Combinatorial Game Theory

SoS EndTerm Report

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Chapter 1

Hello

1.1 Introduction

For roughly the last 3 weeks, I have read up on Combinatorial Game Theory, the analysis of 'games' which correspond to positions and moves and whose gameplay is of strategic and mathematical interest. The goal is to quantify and analyse these games.

1.2 Game Theory?

Due to what I consider a misnomer, *game theory* typically refers to a field of mathematical analysis significantly different from the present discussion. *Game theory* analyses a situation involving 2 or more rational agents, simultaneously making decisions based on certain strategies, which may be based upon varying degrees of information they each possess. A specific application is in *mathematical economics*. This field was initiated by John von Neumann and Oskar Morgenstern, and major contributors include John Nash.

Combinatorial Game Theory refers to the analysis of sequential games. The games involve an idea of position, and a set of fixed rules defining how the position can be changed. They involve positions which are agreed as a 'win' or a 'loss' for one party. Gameplay alternates moves between parties. A defining characteristic of combinatorial games is that every player has complete information about the position of the game, and there is no factor of chance involved in the determination of position. The field was initiated in antiquity, and major modern contributions have come from Sprague, Grundy, John Conway, Elwyn Berkelamp.

1.3 What is a Game?

Combinatorial Game Theory is an important element in recreational mathematics and mathematical dissemination, since most readers have played one or more games and a more mathematical insight is intriguing to anyone who has played a game. But which games are even considered combinatorial games?

The rules for determining whether a game is combinatorial (See WWfyMP p16) are:

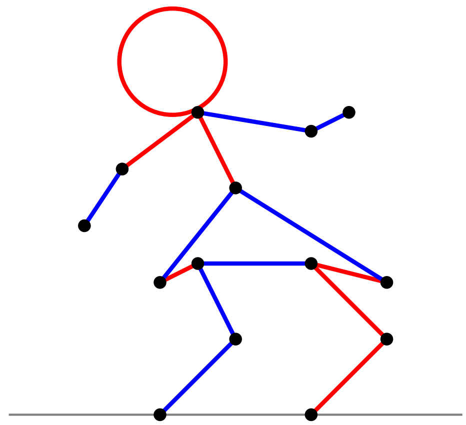

Figure 1.1: A Hackenbush Starting position

1. There are just 2 players, called Left and Right.
2. There are several positions, and usually a *starting position*.
3. There are clearly defined rules specifying the moves each player can make from a position, and the resulting positions are called options (Left options and Right options).
4. Left and Right move alternately in gameplay.
5. In normal play convention, the player unable to move loses. In *misére* play, the player to make the last move loses.
6. The rules must ensure that the game ends and one player wins after a finite number of moves, that there is no stalemate by infinite repetition.
7. Both players have complete information about the present state of the game.
8. There is no move based on chance like rolling a die.

After a move is made from a position A on a game to a position B, the remaining gameplay simply proceeds as if the game had begun at position B. Thus, a game is identified with it's starting position. Additionally, we divide Combinatorial Games into 2 classes: **impartial** games are games whose left and right options are exactly the same, as opposed to **partizan** games whose options may be different.

Chapter 2

Hackenbush

2.1 Blue-Red Hackenbush

Blue-Red Hackenbush (sometimes *Hackenbush Restrained*) is played on a picture of the type fig 1. It consists of Red and Blue edges, with the starting position in which every edge is connected via some edges to the ground. A move for Left consists of deleting a Blue edge, and deleting any edges no longer connected to ground, and for Right by deleting Red edges. The objective is to make the last move to win.

We quantify a *move advantage* in Hackenbush, by indicating how many free moves the winner can win by. For a simple version of Hackenbush such as fig 2, if both players play optimally cutting their trees from the top down, the move advantage that Left has is exactly the number of extra edges he's got.

2.2 Zero Game

What is the move advantage of Left in fig 2? If Left starts, Right will take the last move. If Right starts, Left will take the last move. Thus, from this position, whoever starts will lose. Such positions, where the initial player loses, are called **zero games**. Neither player has an advantage in these games.

In general, for situations like fig 2, where the red and blue trees are completely separate, it is evident who will win by evaluating the number of edges each one has got. Thus, *the outcome of a game can be predicted by determining its move advantage*. This is an important recurring theme, so we call the move advantage of a game, its *value*. A positive value denotes a positive advantage to Left, and a negative value denotes a positive advantage to Right. Here's a question to the reader: Are *all games* positive or negative?

We have just seen the first example of assigning a number to a game. We will soon see how assigning a move advantage to a game divides it into outcome classes and can be subject to mathematical analysis.

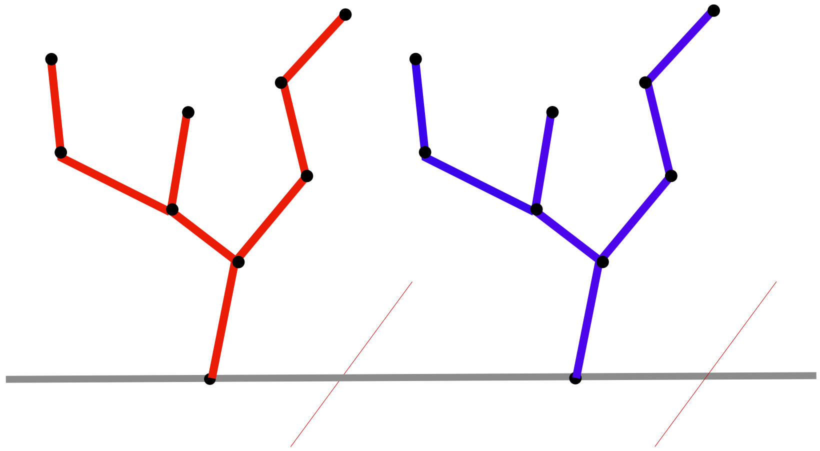

Figure 2.1: A Zero Game

2.3 Half a Move?

Let us analyse the Hackenbush position of fig 3. First, we make a simple observation about Hackenbush games on separate components such as in figures 2 and 3, that the progression of the game in one such component has no effect on the other component. In some sense, the three component games can be 'summed' up independently.

In the first component of figure 3, if Right plays first, he wins by taking the base. If Left plays first, Right still wins after Left's first move. Thus, there is a definitive advantage of Right in that component, i.e the game value is negative. How many moves of advantage does Right have? Is it one move? Consider play when only the second and third components form the starting position. If Right starts, he plays his only move, after which Left uses his spare move in the last component. If Left starts, he plays the leaf atop Right's edge, and wins in his next move. Thus, giving Left an additional move converts the game to a positive one. Hence, in some sense, the component 1 does not give Right a full move's advantage, since giving Left an additional move turns the game in his favour. (You might loosely say that the game value g is between -1 and 0).

Now we consider the complete position of figure 3. If Right starts, he removes a component, after which Left plays his leaf and wins the game next move. If Left starts, Right removes at least one Left edge in his move and wins the game. Thus, we have finally arrived at a zero game. Two copies of the first component exactly balance a free move for Left. So, the game of the first component gives *half a move advantage* to Right!

2.4 Games are Numbers

We just saw how a game could have a value of 1/2. What is the value of the Hackenbush position in fig 4? (*Hint: Compare it with a game of value half.*) Is there a simple way to determine the value of a game?

First, we note that when evaluating a game, we assume optimal play, i.e both players play the best move they've got. Next, we use the principle of recursion, if the values of a game's options are known,

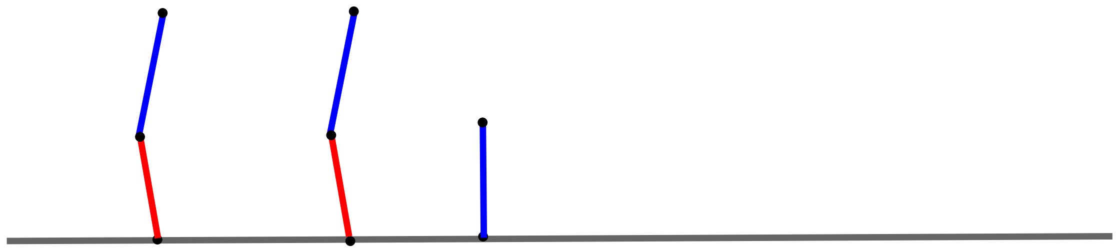

Figure 2.2: Half a Move?

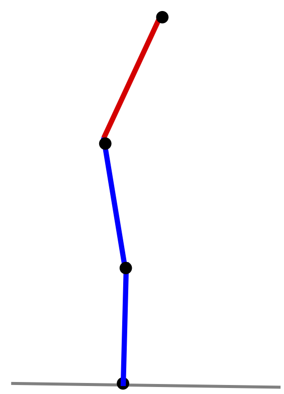

Figure 2.3: Quarter Moves

Left plays into the most positive(greatest) left option, and Right plays into the least positive(least) right option. We denote this in Conway's notation for games (see *On Numbers and Games*):

$$\mathcal{G} = \{\mathcal{G}^L | \mathcal{G}^R\}$$

where $\mathcal{G}^L$ and $\mathcal{G}^R$ are used interchangeably for sets of Left and Right options, or optimal Left and Right options respectively.

The value of the game $\{n|n\}$ is evidently n. What is the value of the game $\{n|n+1\}$, for some natural number n? Consider the Hackenbush position in fig 5, and argue (inductively) that

$$\{n|n+1\} = n + \frac{1}{2}$$

We see that the value of this game is determined by, and in between its optimal option values (*Can this be always true?*). Also, we yet again use the arguably hand-wavy argument of adding game values. All this is made formal in the next section.

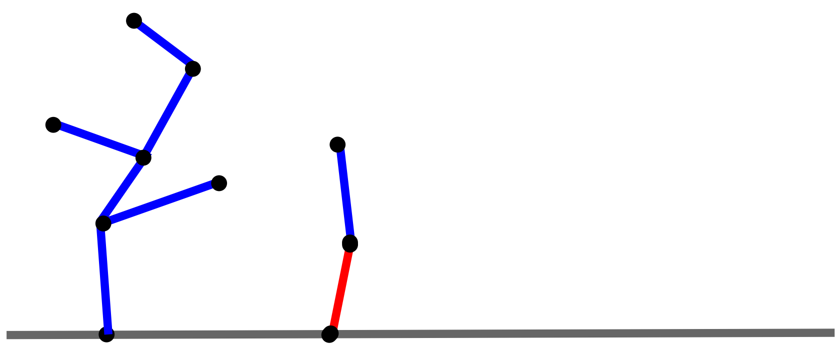

Figure 2.4: A game of value $5\frac{1}{2}$

Chapter 3

All Numbers Big and Small

3.1 Surreal Numbers

Where do the numbers come from? John Conway answers this very question with his amazing construction of the numbers, which was given the apt title of '*Surreal Numbers*' by Knuth. The construction goes as follows:

On the zeroth day of Creation, there are no numbers. Then, J.H.W.H Conway starts building them, following these rules:

1. *Given two sets of existing numbers L and R, with no element of L $\geq$ any element of R, $\{L|R\}$ is a number.*

2. *Denoting the left set and right set of a number x by X_L and X_R, and the typical element x^L and x^R, $x \geq y$ iff $y \ngeq X_R$ and $Y_L \ngeq x$. Also, $x \leq y \iff y \geq x$.*

3. *A number's **birthday** is defined as its day of creation. A number x is said to be **simpler** that y if x is created before y.*

4. *The number $\{\phi|\phi\}$ where ϕ is the empty set, is called* 0 *or 'zero', and it is the only number created on the first day.*

On the second day, we get two more numbers $\{0|\}$ and $\{|0\}$ where $\{0|\} = \{0|\phi\}$. What happens on the *third day*?

Theorem 1. *Given any surreal number x, $x \geq x$ and $x \leq x$.*

Proof. Let x be the simplest number such that $x \ngeq x$. Then, either

1. $\exists x^L \in X_L$ S.T $x^L \geq x$ OR

2. $\exists x^R \in X_R$ S.T $x \geq x^R$

Consider case 1, the definition of $\geq$ implies that $X_L \ngeq x^L$, which implies that $x^L \ngeq x^L$, which is impossible since x^L is simpler than x. Similarly we deal with the case 2. Note that we have proved $X_L \leq x \leq X_R$. $\square$

Theorem 2 (Transitivity)**.** *Given surreal numbers x, y and z, $x \leq y$ and $y \leq z \implies x \leq z$.*

Proof. Once again, let x, y, z be the earliest defined numbers S.T $x \leq y$, $y \leq z$ and $x \nleq z$. Then, by applying the definition of $x \nleq z$, we have the following possibilities:

1. $\exists\, x^L \in X_L$ S.T $x^L \geq z$
2. $\exists\, z^R \in Z_R$ S.T $x \geq z^R$

Consider case 1, $x^L \geq z$, $z \geq y$ and $x^L \ngeq y$, thus we have a simpler triplet, contradicting the hypothesis of extremality. Thus, the relations $x \geq y$ and $x \leq y$ are transitive. □

Theorem 3 (Equivalence)**.** *The relation $x = y$ defined as $x \geq y$ and $x \leq y$ is an equivalence relation on the surreals.*

Proof. We have already proved the Reflexivity and Transitivity, and the Symmetry is obvious. Thus, we have a definitive notion of equality among the surreals. □

We can use this definition of equality, along with the completeness of the relations $x \geq y$ and $x \leq y$, to show that the surreals form a properly ordered set. Next, we proceed to do arithmetic on the surreals.

3.2 An Infinite Digression

So as J.H.W.H Conway keeps creating, it soon becomes clear the infinity of the numbers. On day $\aleph$, we have all the real numbers! And we must ask, what happens on the next day?

We can now define the sets $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{R}$. We also have another number on day $\aleph$, the number:

$$\omega = \{1, 2, 3, 4, 5.....|\phi\}$$

ω is larger than all the natural numbers! It is the first infinite ordinal, the smallest ordinal number represented by an infinite set. We have arrived at the theory of ordinals, the theory of infinities. Tying this up with the rest of mathematics requires some intense set theory, which requires a significant (infinite!) digression from our present topic.

3.3 Addition

What does it mean to add 2 surreals? *Conway says*:

- $x + y = \{(x + Y_L) \cup (y + X_L)|(x + Y_R) \cup (y + X_R)\}$

Let us now get back to our premise, of Combinatorial games, and justify the above formula by summing up game component values, with the following theorem:

Definition (Adding Games)**.** *Consider games G and H. We define the sum $G + H$ as the game in which for every move, the player picks any one game of the two, plays a move. There is no restriction on which component he wishes to play.*

Thus, we can use Surreal Number addition, to easily show symbolically what a sum of games means, and why we can add values of independent games.

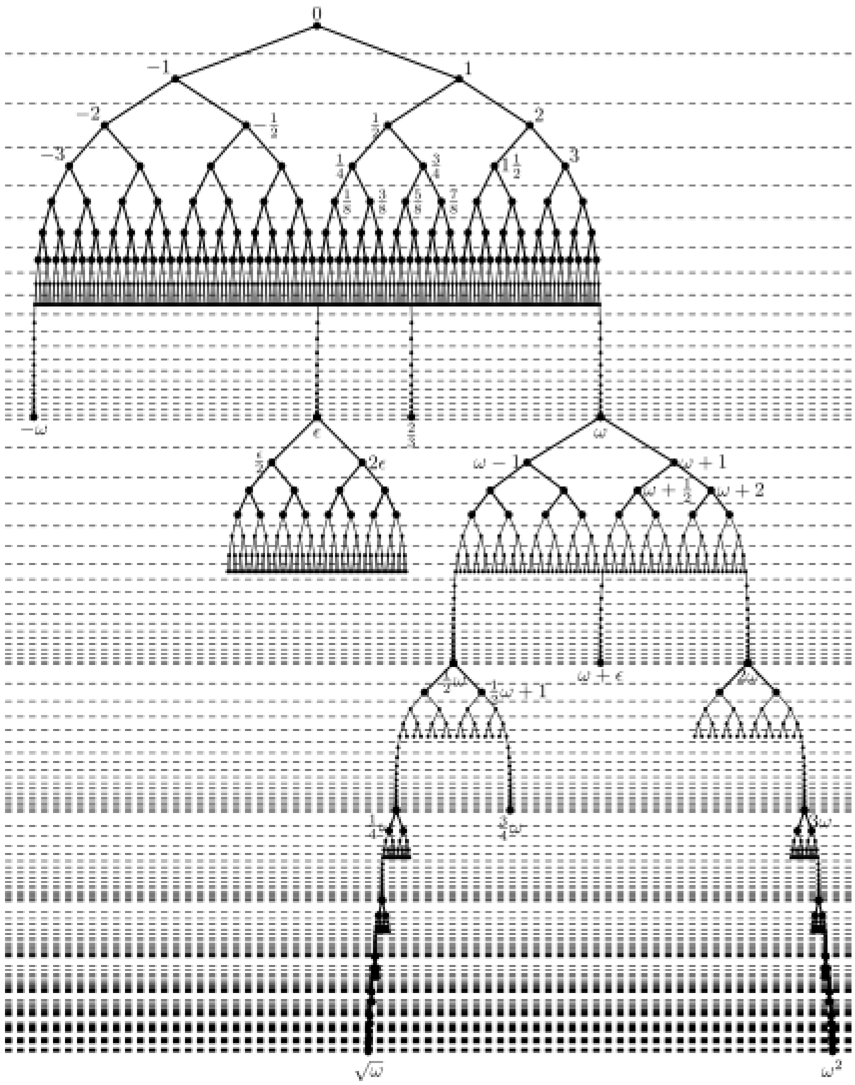

Figure 3.1: The Surreal Number Creation Tree, with Ordinals

Definition. *The negation of a game* $\mathcal{G} = \{\mathcal{G}^L | \mathcal{G}^R\}$ *is the game* $-\mathcal{G} = \{-\mathcal{G}^R | -\mathcal{G}^L\}$.

It is an exercise to the reader to verify the arithmetic of games against the arithmetic of surreal numbers. What is the sum of a game $\mathcal{G}$ and it's negative?

Are all games Surreal Numbers?

3.4 Pseudo-Surreal Numbers

Do we need both of Conway's axioms to make sense of our theorems? What if we remove his first one, and let *any* numbers form the Left and Right sets? The rules for defining arithmetic stay the same as do the definitions of $\leq$ and $\geq$. Consider the pseudo-number $\{1|0\}$. Is it ≥ 0? Is it ≤ 0?. Can you think of a game this 'number' represents?

Definition (Fuzzy Numbers)**.** *Pseudo numbers that are neither* ≥ 0 *nor* ≤ 0 *are said to be fuzzy with 0. This is denoted* $x \,||\, 0$.

These numbers still satisfy the Transitive property, and can be added and subtracted just as any other numbers can. However, they cannot be ordered, since they do not satisfy the trichotomy law.

In fact, there is no restriction on a game that enforces the first of Conway's axioms. Therefore, games can be fuzzy as well! Why haven't we seen any fuzzy games so far? Why can we assume all Hackenbush positions to be surreal numbers? It so happens, that Hackenbush can never have a fuzzy position at all! We shall prove this fact below.

Chapter 4

Games

4.1 Outcome Classes

All combinatorial games can be divided into 4 outcome classes:

- Positive Games: Left always wins regardless of who goes first.
- Negative Games: Right always wins regardless of who goes first.
- Zero Games: The second player always wins, regardless of who goes first.
- Fuzzy Games: The first player always wins, regardless of who goes first.

Why is only the third class of games assigned the value zero, and the fourth one termed 'fuzzy'?We see the following two theorems:

Theorem 4 (Zero Game sum)**.** *For a game $\mathcal{G} = \{\mathcal{G}^L|\mathcal{G}^R\}$ and a zero game $\mathcal{H}$, the outcome of the game $\mathcal{G} + \mathcal{H}$ is the same as the outcome of $\mathcal{G}$.*

Proof. The winning player in $\mathcal{G}$ plays by his strategy in the $\mathcal{G}$ component of the sum, and only plays in $\mathcal{H}$ to reply to a move by his opponent. Since he is the winning player in $\mathcal{G}$ and the second player in $\mathcal{H}$, he will win the sum game $\mathcal{G} + \mathcal{H}$. □

Theorem 5 (Fuzzy Numbers are Fuzzy Games)**.** *A game whose Conway Notation denotes a Fuzzy Number is a Fuzzy Game and vice-versa.*

Proof. As we saw, a pseudo number x is fuzzy iff $\exists\ x^L \in X^L$ & $x^R \in X^R$ ST $x^L \geq 0\ x^R \leq 0$. Thus, Left and Right both have winning moves as the first player and the game is fuzzy. The converse is left to the reader as an exercise. □

How do we prove that Hackenbush is never a fuzzy game? All we need to prove is that for every Hackenbush game $\mathcal{G}$, $\mathcal{G}^L < \mathcal{G} < \mathcal{G}^R$, after which the fact that all Hackenbush positions are Surreal Numbers follows by induction. A proof can be found here.

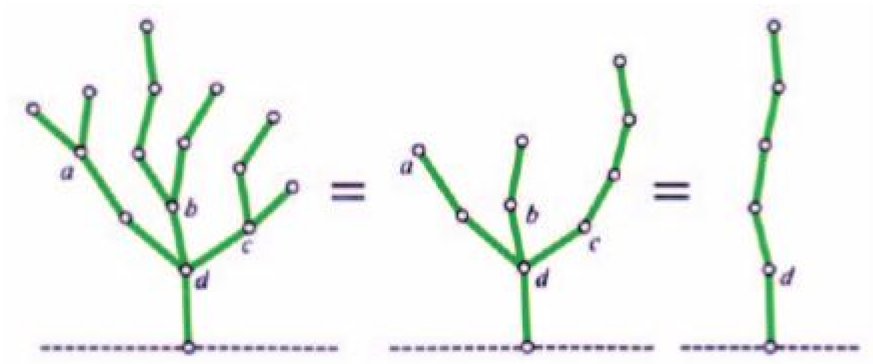

Figure 4.1: Green Hackenbush and its NIM equivalent

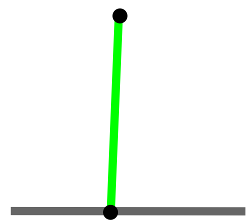

Figure 4.2: $\{0|0\} = *$

4.2 Impartial Games

So far, the game we dealt with has different moves for the two players. In fact with Hackenbush, the starting position itself determined the winner, with no regards to the first player! This class of games, called **partizan games**, is thus prone to be unfair. In general, one's idea of a fair game involves the same kind of move for each player, even though it may or may not be a fair game.

An **Impartial Game** is a game $\mathcal{G}$ in which the Left and Right options are exactly the same, i.e. $\mathcal{G}^{\mathcal{L}} = \mathcal{G}^{\mathcal{R}}$. Thus, positions here cannot give an advantage to either player. Positions for such games can be either zero or fuzzy, depending on whether the first or second player can force a win.

4.2.1 Green Hackenbush

We introduce here a new kind of edge in Hackenbush, coloured Green. This edge can be taken by either Left or Right. A Hackenbush position consisting only of Green edges is an Impartial Game. We can also contruct a difficult to analyse game called *Hackenbush Hotchpotch*, consisting of Blue-Red-Green edges. This is the most general form of Hackenbush, and it is analyzed in WWfyMP chapter 7.

4.2.2 The Star

Consider the simplest Green Hackenbush position, a single green stalk. The game has the value $\{0|0\}$ in Conway Notation. But, there is no number between 0 and 0! Further, we can easily see that it is less than that of $\{0|1\}$, $\{0|\frac{1}{2}\}$, $\{0|\frac{1}{4}\}$ and so on, and **hence the value of this game is less than $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$ $\cdots$ but is simultaneously greater than their negatives!** However, the game's value is NOT 0! Why? Because here, it is the first player's game regardless of who plays first. This is our first encounter with a fuzzy game. Why can we not assign the value 0 to such a game? Consider the game $x + *$, for some number x.

Left can play either to x, or to $x^L + *$. Similarly, Right can play to x or to $x^R + *$. But since $*$ is neither positive nor negative, the best option for both is to simple play to x. Thus, we have

$$\{x|x\} = x + *$$

The value $x + *$ (denoted $x*$, not to be confused with the nimber $*x$) is neither $\leq$ nor $\geq x$, and thus is not $= x$.

4.3 NIM and the Sprague-Grundy Theory

The complete theory of Impartial Games was discovered independently by Sprague and Grundy. The central theme of the theory revolves around NIM, and here we describe it.

Definition (Game of NIM)**.** *The game consists of multiple piles of coins/tokens. A valid move consists of picking a pile and strictly reducing the number of tokens in it. The player to take the last token wins.*

Clearly, the game of NIM is the sum of its constituent piles, and we may thus restrict ourselves to single pile NIM. The value of a NIM heap of size n is denoted $*n$ ($*0 = 0$), and these numbers are called nim-numbers or **nimbers**. Let us take a look at their peculiar properties.

4.3.1 Nimbers

We note that

$$*n = \{0, *1, *2, *3 \ldots |0, *1, *2, *3 \ldots\}$$

and that $*1 = *$.

First, we see that $*x + *x = 0$ for any $x \in \mathbb{N}$, since the 2nd player can simple equalize the 2 heaps. The sum of nimbers in general is quite peculiar, and involves the bitwise XOR operator. More on this is described in the video presentation accompanying the report.

4.3.2 Poker NIM

We can also play NIM with poker chips, where every player gets to keep the chips he has taken from the heaps, and gets to use them to increase any one heap in his move. This is a variant of NIM called poker NIM. However, the outcome of a NIM game is exactly the same as the poker NIM game, since in his turn, the winning player can simply reverse the opponent's last move if it was an increase, and play by his NIM strategy if it was a decrease. Such moves, which do not have an impact on the game since their effect is immediately reverse are called **reversible moves**.

4.3.3 MEX and the Sprague-Grundy Theorem

Consider an impartial game like green hackenbush, whose options are nimbers, denoted $\{*a, *b, *c \cdots\}$. For an example, we consider $\{0, *1, *2, *3, *4, *6, *9\}$. This is equivalent to a single pile NIM with 5 tokens, but with the additional option of increasing the value to 6 or 9. However, these increases are immediately reversible. Thus, the position has value $*5$.

Definition (MEX). *For a set of whole numbers, the minimal excludent, or MEX is defined as the smallest whole number not present in the set. The value of the position $\{*a, *b, *c, *d, \dots\}$ is the MEX of the set $\{a, b, c, d, \dots\}$.*

The proof is evident. Sprague and Grundy showed that *every* impartial game could be shown equivalent to a NIM heap. This result, called the Sprague-Grundy Theorem encapsulates the complete theory of Impartial Games.

Theorem 6 (Sprague-Grundy Theorem). *Every Impartial game in the normal play convention is equivalent to a NIM-heap. In particular, the disjoint sum of various nimbers is a nimber.*

Proof. We prove by induction. Let the theorem be true for all the options of $\mathcal{G}$. Consider the MEX of the values of each of the option heaps, and we have the nimber value of $\mathcal{G}$. $\square$

An example of the reduction from a Green Hackenbush tree to it's equivalent nimber is shown in the figure 4.1.

Appendix A

A.1 References and Further Reading

Many excellent books have been written in the topics we talked about here. We list some of them:

- *Winning Ways for your Mathematical Plays* by Berlekamp, Conway and Guy (WWfyMP)

  A comprehensive research-oriented reference containing multiple modern results on game analysis.

- *On Numbers and Games* by John H Conway (ONAG)

  A well-written concise book, with a bidirectional discussion on games and numbers. A bit hard to follow.

- *Surreal Numbers* by Donald Knuth

  A great 100-page introduction to the surreal numbers.

- *Mathematical Go: Chilling gets the last point* by Berlekamp and Wolfe

  A book on the modern analysis of the game Go.