The Theorem of Erdös Tarski
A complete proof

Alain Finkel

LSV, ENS Paris-Saclay (ex ENS Cachan)

Computer Science Lectures
IITB Mumbai, India
6th + 9th March 2018
The decidability of coverability, in the WSTS theory, was based on the fact that for all wqo’s:

**Upward closed sets in WQO**

Every upward closed set $U = \uparrow U \subseteq X$ is equal to the finite union of elements $\uparrow x$ with $x \in U$. 
Up and down

The decidability of coverability, in the WSTS theory, was based on the fact that for all wqo’s:

**Upward closed sets in WQO**

Every upward closed set $U = \uparrow U \subseteq X$ is equal to the finite union of elements $\uparrow x$ with $x \in U$.

But, unfortunately, every downward closed set $D = \downarrow D \subseteq X$ is not equal to the downward closure of elements $\downarrow x$ with $x \in D$ (think of $\mathbb{N} \neq \bigcup_{i \in F} \downarrow x_i$ where $F$ is finite).

**Downward closed sets in WQO (and weaker domains, wait a little)**

Every downward closed set is equal to a finite union of ideals.
Ideals

\( \emptyset \neq I \subseteq X \) is an ideal if

- downward closed: \( I = \downarrow I \),

...
Ideals

$\emptyset \neq I \subseteq X$ is an ideal if

- downward closed: $I = \downarrow I$,
- directed: $a, b \in I \implies \exists c \in I$ s.t. $a \leq c$ and $b \leq c$. 

![Diagram of a grid with shaded and unshaded squares]
Ideals

\( \emptyset \neq I \subseteq X \) is an ideal if

- downward closed: \( I = \downarrow I \),
- directed: \( a, b \in I \implies \exists c \in I \text{ s.t. } a \leq c \text{ and } b \leq c \).
Ideals

\[ \emptyset \neq I \subseteq X \text{ is an ideal if} \]

- downward closed: \( I = \downarrow I \),
- directed: \( a, b \in I \implies \exists c \in I \text{ s.t. } a \leq c \text{ and } b \leq c \).
An ideal \( I \in \text{Ideals}(\mathbb{N}) \) is either \( \mathbb{N} \) or of the form \( \downarrow x \) for some \( x \in \mathbb{N} \).
\[ \text{Ideals}(\mathbb{N}^d) = \text{Ideals}(\mathbb{N}) \times \text{Ideals}(\mathbb{N}) \times \cdots \times \text{Ideals}(\mathbb{N}) \]

d times

An ideal \( I \in \text{Ideals}(\mathbb{N}) \) is either \( \mathbb{N} \) or of the form \( \downarrow x \) for some \( x \in \mathbb{N} \).

Consider the previous downward closed set:

\[ X = \{(x_1, x_2) \in \mathbb{N}^2 : (x_1 \leq 4) \lor (x_1 \leq 8 \land x_2 \leq 10) \lor (x_2 \leq 5) \}. \]
\[
\text{Ideals}(\mathbb{N}^d) = \text{Ideals}(\mathbb{N}) \times \text{Ideals}(\mathbb{N}) \times \cdots \times \text{Ideals}(\mathbb{N}) \quad d \text{ times}
\]

An ideal \( I \in \text{Ideals}(\mathbb{N}) \) is either \( \mathbb{N} \) or of the form \( \downarrow x \) for some \( x \in \mathbb{N} \).

Consider the previous downward closed set:

\[
X = \{(x_1, x_2) \in \mathbb{N}^2 : (x_1 \leq 4) \lor (x_1 \leq 8 \land x_2 \leq 10) \lor (x_2 \leq 5)\}.
\]

\( X \) = finite union of 3 ideals:

\[
\downarrow 4 \times \mathbb{N} \cup \downarrow 8 \times \downarrow 10 \cup \mathbb{N} \times \downarrow 5
\]
Figure: Decomposition of $X = \{(x_1, x_2) \in \mathbb{N}^2 : (x_1 \leq 4) \lor (x_1 \leq 8 \land x_2 \leq 10) \lor (x_2 \leq 5)\}$ into the three ideals $\downarrow 4 \times \mathbb{N}, \downarrow 8 \times \downarrow 10$ and $\mathbb{N} \times \downarrow 5$
Downward closed sets are everywhere!

- Karp and Miller algorithm (1969) for Petri nets uses a finite representation \((\omega)\) of dc sets.
- The tool TREX (2001) for lossy FIFO systems uses a finite representation of dc sets.
- A forward reachability procedure for Time Petri nets (each token has an age) uses regions generators as a finite representation of dc sets (Abdulla & al. 2004).
Downward closed sets are everywhere!

- Karp and Miller algorithm (1969) for Petri nets uses a finite representation (with $\omega$) of dc sets.
- The tool TREX (2001) for lossy FIFO systems uses a finite representation of dc sets.
- A forward reachability procedure for Time Petri nets (each token has an age) uses regions generators as a finite representation of dc sets (Abdulla & al. 2004).

"Finally, we aim at developing generic methods for building downward closed languages.... This would give a general theory for forward analysis of infinite state systems..." (Abdulla & al. 2004).

In fact, a theory was missing in our mind!

- Raskin & al. (2004) supposed that a finite representation of dc sets exists for ADL with wqo.
- F. & Goubault-Larrecq (2009) proved that it exists for all wqo’s.
■ In fact, Erdös & Tarski proved it in 1943!
- In fact, Erdös & Tarski proved it in 1943!
- hum...not exactly... Erdös & Tarski stated something more general in another context (without statement neither proof)
In fact, Erdös & Tarski proved it in 1943!

hum...not exactly... Erdös & Tarski stated something more general in another context (without statement neither proof)

But, the community of mathematicians (for instance Maurice Pouzet) says that one may deduce it from the Erdös & Tarski theorem.
In fact, Erdös & Tarski proved it in 1943!

hum...not exactly... Erdös & Tarski stated something more general in another context (without statement neither proof)

But, the community of mathematicians (for instance Maurice Pouzet) says that one may deduce it from the Erdös & Tarski theorem.

Their construction is not effective (for constructive decompositions, see "Forward Analysis for WSTS, Part I: Completions", with Goubault-Larrecq, 80 pages, submitted)
In fact, Erdös & Tarski proved it in 1943!

hum...not exactly... Erdös & Tarski stated something more general in another context (without statement neither proof)

But, the community of mathematicians (for instance Maurice Pouzet) says that one may deduce it from the Erdös & Tarski theorem.

Their construction is not effective (for constructive decompositions, see "Forward Analysis for WSTS, Part I: Completions", with Goubault-Larrecq, 80 pages, submitted)

Our abstract

We give a simple and self-contained proof of the fact that every downward closed set decomposes into finitely many ideals iff every antichain is finite.
ON FAMILIES OF MUTUALLY EXCLUSIVE SETS

By P. Erdös and A. Tarski

(Received August 11, 1942)

In this paper we shall be concerned with a certain particular problem from the general theory of sets, namely with the problem of the existence of families of mutually exclusive sets with a maximal power. It will turn out—in a rather unexpected way—that the solution of these problems essentially involves the notion of the so-called "inaccessible numbers." In this connection we shall make some general remarks regarding inaccessible numbers in the last section of our paper.
Definition

Now \((X, \leq)\) is a qo (in short written \(X\) or \(\leq\)).
Definition

- Now \((X, \leq)\) is a qo (in short written \(X\) or \(\leq\))
- \(A \subseteq X\) is an **antichain** if all pairs of \(A\) are incomparable.
Definition

- Now \((X, \leq)\) is a qo (in short written \(X\) or \(\leq\))
- \(A \subseteq X\) is an **antichain** if all pairs of \(A\) are incomparable.
- \(X\) is **Finite AntiChain (FAC)** if all antichains in \(X\) are finite.
Introduction
The basis for Erdös and Tarski Theorem
The Erdös and Tarski Theorem

Definitions
Ideals

**Definition**

- Now \((X, \leq)\) is a qo (in short written \(X\) or \(\leq\))
- \(A \subseteq X\) is an **antichain** if all pairs of \(A\) are incomparable.
- \(X\) is **Finite AntiChain (FAC)** if all antichains in \(X\) are finite.
- \(X\) is **Well Founded (WF)** if all strictly decreasing sequences in \(X\) are finite.
Definition

- Now \((X, \leq)\) is a qo (in short written \(X\) or \(\leq\))
- \(A \subseteq X\) is an **antichain** if all pairs of \(A\) are incomparable.
- \(X\) is **Finite AntiChain (FAC)** if all antichains in \(X\) are finite.
- \(X\) is **Well Founded (WF)** if all strictly decreasing sequences in \(X\) are finite.
- \(X\) is **WQO** if \(X\) is FAC & \(X\) is WF (can be a Theorem).
Definition

- Now \((X, \leq)\) is a qo (in short written \(X\) or \(\leq\))
- \(A \subseteq X\) is an antichain if all pairs of \(A\) are incomparable.
- \(X\) is Finite AntiChain (FAC) if all antichains in \(X\) are finite.
- \(X\) is Well Founded (WF) if all strictly decreasing sequences in \(X\) are finite.
- \(X\) is WQO if \(X\) is FAC & \(X\) is WF (can be a Theorem).

- Every WQO is FAC
Introduction
The basis for Erdős and Tarski Theorem
The Erdős and Tarski Theorem

Definitions
Ideals

Definition

- Now \((X, \leq)\) is a qo (in short written \(X\) or \(\leq\))
- \(A \subseteq X\) is an antichain if all pairs of \(A\) are incomparable.
- \(X\) is Finite AntiChain (FAC) if all antichains in \(X\) are finite.
- \(X\) is Well Founded (WF) if all strictly decreasing sequences in \(X\) are finite.
- \(X\) is WQO if \(X\) is FAC & \(X\) is WF (can be a Theorem).

Every WQO is FAC

The converse is false since \(\mathbb{Z}, \mathbb{Q}, \mathbb{R}\) are FAC but not WF: 
\(13/7, \pi, 3, 2, 1, 0, -1/2, -1, -2, -3, -41, -78695/12, ...\)
Definition

- Now \((X, \leq)\) is a qo (in short written \(X\) or \(\leq\))
- \(A \subseteq X\) is an antichain if all pairs of \(A\) are incomparable.
- \(X\) is Finite AntiChain (FAC) if all antichains in \(X\) are finite.
- \(X\) is Well Founded (WF) if all strictly decreasing sequences in \(X\) are finite.
- \(X\) is WQO if \(X\) is FAC \& \(X\) is WF (can be a Theorem).

- Every WQO is FAC
- The converse is false since \(\mathbb{Z}, \mathbb{Q}, \mathbb{R}\) are FAC but not WF: 
  \(13/7, \pi, 3, 2, 1, 0, -1/2, -1, -2, -3, -41, -78695/12, \ldots\)
- \(\mathbb{Z}^2\) contains infinite antichains, \(A = \{(n, -n) \mid n \in \mathbb{N}\}\), hence the cartesian product of two FAC’s is not necessarily a FAC.
The basis for Erdös and Tarski Theorem

The Erdös and Tarski Theorem

Definitions

Ideals

Figure: Decomposition of \( X = \{(x_1, x_2) \in \mathbb{N}^2 : (x_1 \leq 4) \lor (x_1 \leq 8 \land x_2 \leq 10) \lor (x_2 \leq 5)\} \) into the three ideals \( \downarrow 4 \times \mathbb{N}, \downarrow 8 \times \downarrow 10 \) and \( \mathbb{N} \times \downarrow 5 \)
A **very interesting but unknown** theorem for the verification community.

**Theorem** (Erdős & Tarski’43, Bonnet’75, Fraïssé’86,...)

\[(X, \leq) \text{ FAC } \iff \text{ for all } D = \downarrow D \subseteq X \text{ we have: } D = \bigcup_{\text{finite}} \text{Ideals} \]
A very interesting but unknown theorem for the verification community.

**Theorem** (Erdös & Tarski’43, Bonnet’75, Fraïssé’86,...)

\[(X, \leq) \text{ FAC } \iff \text{ for all } D = \downarrow D \subseteq X \text{ we have: } D = \bigcup \text{ finite ideals}\]

**Corollary** (Blondin, F., McKenzie, ICALP’2014)

Every downward closed set decomposes canonically as the union of its \(\subseteq\)-maximal ideals.
Introduction
The basis for Erdős and Tarski Theorem
The Erdős and Tarski Theorem

Theorem

Let $D = \downarrow D \subseteq X$ and $X$ a WQO. Then $D = I_1 \cup I_2 \cup \cdots \cup I_m$ for some $I_1, I_2, \ldots, I_m \in \text{Ideals}(X)$.

Assume that a bad $D$ (bad = dc set that does not admit a finite (may be empty) decomposition in ideals) exists.
Theorem

Let \( D = \downarrow D \subseteq X \) and \( X \) a WQO. Then \( D = I_1 \cup I_2 \cup \cdots \cup I_m \) for some \( I_1, I_2, \ldots, I_m \in \text{Ideals}(X) \).

Assume that a bad \( D \) (bad = dc set that does not admit a finite (may be empty) decomposition in ideals) exists.

\[ \exists D \text{ bad and minimal for inclusion among bad subsets (strictly decreasing subsequences of dc subsets are finite in a WQO).} \]

- \( D \neq \emptyset \) since \( \emptyset \) is not bad since it is equal to an empty union.
- \( D \neq \{ d \} \) since \( \{ d \} \) is not bad because it is an ideal.
Theorem

Let $D = \downarrow D \subseteq X$ and $X$ a WQO. Then $D = I_1 \cup I_2 \cup \cdots \cup I_m$ for some $I_1, I_2, \ldots, I_m \in \text{Ideals}(X)$.

Assume that a bad $D$ (bad = dc set that does not admit a finite (may be empty) decomposition in ideals) exists.

$\exists D$ bad and minimal for inclusion among bad subsets (strictly decreasing subsequences of dc subsets are finite in a WQO).

- $D \neq \emptyset$ since $\emptyset$ is not bad since it is equal to an empty union.
- $D \neq \{d\}$ since $\{d\}$ is not bad because it is an ideal.
- Hence $D$ contains at least two elements, say $x_1 \neq x_2 \in D$.

Since $D \uparrow x_1$ and $D \uparrow x_2$ are dc and strictly included in $D$, they are not bad (by minimality of $D$).
Theorem

Let \( D = \downarrow D \subseteq X \) and \( X \) a WQO. Then \( D = I_1 \cup I_2 \cup \cdots \cup I_m \) for some \( I_1, I_2, \ldots, I_m \in \text{Ideals}(X) \).

Assume that a bad \( D \) (bad = dc set that does not admit a finite (may be empty) decomposition in ideals) exists.

\( \exists D \) bad and **minimal** for inclusion among bad subsets (**strictly decreasing subsequences of dc subsets are finite in a WQO**).

- \( D \neq \emptyset \) since \( \emptyset \) is not bad since it is equal to an empty union.
- \( D \neq \{d\} \) since \( \{d\} \) is not bad because it is an ideal.
- Hence \( D \) contains at least two elements, say \( x_1 \neq x_2 \in D \).

Since \( D \uparrow x_1 \) and \( D \uparrow x_2 \) are dc and strictly included in \( D \), they are not bad (**by minimality of \( D \)**).

Thus, \( D \uparrow x_1 = \bigcup_{j=1}^n I_j \) and \( D \uparrow x_2 = \bigcup_{j=n+1}^m I_j \) for some ideals \( I_1, I_2, \ldots, I_m \subseteq X \).
Hence

\[ D' = (D \setminus \uparrow x_1) \cup (D \setminus \uparrow x_2) = \bigcup_{j=1}^{m} I_j \]
Hence

\[ D' = (D \setminus \uparrow x_1) \cup (D \setminus \uparrow x_2) = \bigcup_{j=1}^{m} I_j \]

We have:

\[ D' = D \setminus (\uparrow x_1 \cap \uparrow x_2) \]
Hence

\[ D' = (D \setminus \uparrow x_1) \cup (D \setminus \uparrow x_2) = \bigcup_{j=1}^{m} I_j \]

We have:

\[ D' = D \setminus (\uparrow x_1 \cap \uparrow x_2) \]

As \( D' \neq D \) (because \( D' \) is not bad),
Hence

\[ D' = (D \uparrow x_1) \cup (D \uparrow x_2) = \bigcup_{j=1}^{m} I_j \]

We have:

\[ D' = D \setminus (\uparrow x_1 \cap \uparrow x_2) \]

As \( D' \neq D \) (because \( D' \) is not bad), therefore, \( D \cap (\uparrow x_1 \cap \uparrow x_2) \neq \emptyset \)

Thus:
Hence

\[ D' = (D \setminus \uparrow x_1) \cup (D \setminus \uparrow x_2) = \bigcup_{j=1}^{m} I_j \]

We have:

\[ D' = D \setminus (\uparrow x_1 \cap \uparrow x_2) \]

As \( D' \neq D \) (because \( D' \) is not bad), therefore, \( D \cap (\uparrow x_1 \cap \uparrow x_2) \neq \emptyset \)

Thus:

\[ \exists d \in D \cap (\uparrow x_1 \cap \uparrow x_2) \text{ s.t. } x_1 \leq d \text{ and } x_2 \leq d. \]
Hence

\[ D' = (D \setminus \uparrow x_1) \cup (D \setminus \uparrow x_2) = \bigcup_{j=1}^{m} l_j \]

We have:

\[ D' = D \setminus (\uparrow x_1 \cap \uparrow x_2) \]

As \( D' \neq D \) (because \( D' \) is not bad), therefore, \( D \cap (\uparrow x_1 \cap \uparrow x_2) \neq \emptyset \)

Thus:

\[ \exists d \in D \cap (\uparrow x_1 \cap \uparrow x_2) \text{ s.t. } x_1 \leq d \text{ and } x_2 \leq d. \]

Hence \( D \) is directed and therefore \( D \) is an ideal, contradicting our assumption. Thus, \( D \) is equal to a finite union of ideals. \( \square \)
The Erdös & Tarski Theorem

\[(X, \leq) \text{ FAC } \iff \forall D = \downarrow D \subseteq X, \ D = \bigcup_{i=1,\ldots,m} l_i\]

where \(l_i\) are ideals.
The proof of Erdös & Tarski Theorem

(particular case: $X$ countable).

*Only if.* Suppose that $X$, infinite, contains no infinite antichain (FAC). Let any $D = \downarrow D \subseteq X$. The idea is to build a WF dc-equivalent subset $D' \subseteq D$. Let $d_0, d_1, d_2, \ldots, d_n, \ldots$ an infinite enumeration of $D$. 
The proof of Erdös & Tarski Theorem

(particular case: $X$ countable).

*Only if.* Suppose that $X$, infinite, contains no infinite antichain (FAC). Let any $D = \downarrow D \subseteq X$. The idea is to build a WF dc-equivalent subset $D' \subseteq D$. Let $d_0, d_1, d_2, ..., d_n, ...$ an infinite enumeration of $D$.

- $D_0 \overset{\text{def}}{=} D$. 


The proof of Erdös & Tarski Theorem

(particular case: X countable).

*Only if.* Suppose that X, infinite, contains no infinite antichain (FAC). Let any $D = \downarrow D \subseteq X$. The idea is to build a WF dc-equivalent subset $D' \subseteq D$. Let $d_0, d_1, d_2, ..., d_n, ...$ an infinite enumeration of $D$.

- $D_0 \overset{\text{def}}{=} D$.
- For $i \geq 0$, we pick the first element $e_i \in D_i$ (with $e_0 = d_0$), in the enumeration of $D$, and we write:

$$D_{i+1} \overset{\text{def}}{=} D_i \setminus \downarrow e_i \quad (\text{Make a figure !})$$

Each $e_i$ satisfies $e_i \not\leq e_0, e_1, ..., e_{i-1}$. 
The proof of Erdős & Tarski Theorem

(particular case: \( X \) countable).

**Only if.** Suppose that \( X \), infinite, contains no infinite antichain (FAC). Let any \( D = \downarrow D \subseteq X \). The idea is to build a WF dc-equivalent subset \( D' \subseteq D \). Let \( d_0, d_1, d_2, \ldots, d_n, \ldots \) an infinite enumeration of \( D \).

- \( D_0 \overset{\text{def}}{=} D \).
- For \( i \geq 0 \), we pick the first element \( e_i \in D_i \) (with \( e_0 = d_0 \)), in the enumeration of \( D \), and we write:
  \[
  D_{i+1} \overset{\text{def}}{=} D_i \setminus \downarrow e_i \quad \text{(Make a figure !)}
  \]
  Each \( e_i \) satisfies \( e_i \not\leq e_0, e_1, \ldots, e_{i-1} \).

\[
D' \overset{\text{def}}{=} \{ e_i : i \in \mathbb{N} \}\]
Let $\leq'\subseteq \leq$ restricted to $D'$, $\downarrow'$ denotes the $\leq'$-downward closure. Show that $(D', \leq')$ is WQO.
Let $\leq'\leq$ restricted to $D'$, $\downarrow'$ denotes the $\leq'$-downward closure. Show that $(D', \leq')$ is WQO. $(D', \leq')$ is FAC (by hypothesis on $X$).
Let $\leq'\leq$ restricted to $D'$, $\downarrow'$ denotes the $\leq'$-downward closure. Show that $(D', \leq')$ is WQO.

$(D', \leq')$ is FAC (by hypothesis on $X$). Show that $(D', \leq')$ is WF.
Let \( \leq' \) restricted to \( D' \), \( \downarrow' \) denotes the \( \leq' \)-downward closure. Show that \( (D', \leq') \) is WQO. 

\( (D', \leq') \) is FAC (by hypothesis on \( X \)). Show that \( (D', \leq') \) is WF. By construction of \( D' \), the sequence of \( (e_i)_i \) has the following property (1): for every \( i \), each \( e_i \not\leq \) to any \( e_0 = d_0, e_1, e_2, \ldots, e_{i-1} \). Suppose that \( D' \) contains an infinite strictly decreasing sequence:

\[ e_{i_0} > e_{i_1} > \cdots > e_{i_k} > \ldots \]

Since the set of indexes \( \{i_k \mid k \geq 0\} \) is infinite, there necessarily exists an integer \( k \) s.t. \( i_0 < i_k \); with property (1), this implies that \( e_{i_0} \not\geq e_{i_k} \) and this is a contradiction with \( e_{i_0} > e_{i_1} > \cdots > e_{i_k} > \ldots \).
Since \((D', \leq')\) is WQO, \(\exists I_1, I_2, \ldots, I_k \in \text{Ideals}(D', \leq')\) s.t. 
\[ \downarrow' D' = I_1 \cup I_2 \cup \cdots \cup I_k. \]

We claim that \(D = \downarrow D'\). Let us only prove that \(D \subseteq \downarrow D'\): let \(y \in D\), by construction of \(D'\), \(y \leq e_i \in D'\) for some \(i \in \mathbb{N}\), hence \(y \in \downarrow e_i \subseteq \downarrow D'\), hence \(y \in \downarrow D'\) hence \(D \subseteq \downarrow D'\), hence \(D = \downarrow D'\).

Therefore,

\[ D = \downarrow D' = \downarrow (I_1 \cup I_2 \cup \cdots \cup I_k) = \downarrow I_1 \cup \downarrow I_2 \cup \cdots \downarrow I_k. \]

For each \(i\), one has: \(\downarrow I_i \in \text{Ideals}(X, \leq) \iff \downarrow I_i \) is \(\leq\)-directed.
Since \((D', \leq')\) is WQO, \(\exists I_1, I_2, \ldots, I_k \in \text{Ideals}(D', \leq')\) s.t. 
\[ \downarrow' D' = I_1 \cup I_2 \cup \cdots \cup I_k. \]

We claim that \(D = \downarrow D'\). Let us only prove that \(D \subseteq \downarrow D'\):

let \(y \in D\), by construction of \(D'\), \(y \leq e_i \in D'\) for some \(i \in \mathbb{N}\), hence \(y \in \downarrow e_i \subseteq \downarrow D'\), hence \(y \in \downarrow D'\) hence \(D \subseteq \downarrow D'\), hence \(D = \downarrow D'\).

Therefore,

\[ D = \downarrow D' = \downarrow (I_1 \cup I_2 \cup \cdots \cup I_k) = \downarrow I_1 \cup \downarrow I_2 \cup \cdots \downarrow I_k. \]

For each \(i\), one has: \(\downarrow I_i \in \text{Ideals}(X, \leq) \iff \downarrow I_i\) is \(\leq\)-directed.

Let \(a, b \in \downarrow I_i\), there exist \(a', b' \in I_i\) such that \(a \leq a'\) and \(b \leq b'\).
Since \((D', \leq')\) is WQO, \(\exists l_1, l_2, \ldots, l_k \in \text{Ideals}(D', \leq')\) s.t. 
\[\downarrow' D' = l_1 \cup l_2 \cup \cdots \cup l_k.\]
We claim that \(D = \downarrow D'\). Let us only prove that \(D \subseteq \downarrow D'\):

let \(y \in D\), by construction of \(D'\), \(y \leq e_i \in D'\) for some \(i \in \mathbb{N}\), hence 
\(y \in \downarrow e_i \subseteq \downarrow D'\), hence \(y \in \downarrow D'\) hence \(D \subseteq \downarrow D'\), hence \(D = \downarrow D'\).

Therefore,

\[D = \downarrow D' = \downarrow (l_1 \cup l_2 \cup \cdots \cup l_k) = \downarrow l_1 \cup \downarrow l_2 \cup \cdots \downarrow l_k.\]

For each \(i\), one has: \(\downarrow l_i \in \text{Ideals}(X, \leq) \iff \downarrow l_i\) is \(\leq\)-directed.

Let \(a, b \in \downarrow l_i\), there exist \(a', b '\in l_i\) such that \(a \leq a'\) and \(b \leq b'\).

Since \(l_i \in \text{Ideals}(D', \leq')\) is directed, there exists \(c \in l_i\) such that 
\(a' \leq 'c\) and \(b' \leq 'c\). Thus, \(a \leq a' \leq c\) and \(b \leq b' \leq c\).

Hence \(\downarrow l_i\) is directed and then \(\downarrow l_i \in \text{Ideals}(X, \leq).\) □ (Only if)
If. Conversely, suppose that there exists an infinite antichain $A \subseteq X$. 
If. Conversely, suppose that there exists an infinite antichain $A \subseteq X$. We prove that $D \overset{\text{def}}{=} \downarrow A = \bigcup_{a \in A} \downarrow a$ is bad, (i.e. $D$ is not equal to a finite union of ideals). Assume that $D$ is well (not bad).
If. Conversely, suppose that there exists an infinite antichain $A \subseteq X$. We prove that $D \overset{\text{def}}{=} \downarrow A = \bigcup_{a \in A} \downarrow a$ is bad, (i.e. $D$ is not equal to a finite union of ideals). Assume that $D$ is well (not bad).

Then $\exists l_1, l_2, \ldots, l_k \in \text{Ideals}(X)$ s.t. $D = l_1 \cup l_2 \cup \cdots \cup l_k$.

Then $\exists i \leq k$ s.t. $l_i$ contains infinitely many elements from $A$. 
If. Conversely, suppose that there exists an infinite antichain $A \subseteq X$. We prove that $D \overset{\text{def}}{=} \downarrow A = \bigcup_{a \in A} \downarrow a$ is bad, (i.e. $D$ is not equal to a finite union of ideals). Assume that $D$ is well (not bad).

Then $\exists l_1, l_2, \ldots, l_k \in \text{Ideals}(X)$ s.t. $D = l_1 \cup l_2 \cup \cdots \cup l_k$.

Then $\exists i \leq k$ s.t. $l_i$ contains infinitely many elements from $A$. Let $a \neq b \in l_i \cap A$. Since $l_i$ is directed, $\exists c \in l_i$ s.t. $a \leq c$ and $b \leq c$.

Moreover, since $c \in l_i \subseteq D$, $\exists a' \in A$ s.t. $c \leq a'$.

Therefore, $a \leq a'$ and $b \leq a'$.

(be carefull, $a = a'$ or $b = a'$ are possible but not both because...?.)
If. Conversely, suppose that there exists an infinite antichain $A \subseteq X$. We prove that $D \overset{\text{def}}{=} \downarrow A = \bigcup_{a \in A} \downarrow a$ is bad, (i.e. $D$ is not equal to a finite union of ideals). Assume that $D$ is well (not bad).

Then $\exists l_1, l_2, \ldots, l_k \in \text{Ideals}(X)$ s.t. $D = l_1 \cup l_2 \cup \cdots \cup l_k$.

Then $\exists i \leq k$ s.t. $l_i$ contains infinitely many elements from $A$. Let $a \neq b \in l_i \cap A$. Since $l_i$ is directed, $\exists c \in l_i$ s.t. $a \leq c$ and $b \leq c$.

Moreover, since $c \in l_i \subseteq D$, $\exists a' \in A$ s.t. $c \leq a'$.

Therefore, $a \leq a'$ and $b \leq a'$.

(bei carefull, $a = a'$ or $b = a'$ are possible but not both because...?.)

Because $a \neq b$, at least two distinct elements of $A$ are comparable, (i.e. either $a$ and $a'$, or $b$ and $a'$). Hence, $A$ is not an antichain, which is a contradiction. Hence there don’t exist bad sets. □

This theorem can be extended to any non countable ordinal.
Exercises

- Find a direct proof avoiding wqo.
Exercises

- Find a direct proof avoiding wqo.
- For wpo, we define $x < y$ if $x \leq y$ and $x \neq y$. Define $x < y$ when $\leq$ is a wqo but not a wpo.
- Give a definition of the set of minimal elements of $X$: $\text{Min}(X) = \{\ldots\}$.
- Prove that if $(X, \leq)$ is WF then for all $x$ there is a $m \in \text{Min}(X)$ s.t. $x \geq m$.
- For $U = \uparrow U$, prove that $\text{Min}(U)$ is a basis of $U$ when $\leq$ is WF. Why it is not the case if $\leq$ is not WF?
- For $U = \uparrow U$, prove that $\text{Min}(U)/\equiv$ is finite when $\leq$ is FAC.
- Conclude that $\leq$ is wqo iff $\leq$ is WF + FAC.
ON FAMILIES OF MUTUALLY EXCLUSIVE SETS

By P. Erdős and A. Tarski

(Received August 11, 1942)

In this paper we shall be concerned with a certain particular problem from the general theory of sets, namely with the problem of the existence of families of mutually exclusive sets with a maximal power. It will turn out—in a rather unexpected way—that the solution of these problems essentially involves the notion of the so-called "inaccessible numbers." In this connection we shall make some general remarks regarding inaccessible numbers in the last section of our paper.
## Where are the statements and the proofs?

<table>
<thead>
<tr>
<th>Erdős &amp; Tarski Theorem for WQO</th>
</tr>
</thead>
<tbody>
<tr>
<td>Finite decomposition in ideals is in (F. G-L, 2009), simplified in (G-L'2014, unpublished).</td>
</tr>
</tbody>
</table>

Emanuele Frittaion and Alberto Marcone explained how the proof of the Erdős & Tarski Theorem can be seen as a statement and a proof. An almost self-contained proof (we suppose proved for WQO) can be found in “WBTS”: in LMCS’2017.
Where are the statements and the proofs?

- (Erdős & Tarski, 1943) contains no statement neither a proof but only a remark that one could also apply to...!
Where are the statements and the proofs?

- (Erdös & Tarski, 1943) contains no statement neither a proof but only a remark that one could also apply to...!
- (Bonnet’75) only contains the "if" part.
Where are the statements and the proofs?

- (Erdös & Tarski, 1943) contains no statement neither a proof but only a remark that one could also apply to...!
- (Bonnet’75) only contains the "if" part.
- (Fraïssé’86) contains the main arguments (without details) and he refers to (Bonnet’75). Here we followed Fraïssé.
Where are the statements and the proofs?

- (Erdös & Tarski, 1943) contains no statement neither a proof but only a remark that one could also apply to...!
- (Bonnet'75) only contains the "if" part.
- (Fraïsse’86) contains the main arguments (without details) and he refers to (Bonnet’75). Here we followed Fraïsse.
- WQO $\iff$ finite decomposition in ideals is in (F. G-L, 2009), simplified in (G-L’2014, unpublished).
- Emanuele Frittaion and Alberto Marcone explained how the of Theorem Erdös & Tarski can be seen as a statement and a proof.
Where are the statements and the proofs?

- (Erdös & Tarski, 1943) contains no statement neither a proof but only a remark that one could also apply to...!
- (Bonnet’75) only contains the "if" part.
- (Fraïssé’86) contains the main arguments (without details) and he refers to (Bonnet’75). Here we followed Fraïssé.
- WQO $\implies$ finite decomposition in ideals is in (F. G-L, 2009), simplified in (G-L’2014, unpublished).
- Emanuele Frittaion and Alberto Marcone explained how the of Theorem Erdös & Tarski can be seen as a statement and a proof.

An almost self-contained proof (we suppose proved for wqo)

Can be found in "WBTS": in LMCS’2017.

R. Bonnet, "On the cardinality of the set of initial intervals of a partially ordered set", in "Infinite and finite sets: to Paul Erdős on his 60th birthday", North-Holland, 1975.


Alain Finkel and Jean Goubault-Larrecq, "Forward Analysis for WSTS, Part I: Completions, STACS'2009.