The Theorem of Erdös Tarski
A complete proof

Alain Finkel

LSV, ENS Paris-Saclay (ex ENS Cachan)

Computer Science Lectures
IITB Mumbai, India
$6th + 9th$ March 2018

Up and down

The decidability of coverability, in the WSTS theory, was based on the fact that for all wqo's:

Upward closed sets in WQO

Every upward closed set $U = \uparrow U \subseteq X$ is equal to the finite union of elements $\uparrow x$ with $x \in U$.

Up and down

The decidability of coverability, in the WSTS theory, was based on the fact that for all wqo's:

Upward closed sets in WQO

Every upward closed set $U = \uparrow U \subseteq X$ is equal to the finite union of elements $\uparrow x$ with $x \in U$.

But, unfortunately, every downward closed set $D = \downarrow D \subseteq X$ is not equal to the downward closure of elements $\downarrow x$ with $x \in D$ (think of $\mathbb{N} \neq \bigcup_{i \in F} \downarrow x_i$ where F is finite).

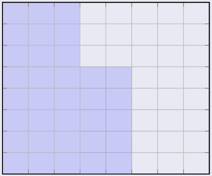
Downward closed sets in WQO (and weaker domains, wait a little)

Every downward closed set is equal to a finite union of ideals

Ideals

$\emptyset \neq I \subseteq X$ is an *ideal* if

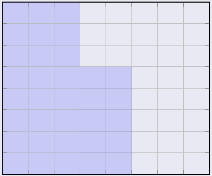
- downward closed: $I = \downarrow I$,



Ideals

$\emptyset \neq I \subseteq X$ is an *ideal* if

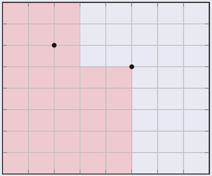
- downward closed: $I = \downarrow I$,
- directed: $a, b \in I \implies \exists c \in I$ s.t. $a \leq c$ and $b \leq c$.



Ideals

$\emptyset \neq I \subseteq X$ is an *ideal* if

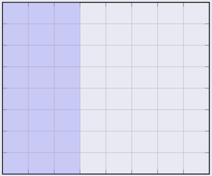
- downward closed: $I = \downarrow I$,
- directed: $a, b \in I \implies \exists c \in I$ s.t. $a \leq c$ and $b \leq c$.



Ideals

$\emptyset \neq I \subseteq X$ is an *ideal* if

- downward closed: $I = \downarrow I$,
- directed: $a, b \in I \implies \exists c \in I$ s.t. $a \leq c$ and $b \leq c$.



$$\text{Ideals}(\mathbb{N}^d) = \underbrace{\text{Ideals}(\mathbb{N}) \times \text{Ideals}(\mathbb{N}) \times \cdots \times \text{Ideals}(\mathbb{N})}_{d \text{ times}}$$

An ideal $I \in \text{Ideals}(\mathbb{N})$ is either $\mathbb{N}$ or of the form $\downarrow x$ for some $x \in \mathbb{N}$.

$$\text{Ideals}(\mathbb{N}^d) = \underbrace{\text{Ideals}(\mathbb{N}) \times \text{Ideals}(\mathbb{N}) \times \cdots \times \text{Ideals}(\mathbb{N})}_{d \text{ times}}$$

An ideal $I \in \text{Ideals}(\mathbb{N})$ is either $\mathbb{N}$ or of the form $\downarrow x$ for some $x \in \mathbb{N}$.

Consider the previous downward closed set:

$$X = \{(x_1, x_2) \in \mathbb{N}^2 : (x_1 \leq 4) \vee (x_1 \leq 8 \wedge x_2 \leq 10) \vee (x_2 \leq 5)\}.$$

$$\mathrm{Ideals}(\mathbb{N}^d) = \underbrace{\mathrm{Ideals}(\mathbb{N}) \times \mathrm{Ideals}(\mathbb{N}) \times \cdots \times \mathrm{Ideals}(\mathbb{N})}_{d \text{ times}}$$

An ideal $I \in \mathrm{Ideals}(\mathbb{N})$ is either $\mathbb{N}$ or of the form $\downarrow x$ for some $x \in \mathbb{N}$.

Consider the previous downward closed set:

$$X = \{(x_1, x_2) \in \mathbb{N}^2 : (x_1 \leq 4) \vee (x_1 \leq 8 \wedge x_2 \leq 10) \vee (x_2 \leq 5)\}.$$

$X =$ finite union of 3 ideals:

$$\downarrow 4 \times \mathbb{N} \;\cup\; \downarrow 8 \times \downarrow 10 \;\cup\; \mathbb{N} \times \downarrow 5$$

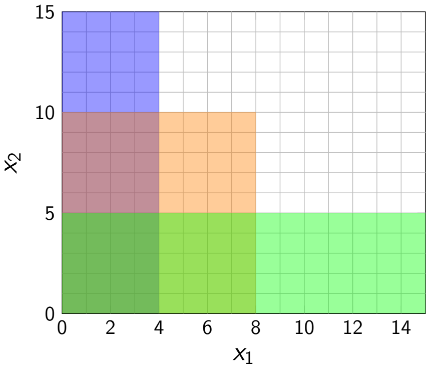


Figure: Decomposition of $X = \{(x_1, x_2) \in \mathbb{N}^2 : (x_1 \leq 4) \vee (x_1 \leq 8 \wedge x_2 \leq 10) \vee (x_2 \leq 5)\}$ into the three ideals $\downarrow 4 \times \mathbb{N}, \downarrow 8 \times \downarrow 10$ and $\mathbb{N} \times \downarrow 5$

Downward closed sets are everywhere !

- Karp and Miller algorithm (1969) for Petri nets uses a finite representation (with ω) of dc sets.
- The tool TREX (2001) for lossy FIFO systems uses a finite representation of dc sets.
- A forward reachability procedure for Time Petri nets (each token has an age) uses regions generators as a finite representation of dc sets (Abdulla & al. 2004).

Downward closed sets are everywhere !

- Karp and Miller algorithm (1969) for Petri nets uses a finite representation (with ω) of dc sets.
- The tool TREX (2001) for lossy FIFO systems uses a finite representation of dc sets.
- A forward reachability procedure for Time Petri nets (each token has an age) uses regions generators as a finite representation of dc sets (Abdulla & al. 2004).

"Finally, we aim at developing generic methods for building downward closed languages....This would give a general theory for forward analysis of infinite state systems..." (Abdulla & al. 2004).
In fact, a theory was missing in our mind !

- Raskin & al. (2004) supposed that a finite representation of dc sets exists for ADL with wqo.
- F. & Goubault-Larrecq (2009) proved that it exists for all wqo's.

- In fact, Erdös & Tarski proved it in 1943 !

- In fact, Erdös & Tarski proved it in 1943 !
- hum...not exactly... Erdös & Tarski stated something more general in another context (without statement neither proof)

- In fact, Erdös & Tarski proved it in 1943 !
- hum...not exactly... Erdös & Tarski stated something more general in another context (without statement neither proof)
- But, the community of mathematicians (for instance Maurice Pouzet) says that one may deduce it from the Erdös & Tarski theorem.

- In fact, Erdös & Tarski proved it in 1943 !
- hum...not exactly... Erdös & Tarski stated something more general in another context (without statement neither proof)
- But, the community of mathematicians (for instance Maurice Pouzet) says that one may deduce it from the Erdös & Tarski theorem.
- Their construction is not effective (for constructive decompositions, see "Forward Analysis for WSTS, Part I: Completions", with Goubault-Larrecq, 80 pages, submitted)

- In fact, Erdös & Tarski proved it in 1943 !
- hum...not exactly... Erdös & Tarski stated something more general in another context (without statement neither proof)
- But, the community of mathematicians (for instance Maurice Pouzet) says that one may deduce it from the Erdös & Tarski theorem.
- Their construction is not effective (for constructive decompositions, see "Forward Analysis for WSTS, Part I: Completions", with Goubault-Larrecq, 80 pages, submitted)

Our abstract

We give a simple and self-contained proof of the fact that every downward closed set decomposes into finitely many ideals iff every antichain is finite.

Annals of Mathematics
Vol. 44, No. 2, April, 1943

ON FAMILIES OF MUTUALLY EXCLUSIVE SETS

By P. Erdös and A. Tarski

(Received August 11, 1942)

In this paper we shall be concerned with a certain particular problem from the general theory of sets, namely with the problem of the existence of families of mutually exclusive sets with a maximal power. It will turn out—in a rather unexpected way—that the solution of these problems essentially involves the notion of the so-called "inaccessible numbers." In this connection we shall make some general remarks regarding inaccessible numbers in the last section of our paper.

Definition

- Now $(X, \leq)$ is a qo (in short written X or $\leq$)

Definition

- Now $(X, \leq)$ is a qo (in short written X or $\leq$)
- $A \subseteq X$ is an antichain if all pairs of A are incomparable.

Definition

- Now $(X, \leq)$ is a qo (in short written X or $\leq$)
- $A \subseteq X$ is an antichain if all pairs of A are incomparable.
- X is Finite AntiChain (FAC) if all antichains in X are finite.

Definition

- Now $(X, \leq)$ is a qo (in short written X or $\leq$)
- $A \subseteq X$ is an antichain if all pairs of A are incomparable.
- X is Finite AntiChain (FAC) if all antichains in X are finite.
- X is Well Founded (WF) if all strictly decreasing sequences in X are finite.

Definition

- Now $(X, \leq)$ is a qo (in short written X or $\leq$)
- $A \subseteq X$ is an antichain if all pairs of A are incomparable.
- X is Finite AntiChain (FAC) if all antichains in X are finite.
- X is Well Founded (WF) if all strictly decreasing sequences in X are finite.
- X is WQO if X is FAC & X is WF (can be a Theorem).

Definition

- Now $(X, \leq)$ is a qo (in short written X or $\leq$)
- $A \subseteq X$ is an antichain if all pairs of A are incomparable.
- X is Finite AntiChain (FAC) if all antichains in X are finite.
- X is Well Founded (WF) if all strictly decreasing sequences in X are finite.
- X is WQO if X is FAC & X is WF (can be a Theorem).

- Every WQO is FAC

Definition

- Now $(X, \leq)$ is a qo (in short written X or $\leq$)
- $A \subseteq X$ is an antichain if all pairs of A are incomparable.
- X is Finite AntiChain (FAC) if all antichains in X are finite.
- X is Well Founded (WF) if all strictly decreasing sequences in X are finite.
- X is WQO if X is FAC & X is WF (can be a Theorem).

- Every WQO is FAC
- The converse is false since $\mathbb{Z}, \mathbb{Q}, \mathbb{R}$ are FAC but not WF: $13/7, \pi, 3, 2, 1, 0, -1/2, -1, -2, -3, -41, -78695/12, \ldots$

Definition

- Now $(X, \leq)$ is a qo (in short written X or $\leq$)
- $A \subseteq X$ is an antichain if all pairs of A are incomparable.
- X is Finite AntiChain (FAC) if all antichains in X are finite.
- X is Well Founded (WF) if all strictly decreasing sequences in X are finite.
- X is WQO if X is FAC & X is WF (can be a Theorem).

- Every WQO is FAC
- The converse is false since $\mathbb{Z}, \mathbb{Q}, \mathbb{R}$ are FAC but not WF: $13/7, \pi, 3, 2, 1, 0, -1/2, -1, -2, -3, -41, -78695/12, \ldots$
- $\mathbb{Z}^2$ contains infinite antichains, $A = \{(n, -n) \mid n \in \mathbb{N}\}$, hence the cartesian product of two FAC's is not necessarly a FAC.

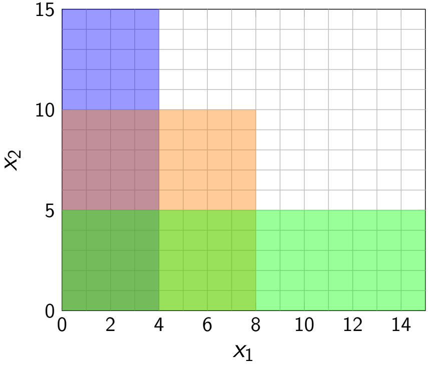
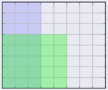


Figure: Decomposition of $X = \{(x_1, x_2) \in \mathbb{N}^2 : (x_1 \leq 4) \vee (x_1 \leq 8 \wedge x_2 \leq 10) \vee (x_2 \leq 5)\}$ into the three ideals $\downarrow 4 \times \mathbb{N}, \downarrow 8 \times \downarrow 10$ and $\mathbb{N} \times \downarrow 5$

A very interesting but unknown theorem for the verification community.

Theorem (Erdös & Tarski'43, Bonnet'75, Fraïsse'86,...)

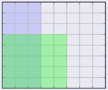
$$(X,\leq)\text{ FAC} \iff \text{for all } D = \downarrow D \subseteq X \text{ we have: } D = \bigcup_{\text{finite}} \text{Ideals}$$



A very interesting but unknown theorem for the verification community.

Theorem (Erdös & Tarski'43, Bonnet'75, Fraïsse'86,...)

$$(X, \leq) \text{ FAC} \iff \text{for all } D = \downarrow D \subseteq X \text{ we have: } D = \bigcup_{\text{finite}} \text{Ideals}$$



Corollary (Blondin, F,. McKenzie, ICALP'2014)

Every downward closed set decomposes canonically as the union of its $\subseteq$-maximal ideals.

Theorem

Let $D = \downarrow D \subseteq X$ and X a WQO. Then $D = I_1 \cup I_2 \cup \cdots \cup I_m$ for some $I_1, I_2, \ldots, I_m \in Ideals(X)$.

Assume that a bad D (bad = dc set that does not admit a finite (may be empty) decomposition in ideals) exists.

Theorem

Let $D = \downarrow D \subseteq X$ and X a WQO. Then $D = I_1 \cup I_2 \cup \cdots \cup I_m$ for some $I_1, I_2, \ldots, I_m \in Ideals(X)$.

Assume that a bad D (bad = dc set that does not admit a finite (may be empty) decomposition in ideals) exists.

$\exists D$ bad and minimal for inclusion among bad subsets (strictly decreasing subsequences of dc subsets are finite in a WQO).

- $D \neq \emptyset$ since $\emptyset$ is not bad since it is equal to an empty union.
- $D \neq \{d\}$ since $\{d\}$ is not bad because it is an ideal.

Theorem

Let $D = \downarrow D \subseteq X$ and X a WQO. Then $D = I_1 \cup I_2 \cup \cdots \cup I_m$ for some $I_1, I_2, \ldots, I_m \in Ideals(X)$.

Assume that a bad D (bad = dc set that does not admit a finite (may be empty) decomposition in ideals) exists.

$\exists D$ bad and minimal for inclusion among bad subsets (strictly decreasing subsequences of dc subsets are finite in a WQO).

- $D \neq \emptyset$ since $\emptyset$ is not bad since it is equal to an empty union.
- $D \neq \{d\}$ since $\{d\}$ is not bad because it is an ideal.
- Hence D contains at least two elements, say $x_1 \neq x_2 \in D$.

Since $D \setminus \uparrow x_1$ and $D \setminus \uparrow x_2$ are dc and strictly included in D, they are not bad (by minimality of D).

Theorem

Let $D = \downarrow D \subseteq X$ and X a WQO. Then $D = I_1 \cup I_2 \cup \cdots \cup I_m$ for some $I_1, I_2, \ldots, I_m \in Ideals(X)$.

Assume that a bad D (bad = dc set that does not admit a finite (may be empty) decomposition in ideals) exists.

$\exists D$ bad and minimal for inclusion among bad subsets (strictly decreasing subsequences of dc subsets are finite in a WQO).

- $D \neq \emptyset$ since $\emptyset$ is not bad since it is equal to an empty union.
- $D \neq \{d\}$ since $\{d\}$ is not bad because it is an ideal.
- Hence D contains at least two elements, say $x_1 \neq x_2 \in D$.

Since $D \setminus \uparrow x_1$ and $D \setminus \uparrow x_2$ are dc and strictly included in D, they are not bad (by minimality of D).

Thus, $D \setminus \uparrow x_1 = \bigcup_{j=1}^{n} I_j$ and $D \setminus \uparrow x_2 = \bigcup_{j=n+1}^{m} I_j$ for some ideals $I_1, I_2, \ldots, I_m \subseteq X$.

Hence

$$D' = (D \setminus \uparrow x_1) \cup (D \setminus \uparrow x_2) = \bigcup_{j=1}^{m} I_j$$

Hence

$$D' = (D \setminus \uparrow x_1) \cup (D \setminus \uparrow x_2) = \bigcup_{j=1}^{m} I_j$$

We have:

$$D' = D \setminus (\uparrow x_1 \cap \uparrow x_2)$$

Hence

$$D' = (D \setminus \uparrow x_1) \cup (D \setminus \uparrow x_2) = \bigcup_{j=1}^{m} I_j$$

We have:

$$D' = D \setminus (\uparrow x_1 \cap \uparrow x_2)$$

As $D' \neq D$ (because D' is not bad),

Hence

$$D' = (D \setminus \uparrow x_1) \cup (D \setminus \uparrow x_2) = \bigcup_{j=1}^{m} I_j$$

We have:

$$D' = D \setminus (\uparrow x_1 \cap \uparrow x_2)$$

As $D' \neq D$ (because D' is not bad), therefore, $D \cap (\uparrow x_1 \cap \uparrow x_2) \neq \emptyset$

Thus:

Hence

$$D' = (D \setminus \uparrow x_1) \cup (D \setminus \uparrow x_2) = \bigcup_{j=1}^{m} I_j$$

We have:

$$D' = D \setminus (\uparrow x_1 \cap \uparrow x_2)$$

As $D' \neq D$ (because D' is not bad), therefore, $D \cap (\uparrow x_1 \cap \uparrow x_2) \neq \emptyset$

Thus:

$\exists d \in D \cap (\uparrow x_1 \cap \uparrow x_2)$ s.t. $x_1 \leq d$ and $x_2 \leq d$.

Hence

$$D' = (D \setminus \uparrow x_1) \cup (D \setminus \uparrow x_2) = \bigcup_{j=1}^{m} I_j$$

We have:

$$D' = D \setminus (\uparrow x_1 \cap \uparrow x_2)$$

As $D' \neq D$ (because D' is not bad), therefore, $D \cap (\uparrow x_1 \cap \uparrow x_2) \neq \emptyset$

Thus:

$\exists d \in D \cap (\uparrow x_1 \cap \uparrow x_2)$ s.t. $x_1 \leq d$ and $x_2 \leq d$.

Hence D is directed and therefore D is an ideal, contradicting our assumption. Thus, D is equal to a finite union of ideals. □

The Erdös & Tarski Theorem

$$(X, \leq) \text{ FAC } \iff \forall D = \downarrow D \subseteq X,\ D = \bigcup_{i=1,\ldots,m} I_i$$

where I_i are ideals.

The proof of Erdös & Tarski Theorem

(particular case: X countable).
Only if. Suppose that X, infinite, contains no infinite antichain (FAC). Let any $D = \downarrow D \subseteq X$. The idea is to build a WF dc-equivalent subset $D' \subseteq D$. Let $d_0, d_1, d_2, ..., d_n, ...$ an infinite enumeration of D.

The proof of Erdös & Tarski Theorem

(particular case: X countable).
Only if. Suppose that X, infinite, contains no infinite antichain (FAC). Let any $D = \downarrow D \subseteq X$. The idea is to build a WF dc-equivalent subset $D' \subseteq D$. Let $d_0, d_1, d_2, \ldots, d_n, \ldots$ an infinite enumeration of D.

- $D_0 \stackrel{\text{def}}{=} D$.

The proof of Erdös & Tarski Theorem

(particular case: X countable).
Only if. Suppose that X, infinite, contains no infinite antichain (FAC). Let any $D = \downarrow D \subseteq X$. The idea is to build a WF dc-equivalent subset $D' \subseteq D$. Let $d_0, d_1, d_2, \ldots, d_n, \ldots$ an infinite enumeration of D.

- $D_0 \stackrel{\text{def}}{=} D$.
- For $i \geq 0$, we pick the first element $e_i \in D_i$ (with $e_0 = d_0$), in the enumeration of D, and we write:

$$D_{i+1} \stackrel{\text{def}}{=} D_i \setminus \downarrow e_i \quad \text{(Make a figure !)}$$

Each e_i satisfies $e_i \not\leq e_0, e_1, \ldots, e_{i-1}$.

The proof of Erdös & Tarski Theorem

(particular case: X countable).
Only if. Suppose that X, infinite, contains no infinite antichain (FAC). Let any $D = \downarrow D \subseteq X$. The idea is to build a WF dc-equivalent subset $D' \subseteq D$. Let $d_0, d_1, d_2, \dots, d_n, \dots$ an infinite enumeration of D.

- $D_0 \stackrel{\text{def}}{=} D$.
- For $i \geq 0$, we pick the first element $e_i \in D_i$ (with $e_0 = d_0$), in the enumeration of D, and we write:

$$D_{i+1} \stackrel{\text{def}}{=} D_i \setminus \downarrow e_i \quad \text{(Make a figure !)}$$

Each e_i satisfies $e_i \not\leq e_0, e_1, \dots, e_{i-1}$.

$$D' \stackrel{\text{def}}{=} \{e_i : i \in \mathbb{N}\}$$

Let $\leq' = \leq$ restricted to D', $\downarrow'$ denotes the $\leq'$-downward closure.
Show that $(D', \leq')$ is WQO.

Let $\leq' = \leq$ restricted to D', $\downarrow'$ denotes the $\leq'$-downward closure.
Show that $(D', \leq')$ is WQO.
$(D', \leq')$ is FAC (by hypothesis on X).

Let $\leq' = \leq$ restricted to D', $\downarrow'$ denotes the $\leq'$-downward closure.
Show that $(D', \leq')$ is WQO.
$(D', \leq')$ is FAC (by hypothesis on X). Show that $(D', \leq')$ is WF.

Let $\leq' = \leq$ restricted to D', $\downarrow'$ denotes the $\leq'$-downward closure.
Show that $(D', \leq')$ is WQO.
$(D', \leq')$ is FAC (by hypothesis on X). Show that $(D', \leq')$ is WF.
By construction of D', the sequence of $(e_i)_i$ has the following property (1): for every i, each $e_i \not\leq$ to any $e_0 = d_0, e_1, e_2, \ldots, e_{i-1}$.
Suppose that D' contains an infinite strictly decreasing sequence:

$$e_{i_0} > e_{i_1} > \cdots > e_{i_k} > \ldots$$

. Since the set of indexes $\{i_k \mid k \geq 0\}$ is infinite, there necessarily exists an integer k s.t. $i_0 < i_k$; with property (1), this implies that $e_{i_0} \not\geq e_{i_k}$ and this is a contradiction with $e_{i_0} > e_{i_1} > \cdots > e_{i_k} > \ldots$.

Since $(D', \leq')$ is WQO, $\exists I_1, I_2, \ldots, I_k \in \text{Ideals}(D', \leq')$ s.t. $\downarrow' D' = I_1 \cup I_2 \cup \cdots \cup I_k$.
We claim that $D = \downarrow D'$. Let us only prove that $D \subseteq \downarrow D'$:
let $y \in D$, by construction of D', $y \leq e_i \in D'$ for some $i \in \mathbb{N}$, hence $y \in \downarrow e_i \subseteq \downarrow D'$, hence $y \in \downarrow D'$ hence $D \subseteq \downarrow D'$, hence $D = \downarrow D'$.
Therefore,

$$D = \downarrow D' = \downarrow (I_1 \cup I_2 \cup \cdots \cup I_k) = \downarrow I_1 \cup \downarrow I_2 \cup \ldots \downarrow I_k \ .$$

For each i, one has: $\downarrow I_i \in \text{Ideals}(X, \leq) \Leftrightarrow \downarrow I_i$ is $\leq$-directed.

Since $(D', \leq')$ is WQO, $\exists I_1, I_2, \ldots, I_k \in \text{Ideals}(D', \leq')$ s.t. $\downarrow' D' = I_1 \cup I_2 \cup \cdots \cup I_k$.
We claim that $D = \downarrow D'$. Let us only prove that $D \subseteq \downarrow D'$:
let $y \in D$, by construction of D', $y \leq e_i \in D'$ for some $i \in \mathbb{N}$, hence $y \in \downarrow e_i \subseteq \downarrow D'$, hence $y \in \downarrow D'$ hence $D \subseteq \downarrow D'$, hence $D = \downarrow D'$.
Therefore,

$$D = \downarrow D' = \downarrow (I_1 \cup I_2 \cup \cdots \cup I_k) = \downarrow I_1 \cup \downarrow I_2 \cup \ldots \downarrow I_k \ .$$

For each i, one has: $\downarrow I_i \in \text{Ideals}(X, \leq) \Leftrightarrow \downarrow I_i$ is $\leq$-directed.

Let $a, b \in \downarrow I_i$, there exist $a', b' \in I_i$ such that $a \leq a'$ and $b \leq b'$.

Since $(D', \leq')$ is WQO, $\exists I_1, I_2, \ldots, I_k \in \mathrm{Ideals}(D', \leq')$ s.t. $\downarrow' D' = I_1 \cup I_2 \cup \cdots \cup I_k$.
We claim that $D = \downarrow D'$. Let us only prove that $D \subseteq \downarrow D'$:
let $y \in D$, by construction of D', $y \leq e_i \in D'$ for some $i \in \mathbb{N}$, hence $y \in \downarrow e_i \subseteq \downarrow D'$, hence $y \in \downarrow D'$ hence $D \subseteq \downarrow D'$, hence $D = \downarrow D'$.
Therefore,

$$D = \downarrow D' = \downarrow (I_1 \cup I_2 \cup \cdots \cup I_k) = \downarrow I_1 \cup \downarrow I_2 \cup \ldots \downarrow I_k \ .$$

For each i, one has: $\downarrow I_i \in \mathrm{Ideals}(X, \leq) \Leftrightarrow \downarrow I_i$ is $\leq$-directed.

Let $a, b \in \downarrow I_i$, there exist $a', b' \in I_i$ such that $a \leq a'$ and $b \leq b'$.

Since $I_i \in \mathrm{Ideals}(D', \leq')$ is directed, there exists $c \in I_i$ such that $a' \leq' c$ and $b' \leq' c$. Thus, $a \leq a' \leq c$ and $b \leq b' \leq c$.
Hence $\downarrow I_i$ is directed and then $\downarrow I_i \in \mathrm{Ideals}(X, \leq)$. $\square$(Only if)

If. Conversely, suppose that there exists an infinite antichain $A \subseteq X$.

If. Conversely, suppose that there exists an infinite antichain $A \subseteq X$. We prove that $D \stackrel{\text{def}}{=} \downarrow A = \bigcup_{a \in A} \downarrow a$ is bad, (i.e. D is not equal to a finite union of ideals). Assume that D is well (not bad).

If. Conversely, suppose that there exists an infinite antichain $A \subseteq X$. We prove that $D \stackrel{\text{def}}{=} \downarrow A = \bigcup_{a \in A} \downarrow a$ is bad, (i.e. D is not equal to a finite union of ideals). Assume that D is well (not bad).

Then $\exists I_1, I_2, \ldots, I_k \in \text{Ideals}(X)$ s.t. $D = I_1 \cup I_2 \cup \cdots \cup I_k$.

Then $\exists i \leq k$ s.t. I_i contains infinitely many elements from A.

If. Conversely, suppose that there exists an infinite antichain $A \subseteq X$. We prove that $D \stackrel{\text{def}}{=} \downarrow A = \bigcup_{a \in A} \downarrow a$ is bad, (i.e. D is not equal to a finite union of ideals). Assume that D is well (not bad).

Then $\exists I_1, I_2, \ldots, I_k \in \text{Ideals}(X)$ s.t. $D = I_1 \cup I_2 \cup \cdots \cup I_k$.

Then $\exists i \leq k$ s.t. I_i contains infinitely many elements from A. Let $a \neq b \in I_i \cap A$. Since I_i is directed, $\exists c \in I_i$ s.t. $a \leq c$ and $b \leq c$.

Moreover, since $c \in I_i \subseteq D$, $\exists a' \in A$ s.t. $c \leq a'$.
Therefore, $a \leq a'$ and $b \leq a'$.
(be carefull, $a = a'$ or $b = a'$ are possible but not both because...?.)

If. Conversely, suppose that there exists an infinite antichain $A \subseteq X$. We prove that $D \stackrel{\text{def}}{=} \downarrow A = \bigcup_{a \in A} \downarrow a$ is bad, (i.e. D is not equal to a finite union of ideals). Assume that D is well (not bad).

Then $\exists I_1, I_2, \ldots, I_k \in \text{Ideals}(X)$ s.t. $D = I_1 \cup I_2 \cup \cdots \cup I_k$.

Then $\exists i \leq k$ s.t. I_i contains infinitely many elements from A. Let $a \neq b \in I_i \cap A$. Since I_i is directed, $\exists c \in I_i$ s.t. $a \leq c$ and $b \leq c$.

Moreover, since $c \in I_i \subseteq D$, $\exists a' \in A$ s.t. $c \leq a'$.
Therefore, $a \leq a'$ and $b \leq a'$.
(be carefull, $a = a'$ or $b = a'$ are possible but not both because...?.)

Because $a \neq b$, at least two distinct elements of A are comparable, (i.e. either a and a', or b and a'). Hence, A is not an antichain, which is a contradiction. Hence there don't exist bad sets. $\square$
This theorem can be extended to any non countable ordinal.

Exercises

- Find a direct proof avoiding wqo.

Exercises

- Find a direct proof avoiding wqo.
- For wpo, we define $x < y$ if $x \leq y$ and $x \neq y$.
  Define $x < y$ when $\leq$ is a wqo but not a wpo.
- Give a definition of the set of minimal elements of X:
  $Min(X) = \{.....$
- Prove that if $(X, \leq)$ is WF then for all x there is a $m \in Min(X)$ s.t. $x \geq m$.
- For $U = \uparrow U$, prove that $Min(U)$ is a basis of U when $\leq$ is WF. Why it is not the case if $\leq$ is not WF ?
- For $U = \uparrow U$, prove that $Min(U)/\equiv$ is finite when $\leq$ is FAC.
- Conclude that $\leq$ is wqo iff $\leq$ is WF + FAC.

Annals of Mathematics
Vol. 44, No. 2, April, 1943

ON FAMILIES OF MUTUALLY EXCLUSIVE SETS

By P. Erdös and A. Tarski

(Received August 11, 1942)

In this paper we shall be concerned with a certain particular problem from the general theory of sets, namely with the problem of the existence of families of mutually exclusive sets with a maximal power. It will turn out—in a rather unexpected way—that the solution of these problems essentially involves the notion of the so-called "inaccessible numbers." In this connection we shall make some general remarks regarding inaccessible numbers in the last section of our paper.

Where are the statements and the proofs ?

Where are the statements and the proofs ?

- (Erdös & Tarski, 1943) contains no statement neither a proof but only a remark that one could also apply to...!

Where are the statements and the proofs ?

- (Erdös & Tarski, 1943) contains no statement neither a proof but only a remark that one could also apply to...!
- (Bonnet'75) only contains the "if" part.

Where are the statements and the proofs ?

- (Erdös & Tarski, 1943) contains no statement neither a proof but only a remark that one could also apply to...!
- (Bonnet'75) only contains the "if" part.
- (Fraïsse'86) contains the main arguments (without details) and he refers to (Bonnet'75). Here we followed Fraïsse.

Where are the statements and the proofs ?

- (Erdös & Tarski, 1943) contains no statement neither a proof but only a remark that one could also apply to...!
- (Bonnet'75) only contains the "if" part.
- (Fraïsse'86) contains the main arguments (without details) and he refers to (Bonnet'75). Here we followed Fraïsse.
- WQO $\implies$ finite decomposition in ideals is in (F. G-L, 2009), simplified in (G-L'2014, unpublished).
- Emanuele Frittaion and Alberto Marcone explained how the of Theorem Erdös & Tarski can be seen as a statement and a proof.

Where are the statements and the proofs ?

- (Erdös & Tarski, 1943) contains no statement neither a proof but only a remark that one could also apply to...!
- (Bonnet'75) only contains the "if" part.
- (Fraïsse'86) contains the main arguments (without details) and he refers to (Bonnet'75). Here we followed Fraïsse.
- WQO $\implies$ finite decomposition in ideals is in (F. G-L, 2009), simplified in (G-L'2014, unpublished).
- Emanuele Frittaion and Alberto Marcone explained how the of Theorem Erdös & Tarski can be seen as a statement and a proof.

An almost self-contained proof (we suppose proved for wqo)

Can be found in "WBTS": in LMCS'2017.

- P. Erdős and A. Tarski, "On families of mutually exclusive sets", in Ann. of Math., 1943.
- R. Bonnet, "On the cardinality of the set of initial intervals of a partially ordered set", in "Infinite and finite sets: to Paul Erdős on his 60$^{\text{th}}$ birthday", North-Holland, 1975.
- R. Fraïssé, "Theory of relations", in Studies in Logic and the Foundations of Mathematics, 1986.
- M. Kabil and M. Pouzet, "Une extension d'un théorème de P. Jullien sur les âges de mots", in ITA 26, 1992.
- Emanuele Frittaion and Alberto Marcone, "Reverse mathematics and initial intervals", in Ann. Pure Appl. Logic, 2014.
- Alain Finkel and Jean Goubault-Larrecq, "Forward Analysis for WSTS, Part I: Completions, STACS'2009.
- Blondin, Michael and Finkel, Alain and McKenzie, Pierre, "Well Behaved Transition Systems" in Logical Methods in Computer Science, 2017.