable?

- 1. $\{\langle M \rangle \mid M \text{ runs for 5 steps on word 010}\}$. No. Property of TMs.
- 2. $\{\langle M \rangle \mid M \text{ has at most 25 states.}\}$. No. Property of TMs.
- 3. { $\langle M \rangle \mid L(M)$ is recognized by a TM with at least 25 states.}. No. Trivial property.
- 4. $\{\langle M \rangle \mid L(M) \text{ is recognized by a TM with at most 25 states and tape alphabet at most 10.}\}$. Yes.
- 5. $\{\langle M \rangle \mid L(M) \text{ is infinite.}\}$. Yes.
- {⟨M⟩ | M with alphabet {0,1, ⊔} ever prints three consecutive 1's on the tape}. No. Property of TMs, but undecidable!



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- ▶ For No answers, language can still be decidable or undecidable.
- ► If Rice's theorem does not apply, fall back on reductions!



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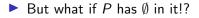


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 - Thus M' either acc \emptyset or L depending on if M acc w.
- ▶ Thus $\langle M' \rangle \in \mathcal{L}_P$ iff $L(M') \in P$ iff L(M') = L iff M acc w.
- This gives an algo for A_{TM} so contradiction!





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- ▶ But what if P has Ø in it!?
- ► Take <u>P</u>.
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- Apply proof to get undecidability of $\mathcal{L}_{\overline{P}}$.
- Conclude undecidability of \mathcal{L}_P .

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Computers, C-programs, counter machines



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Computers, C-programs, counter machines But these are just Turing machines?



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- Computers, C-programs, counter machines
- Problems on CFLs: Given CFG G, is $L(G) = \Sigma^*$?
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- ▶ Problems on CFLs: Given CFG *G*, is $L(G) = \Sigma^*$?
- Problems on Tiling
- Problems on String Matching



A string matching problem

Given two lists $A = \{s_1, \ldots, s_n\}$ and $B = \{t_1, \ldots, t_n\}$, over the same alphabet, is there a sequence of combining elements that produces the same string in both lists?



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Consider the following lists

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$$A = \{110, 0011, 0110\}$$
 and $B = \{110110, 00, 110\}$



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- $A = \{0011, 11, 1101\}$ and $B = \{101, 011, 110\}$



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$$A = \{100, 0, 1\}$$
 and $B = \{1, 100, 0\}$

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Can you write an algorithm for solving this?

