On Synthesizing Computable Skolem functions for FO logic

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Skolem functions

Given a FOL formula $\varphi(X, Y)$ over (inputs) $X$ and (outputs) $Y$, $F(\cdot)$ is a Skolem function iff

$$\forall X (\exists Y \varphi(X, Y) \iff \varphi(X, F(X)))$$
Introduction

Skolem functions

Given a FOL formula $\varphi(X, Y)$ over (inputs) $X$ and (outputs) $Y$, $F(\cdot)$ is a Skolem function iff

$$\forall X \left( \exists Y \varphi(X, Y) \iff \varphi(X, F(X)) \right)$$

- Classical concept arising from quantifier elimination in FOL.
- Known to always exist! But,
  - Is the function computable?
  - Can we effectively compute/synthesize such a function?
A storied history

Skolem functions play an important role in first order logic

- Getting rid of existential quantifiers
- Seminal work by Thoralf Skolem 1920s and Jacques Herbrand 1930s.
- Skolemization and “Skolem-Normal form”
- Focus on existence of form, NOT computability.
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We can trace this history even further back

- Existence and construction of Boolean unifiers
  - Boole ‘1847, Lowenheim ‘1908.
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- Existence and construction of Boolean unifiers
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Why should we be interested in synthesizability of Skolem functions?

- Heart of Automated Program Synthesis and repair.

\[
g(x_1, x_2) \geq x_1 \text{ and } \\
g(x_1, x_2) \geq x_2 \text{ and } \\
(g(x_1, x_2) \equiv x_1 \text{ or } \\
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Synthesize program for \( g \)
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g(x_1, x_2) &\geq x_1 \text{ and } \\
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y_1 &\geq x_1 \text{ and } \\
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Golia et al, IJCAI'21
Applications

**Why should we be interested in synthesizability of Skolem functions?**

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Synthesize program for \( g \) | Synthesize program for \( y_1 \)

---

**Prior work**

- Propositional setting: Akshay et al.'17,'18,'19,'20,'21, Rabe et al. '17,'18, Golia et al.'20,'21, etc., Fried et al.'16, John et al.'15, Heule et al.'14, etc.

- Beyond Propositional setting:
  - Results on specific theories: Linear rational arithmetic Kuncak et al.'10, Bit vectors Spielman et al., Priener et al.
  - Partial approach for Quantifier Elimination Jiang'09.
Skolem functions beyond terms

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- What is a Skolem function for $x$? $F(x) = y + z$
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- Consider the formula
  \[ \forall y \forall z \exists x ((y > 0) \rightarrow (x > z)) \]
- What is a Skolem function for $x$? $F(x) = y + z$, which is a term in the logic.
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Consider Presburger arithmetic, integers over vocabulary $\mathcal{V} = \{<, +, =, 0, 1\}$

• However, suppose we have

$$\forall y \forall z \exists x (((x = y) \lor (x = z)) \land ((x \geq y) \land (x \geq z)))$$

• No term can serve as a Skolem function for $x$ (all terms are linear functions).

• But $F(x) = \max(y, z)$ is clearly a Skolem function, which can be written as a program:

"input(y, z); if $y \geq z$ then return $y$ else return $z$"

• In fact, for ANY formula in this theory, Skolem functions can be written this way!

Idea of going beyond terms not new: Skolem functions as set of conditional statements [Jiang'09]
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The thesis of this paper

For computability/synthesis, Skolem functions should be seen as programs aka Turing machines!
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- Is there a theory where even programs fail? A theory where there is a formula for which there is no Skolem function as a program?
- Unfortunately yes.

Natural numbers over $\mathbb{N} = \{=, +, \ast, 0, 1\}$.

Follows from the classical Matiyasevich-Robinson-Davis-Putnam (MRDP) theorem!
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- Follows from the classical Matiyasevich-Robinson-Davis-Putnam (MRDP) theorem!
The problem statements

Given a vocabulary $\mathcal{V}$ and a $\mathcal{V}$-structure $\mathcal{M}$. 

Questions of concern

1. For every $\mathcal{V}$-formula $\xi = \forall X \exists Y \varphi(X, Y)$, does there exist a Turing Machine $TM_\xi$, $\mathcal{M}$ that serves as a Skolem function for $Y$ in $\xi$, when evaluated over $\mathcal{M}$? (SkExist)

2. Is there an algorithm $A_\mathcal{M}$ that takes $\xi$ as input and returns $TM_\xi$, $\mathcal{M}$? (SkSyn)

Question 1

• Can SkExist ever return No?
• Is SkExist decidable?

Note: We assume structures to be "computable": predicates/functions are effectively computable.

Question 2

When SkExist returns Yes, then

• can SkSyn return No?
• can we characterize precisely when SkSyn returns Yes?

• Moreover, can we explicitly construct $A_\mathcal{M}$?
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Main results

Negative results

1. Depending on $\mathcal{M}$, $\text{SkExist}$ can return Yes as well as No.

2. $\text{SkExist}$ is undecidable, even when $V$ has a single binary predicate and a single constant, even for $\xi$ in quantifier prefix classes $\exists\forall\exists$ and $\forall\exists\exists$ (but not $\exists^+\forall^*$).

3. There are instances where $\text{SkExist}$ has Yes answer but not $\text{SkSyn}$.

But we know many theories where Skolem functions can be synthesized for all formulas. So what makes them decidable?

A characterization for Synthesis

Let $\mathcal{M}$ be a computable $V$-structure for vocabulary $V$.

• $\text{SkSyn}$ has a positive answer for $\mathcal{M}$ iff the "elementary diagram" of $\mathcal{M}$ is decidable.
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A brief detour into Model theory

So what is the elementary diagram of $\mathcal{M}$?
Vocabulary \( \mathcal{V} \)

E.g., \( \{<, =, +, 0, 1\} \)
A brief detour into Model theory

- **Vocabulary** $\mathcal{V}$
  e.g., $\{<, =, +, 0, 1\}$

- **Structure** $\mathcal{M}$
  - Universe $\mathbb{Z}$
  - $\prec: (0, 1), (-1, 0), (5, 7), \ldots$
  - $=: (0, 0), \ldots$
  - $+: (0, 1) \rightarrow 1, (-3, 2) \rightarrow -1, \ldots$
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• Expansion of Vocabulary $\mathcal{V}(\mathcal{M})$
  $\{<, =, +, 0, 1, c_0, c_1, c_{-1}, \ldots\}$

Also called elementary diagram of $\mathcal{M}$, $\text{ED}(\mathcal{M})$. 
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- **Expansion of Structure** $\mathcal{M}_{exp}$
  Universe $\mathbb{Z}$
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Also called **elementary diagram of** $\mathcal{M}$, ED($\mathcal{M}$).
A brief detour into Model theory

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- **Expansion of Vocabulary** $\mathcal{V}(\mathcal{M})$
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  $\{<, =, +, 0, 1, c_0, c_1, c_{-1}, \ldots\}$

- **Expansion of Structure** $\mathcal{M}_{\text{exp}}$
  
  **Universe** $\mathbb{Z}$
  
  $<: (0, 1), (-1, 0), (5, 7), \ldots,$
  
  $=: (0, 0), \ldots$
  
  $+: (0, 1) \rightarrow 1, (-3, 2) \rightarrow -1, \ldots$
  
  $0: 0, 1: 1$
  
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- $\text{Th}(\mathcal{M}_{\text{exp}})$ is the set of all true sentences in $\mathcal{M}_{\text{exp}}$. Also called **elementary diagram of** $\mathcal{M}$, $\text{ED}(\mathcal{M})$. 
A brief detour into Model theory

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Elementary diagram is said to be decidable if given any sentence $\varphi$ in $\mathcal{V}(\mathcal{M})$, we can algorithmically decide if $\varphi \in ED(\mathcal{M})$. 

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A brief detour into Model theory

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Elementary diagram is said to be decidable if given any sentence \( \varphi \) in \( \mathcal{V}(\mathcal{M}) \), we can algorithmically decide if \( \varphi \in \text{ED}(\mathcal{M}) \).

This is the necessary and sufficient condition for synthesis!
Theorem

\textbf{SkSyn} has a positive answer for $\mathcal{M}$ iff the “elementary diagram” of $\mathcal{M}$ is decidable.
Consequences and more!

Theorem

\textbf{SkSyn} has a positive answer for \( M \) and we can effectively synthesize Skolem functions as halting Turing machines for \( M \) iff the “elementary diagram” of \( M \) is decidable.
Theorem

SkSyn has a positive answer for $M$ and we can effectively synthesize Skolem functions as halting Turing machines for $M$ iff the “elementary diagram” of $M$ is decidable.

Consequences

1. SkSyn has a negative answer for $(\mathbb{N}, <, =, +, *, 0, 1)$.
2. SkSyn has a positive answer and we can effectively synthesize Skolem functions for
   1. Presburger arithmetic
   2. Linear rational arithmetic
   3. Real algebraic numbers
   4. Dense linear orders without endpoints
Consequences and more!

**Theorem**

\( \text{SkSyn} \) has a positive answer for \( \mathcal{M} \) and we can effectively synthesize Skolem functions as halting Turing machines for \( \mathcal{M} \) iff the “elementary diagram” of \( \mathcal{M} \) is decidable.

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**Theorem**

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- Not true in general! There exist $\mathcal{M}$ s.t. $Th(\mathcal{M})$ is decidable but $ED(\mathcal{M})$ is not (see paper).
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Complexity

- Lower bound follows from complexity of deciding theory.
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**Complexity**

- Lower bound follows from complexity of deciding theory.
- If theory admits effective constraint solving, then can give upper bounds! (see paper)
A framework to the study algorithmic computation of Skolem functions.

- Skolem functions as Turing machines/programs.
- A characterization resulting in strong positive and negative results.
A framework to the study algorithmic computation of Skolem functions.

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Other results in paper

- e.g., what happens if you fix the formula and vary the structure?
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The future
- Synthesizing succinct Skolem functions and algorithms with better complexity.
- Characterization of when terms are sufficient.
- Implementation for certain theories?
Conclusion - A beginning

A framework to the study algorithmic computation of Skolem functions.

• Skolem functions as Turing machines/programs.
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The future

• Synthesizing succinct Skolem functions and algorithms with better complexity.
• Characterization of when terms are sufficient.
• Implementation for certain theories? Work in progress!
Thank you!
A Short Sketch of proof

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SkSyn has a positive answer for M iff the “elementary diagram” of M is decidable.
A Short Sketch of proof

**Theorem**

\textbf{SkSyn} has a positive answer for \( \mathcal{M} \) iff the “elementary diagram” of \( \mathcal{M} \) is decidable.

\( \iff \) Program/TM for Skolem function for \( Y \) in \( \forall X \exists Y \varphi(X, Y) \) is as follows:

1. Given value of \( X \), say \( \sigma \), construct \( \Psi_{\sigma} = \exists Y \varphi(\sigma, Y) \).
2. Use dec proc for \( \text{ED}(\mathcal{M}) \) on this.
   - If false, output arbitrary value.
   - If true, for each elt \( \rho \) in \( \text{dom}(Y) \), do:
     1. Construct \( \varphi(\sigma, \rho) \).
     2. Apply dec proc for \( \text{ED}(\mathcal{M}) \) on this.
3. If true, output \( \rho \), quit loop, else goto next elt.
A Short Sketch of proof

Theorem

SkSyn has a positive answer for $\mathcal{M}$ iff the “elementary diagram” of $\mathcal{M}$ is decidable.

$(\implies)$ Program/TM for Skolem function for $Y$ in $\forall X \exists Y \varphi(X, Y)$ is as follows:

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1. Given value of $X$, say $\sigma$, construct $\Psi_\sigma = \exists Y \phi(\sigma, Y)$
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A Short Sketch of proof

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SkSyn has a positive answer for $M$ iff the “elementary diagram” of $M$ is decidable.

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A Short Sketch of proof

Theorem

SkSyn has a positive answer for $\mathcal{M}$ iff the “elementary diagram” of $\mathcal{M}$ is decidable.

($\Leftarrow$) Dec proc for $ED(\mathcal{M})$ using Sk fn generator for formulas over $\mathcal{M}$. 

1. Given $V(\mathcal{M})$ sentence $\phi$ with constant $c \in V(\mathcal{M})$, construct $\phi'$ where $c$ replaced by fresh var.

2. For fresh var, $z_1, z_2$ define $\psi = \forall y \forall z_1 \forall z_2 \exists x ((x = z_1 \land \phi') \lor (x = z_2 \land \neg \phi'))$ (note: this is a valid formula!)

3. Use Sk fn gen on $\psi$ to synthesize Sk fn for $x$, $F(y, z_1, z_2)$.

4. For two distinct elements $d, e \in \mathcal{M}$, evaluate $F(c, d, e)$.
   - if $F(c, d, e) = d$, then $\phi$ is valid.
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A Short Sketch of proof

**Theorem**

\( \text{SkSyn} \) has a positive answer for \( M \) iff the “elementary diagram” of \( M \) is decidable.

(\( \iff \)) Dec proc for \( ED(M) \) using Sk fn generator for formulas over \( M \).

1. Given \( \mathcal{V}(M) \) sentence \( \phi \) with constant \( c \in \mathcal{V}(M) \), construct \( \phi'(y) \) where \( c \) replaced by fresh var \( y \).
Theorem

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