CS 208: Automata Theory and Logic
Part II, Lecture 3: Reductions

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Summary of previous lecture

Regular ⊊ Decidable ⊊ Recursively Enumerable ⊊ All languages
DFA/NFA < Algorithms/Halting TM < Semi-algorithms/TM

Properties

1. There exist languages that are not R.E.
2. There exist languages that are R.E but are undecidable.
   Eg. universal TM lang \( L^A_{TM} = \{ \langle M, w \rangle \mid M \text{ is a TM and } M \text{ accepts } w \} \)
3. Decidable languages are closed under complementation.
4. \( L \) is decidable iff \( L \) is R.E and \( \bar{L} \) is also R.E.
The halting problem

The halting problem for Turing Machines is undecidable

Does a given Turing machine halt on a given input?

- \( L_{TM}^{HALT} = \{ \langle M, w \rangle \mid M \text{ is a TM and } M \text{ halts on input } w \} \).
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– \( L_{TM}^{\text{HALT}} = \{ \langle M, w \rangle \mid M \text{ is a TM and } M \text{ halts on input } w \} \).

Proof: Suppose there exists TM \( H \) deciding \( L_{TM}^{\text{HALT}} \), then construct a TM \( D \) s.t., on input \( \langle M, w \rangle \):

– runs TM \( H \) on input \( \langle M, w \rangle \)
– if \( H \) rejects then reject.
– if \( H \) accepts, then simulate \( M \) on \( w \) until it halts.
– if at halting \( M \) has accepted \( w \), accept, else reject.

But \( D \) decides \( L_{TM}^{A} \) which is undecidable. A contradiction.
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This proof strategy is called a reduction.
Reduction from the acceptance problem

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Some more undecidable problems

The emptiness problem for TMs

Does a given Turing machine accept any word?

- \( L_{\emptyset}^{TM} = \{ \langle M \rangle \mid M \text{ is a TM and } L(M) = \emptyset \} \).
Some more undecidable problems

### The emptiness problem for TMs

Does a given Turing machine accept any word?

\[ L_{TM}^0 = \{ \langle M \rangle \mid M \text{ is a TM and } L(M) = \emptyset \}. \]

### The regularity problem for TMs

Does a given Turing machine accept a regular language?

\[ L_{TM}^{REG} = \{ \langle M \rangle \mid M \text{ is a TM and } L(M) \text{ is a regular language} \}. \]
# Some more undecidable problems

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## Rice’s Theorem

Any “non-trivial” property of R.E languages is undecidable!
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### Rice’s theorem (1953)

Any non-trivial property of R.E languages is undecidable!

- Property $P \equiv$ set of languages (i.e., their TM encodings) satisfying $P$
- Property of r.e languages: membership of $M$ in $P$ depends only on the language of $M$. If $L(M) = L(M')$, then $\langle M \rangle \in P$ iff $\langle M' \rangle \in P$.
- Non-trivial: It holds for some but not all TMs.
Rice’s theorem

Rice’s theorem (1953)

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