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Note

Bounds for the Graham–Pollak theorem for hypergraphs

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ABSTRACT

Let $f_r(n)$ represent the minimum number of complete r-partite r-graphs required to partition the edge set of the complete r-uniform hypergraph on n vertices. The Graham–Pollak theorem states that $f_2(n) = n - 1$. An upper bound of $(1 + o(1))\binom{n}{\lfloor \frac{r}{2} \rfloor}$ was known. Recently this was improved to $\frac{14}{15}(1+o(1))\binom{n}{\lfloor \frac{r}{2} \rfloor}$ for even $r \geq 4$. A bound of $\left[\frac{r}{2}(\frac{14}{15})^{\frac{r}{4}} + o(1)\right](1+o(1))\binom{n}{\lfloor \frac{r}{2} \rfloor}$ was also proved recently. Let c_r be the limit of $\frac{f_r(n)}{\binom{n}{\lfloor \frac{r}{2} \rfloor}}$ as $n \to \infty$. The smallest odd r for which $c_r < 1$ that was known was for $r = 295$. In this note we improve this to $c_{113} < 1$ and also give better upper bounds for $f_r(n)$, for small values of even r.

1. Introduction

An r-uniform hypergraph H (also referred to as an r-graph) is said to be r-partite if its vertex set $V(H)$ can be partitioned into sets $V_1, V_2, \ldots, V_r$, so that every edge in the edge set $E(H)$ of H consists of choosing precisely one vertex from each set V_i. That is, $E(H) \subseteq V_1 \times V_2 \times \cdots \times V_r$. Let $f_r(n)$ be the minimum number of complete r-partite r-graphs needed to partition the edge set of the complete r-uniform hypergraph on n vertices. The problem of determining $f_r(n)$ for $r > 2$ was proposed by Aharoni and Linial [1]. For $r = 2$, $f_2(n)$ is the minimum number of bipartite subgraphs required to partition the edge set of the complete graph. Graham and Pollak ([5,6] see also [4]) proved that at least $n - 1$ bipartite graphs are required to cover the complete graph on n vertices. Other proofs were found by Tverberg [10], Peck [9] and Vishwanathan [11,12].

For a general r, constructions due to Alon [1] and later Cioabă, Kündgen and Verstraëte [2] give an upper bound for $f_r(n)$. Cioabă et al. showed that by ordering the vertices, the collection of r-graphs whose even positions are fixed partitions the edge set of the complete r-uniform hypergraph. The cardinality of the collection of r-graphs obtained is $\binom{n-(r+1)/2}{(r-1)/2}$ for odd r, and $\binom{n-r/2}{r/2}$ for even r. The upper bound described below is from the above construction and the lower bound is obtained using the ideas from linear algebra by Alon [1].

$$\frac{2}{\binom{2\lfloor r/2 \rfloor}{\lfloor r/2 \rfloor}}(1 + o(1))\binom{n}{\lfloor \frac{r}{2} \rfloor} \leq f_r(n) \leq (1 - o(1))\binom{n}{\lfloor \frac{r}{2} \rfloor}.$$

Alon also proved that $f_3(n) = n - 2$ [1]. Cioabă and Tait [3] showed that the construction is not tight in general but there was no asymptotic improvement to Alon's upper bound. In a breakthrough paper, Leader, Milićević and Tan [7] showed that $f_4(n) \leq (\frac{14}{15})(1 + o(1))\binom{n}{2}$. Using this they observed that $f_r(n) \leq (\frac{14}{15})(1 + o(1))\binom{n}{\frac{r}{2}}$ for even r. Let c_r be the smallest

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c such that $f_r(n) \leq c(1+o(1))\binom{n}{\lfloor \frac{r}{2} \rfloor}$. Later, Leader and Tan [8] showed that for a general $r \geq 4$, $c_r \leq \frac{r}{2}(\frac{14}{15})^{\frac{r}{4}} + o(1)$ and as a direct consequence showed that $c_{295} < 1$ [8]. The smallest odd r_0 for which $c_{r_0} < 1$ is important since this implies that $c_r < 1$ for all $r > r_0$. In this note we improve the smallest known odd r, for which $c_r < 1$ to $r = 113$. We also give an improved upper bound for $f_r(n)$ for even r and $8 \leq r \leq 1096$ which is used in the above result. We show that for all even $r \geq 6$,

$$f_r(n) \leq \left(\frac{14}{15}\right)^{\frac{r}{6}}(1+o(1))\binom{n}{\frac{r}{2}}.$$

2. The main result

Let S and T be two disjoint sets. Let $\binom{S}{a} \times \binom{T}{b}$ denote all subsets X of $S \cup T$ such that $|X \cap S| = a$ and $|X \cap T| = b$.

A set Γ of complete r-partite r-graphs over $S \cup T$ is said to *exactly cover* a hypergraph F, if the hypergraphs in Γ are edge-disjoint and the union of the edges of the hypergraphs in Γ is F. A complete r-partite r-graph is also referred to as a *block*.

So $f_r(n)$ is the minimum number of complete r-partite r-graphs required to exactly cover the edge set of the complete r-uniform hypergraph on n vertices.

Theorem 1. *For even $r \geq 6$, $f_r(n) \leq (\frac{14}{15})^{\frac{r}{6}}(1+o(1))\binom{n}{\frac{r}{2}}$.*
(Here the o(1) term is as $n \to \infty$ with r fixed.)

Proof. We show that for even $m \geq 8$, and $n \geq m$,

$$f_m(n) \leq \left(\frac{14}{15}\right)^{\frac{m}{6}} \frac{n^{\frac{m}{2}}}{(\frac{m}{2})!} + n^{\frac{m}{2}-1}\log n.$$

The proof is by induction on m and n.

We use the following known bounds: $f_2(n) \leq n-1, f_3(n) \leq n-2$.

By dividing the set $[n]$ into two parts of size $\frac{n}{2}$ each, we get the following recurrence for $f_m(n)$.

$$\begin{aligned} f_m(n) \leq\ & 2 \cdot f_m\left(\frac{n}{2}\right) + 2 \cdot f_{m-1}\left(\frac{n}{2}\right) + 2 \cdot f_{m-2}\left(\frac{n}{2}\right) f_2\left(\frac{n}{2}\right) + 2 \cdot f_{m-3}\left(\frac{n}{2}\right) \cdot f_3\left(\frac{n}{2}\right) \\ & + \cdots + 2 \cdot f_{\frac{m}{2}+1}\left(\frac{n}{2}\right) \cdot f_{\frac{m}{2}-1}\left(\frac{n}{2}\right) + [f_{\frac{m}{2}}\left(\frac{n}{2}\right)]^2. \end{aligned} \tag{1}$$

The bound $f_4(n) \leq (\frac{14}{15})\frac{n^2}{2!} + n\log n$ follows from [7]. We prove that $f_6(n) \leq (\frac{14}{15})\frac{n^3}{3!} + n^2 \log n$ first. The base case for $f_6(n)$ holds since $f_6(6) = 1$. Assume it is true for all values less than n.

$$\begin{aligned} f_6(n) &\leq 2f_6\left(\frac{n}{2}\right) + 2f_5\left(\frac{n}{2}\right) + 2f_2\left(\frac{n}{2}\right)f_4\left(\frac{n}{2}\right) + f_3\left(\frac{n}{2}\right)f_3\left(\frac{n}{2}\right) \\ &\leq 2\left[\left(\frac{14}{15}\right)\frac{n^3}{8 \cdot 3!} + \frac{n^2\log(\frac{n}{2})}{4}\right] + 2\frac{n^2}{4 \cdot 2!} + 2\frac{n}{2}\left[\left(\frac{14}{15}\right)\frac{n^2}{4 \cdot 2!} + \frac{n\log(\frac{n}{2})}{2}\right] + \frac{n^2}{4} \\ &\leq \left(\frac{14}{15}\right)\frac{n^3}{3!} + n^2 \log n \end{aligned}$$

Now we prove that $f_m(n) \leq (\frac{14}{15})^{\frac{m}{6}} \frac{n^{\frac{m}{2}}}{(\frac{m}{2})!} + n^{\frac{m}{2}-1}\log n$. The base case for $f_m(n)$ holds since $f_m(m) = 1$. Assume it is true for all values less than n. as stated earlier. Suppose m is a multiple of 4. By rearranging the terms in (1) according to even and odd indices we have,

$$\begin{aligned} f_m(n) \leq\ & 2 \cdot f_m\left(\frac{n}{2}\right) + 2 \cdot f_{m-2}\left(\frac{n}{2}\right) f_2\left(\frac{n}{2}\right) + \cdots + [f_{\frac{m}{2}}\left(\frac{n}{2}\right)]^2 \\ & + 2 \cdot f_{m-1}\left(\frac{n}{2}\right) + 2 \cdot f_{m-3}\left(\frac{n}{2}\right) \cdot f_3\left(\frac{n}{2}\right) + \cdots + 2 \cdot f_{\frac{m}{2}+1}\left(\frac{n}{2}\right) \cdot f_{\frac{m}{2}-1}\left(\frac{n}{2}\right) \end{aligned}$$

Substituting for $f_2(\frac{n}{2})$, $f_4(\frac{n}{2})$ and $f_6(\frac{n}{2})$ and using the inductive hypothesis for all even $i \geq 8$ and $f_i(\frac{n}{2}) = \binom{\frac{n}{2}}{\lfloor \frac{i}{2} \rfloor}$ for all odd $i \geq 3$, we get

$$\begin{aligned} f_m(n) \leq\ & 2\left[\left(\frac{14}{15}\right)^{\frac{m}{6}} \frac{n^{\frac{m}{2}}}{2^{\frac{m}{2}}(\frac{m}{2})!} + \frac{n^{\frac{m}{2}-1}\log(\frac{n}{2})}{2^{\frac{m}{2}-1}}\right] + 2\binom{n}{2}\left[\left(\frac{14}{15}\right)^{\frac{m-2}{6}} \frac{n^{\frac{m}{2}-1}}{2^{\frac{m}{2}-1}(\frac{m}{2}-1)!} + \frac{n^{\frac{m}{2}-2}\log(\frac{n}{2})}{2^{\frac{m}{2}-2}}\right] \\ & + 2\left[\left(\frac{14}{15}\right)\frac{n^2}{2^2\cdot 2!} + \frac{n\log(\frac{n}{2})}{2}\right]\left[\left(\frac{14}{15}\right)^{\frac{m-4}{6}} \frac{n^{\frac{m}{2}-2}}{2^{\frac{m}{2}-2}(\frac{m}{2}-2)!} + \frac{n^{\frac{m}{2}-3}\log(\frac{n}{2})}{2^{\frac{m}{2}-3}}\right] \\ & + 2\left[\left(\frac{14}{15}\right)\frac{n^3}{2^3\cdot 3!} + \frac{n^2\log(\frac{n}{2})}{2^2}\right]\left[\left(\frac{14}{15}\right)^{\frac{m}{6}-1} \frac{n^{\frac{m}{2}-3}}{2^{\frac{m}{2}-3}(\frac{m}{2}-3)!} + \frac{n^{\frac{m}{2}-4}\log(\frac{n}{2})}{2^{\frac{m}{2}-4}}\right] + \cdots \\ & + \left[\left(\frac{14}{15}\right)^{\frac{m}{12}} \frac{n^{\frac{m}{4}}}{2^{\frac{m}{4}}(\frac{m}{4})!} + \frac{n^{\frac{m}{4}-1}\log(\frac{n}{2})}{2^{\frac{m}{4}-1}}\right]\left[\left(\frac{14}{15}\right)^{\frac{m}{12}} \frac{n^{\frac{m}{4}}}{2^{\frac{m}{4}}(\frac{m}{4})!} + \frac{n^{\frac{m}{4}-1}\log(\frac{n}{2})}{2^{\frac{m}{4}-1}}\right] \\ & + \left[2\binom{\frac{n}{2}}{\frac{m}{2}-1} + 2\binom{\frac{n}{2}}{\frac{m}{2}-2}\binom{\frac{n}{2}}{1} + \cdots + 2\binom{\frac{n}{2}}{\frac{m}{4}}\binom{\frac{n}{2}}{\frac{m}{4}-1}\right] \end{aligned}$$

Grouping terms according to their asymptotic behavior of n and since there are at most $\frac{m}{4}$ terms, each contributing $\frac{2n^{\frac{m}{2}-2}\log^2(\frac{n}{2})}{2^{\frac{m}{2}-2}}$, we have

$$\begin{aligned} f_m(n) \leq & \Bigg[2\left(\frac{14}{15}\right)^{\frac{m}{6}} \frac{n^{\frac{m}{2}}}{2^{\frac{m}{2}}(\frac{m}{2})!} + 2\left(\frac{14}{15}\right)^{\frac{m-2}{6}} \frac{n^{\frac{m}{2}}}{2^{\frac{m}{2}}(\frac{m}{2}-1)!} + 2\left(\frac{14}{15}\right)^{\frac{m+2}{6}} \frac{n^{\frac{m}{2}}}{2^{\frac{m}{2}}\cdot 2!\cdot(\frac{m}{2}-2)!} + \cdots \\ & + 2\left(\frac{14}{15}\right)^{\frac{m}{6}} \frac{n^{\frac{m}{2}}}{2^{\frac{m}{2}}\cdot(\frac{i}{2})!(\frac{m}{2}-\frac{i}{2})!} + \cdots + \left(\frac{14}{15}\right)^{\frac{m}{6}} \frac{n^{\frac{m}{2}}}{2^{\frac{m}{2}}(\frac{m}{4})!(\frac{m}{4})!} \Bigg] \\ & + \frac{2n^{\frac{m}{2}-1}\log(\frac{n}{2})}{2^{\frac{m}{2}-1}} + \frac{2n^{\frac{m}{2}-1}\log(\frac{n}{2})}{2^{\frac{m}{2}-1}} + 2\left[\left(\frac{14}{15}\right)\frac{n^{\frac{m}{2}-1}\log(\frac{n}{2})}{2^{\frac{m}{2}-1}\cdot 2!} + \left(\frac{14}{15}\right)^{\frac{m-4}{6}} \frac{n^{\frac{m}{2}-1}\log(\frac{n}{2})}{2^{\frac{m}{2}-1}\cdot(\frac{m}{2}-2)!}\right] \\ & + 2\left[\left(\frac{14}{15}\right)\frac{n^{\frac{m}{2}-1}\log(\frac{n}{2})}{2^{\frac{m}{2}-1}\cdot 3!} + \left(\frac{14}{15}\right)^{\frac{m-6}{6}} \frac{n^{\frac{m}{2}-1}\log(\frac{n}{2})}{2^{\frac{m}{2}-1}\cdot(\frac{m}{2}-3)!}\right] + \cdots \\ & + \left[\left(\frac{14}{15}\right)^{\frac{m}{12}} \frac{n^{\frac{m}{2}-1}\log(\frac{n}{2})}{2^{\frac{m}{2}-1}\cdot(\frac{m}{4})!} + \left(\frac{14}{15}\right)^{\frac{m}{12}} \frac{n^{\frac{m}{2}-1}\log(\frac{n}{2})}{2^{\frac{m}{2}-1}\cdot(\frac{m}{4})!}\right] \\ & + \frac{2n^{\frac{m}{2}-2}\log^2(\frac{n}{2})}{2^{\frac{m}{2}-2}}\frac{m}{4} \\ & + \left[2\frac{n^{\frac{m}{2}-1}}{2^{\frac{m}{2}-1}\cdot(\frac{m}{2}-1)!} + 2\frac{n^{\frac{m}{2}-1}}{2^{\frac{m}{2}-1}\cdot(\frac{m}{2}-2)!} + \cdots + 2\frac{n^{\frac{m}{2}-1}}{2^{\frac{m}{2}-1}\cdot(\frac{m}{4})!(\frac{m}{4}-1)!}\right] \end{aligned}$$

Since $(\frac{14}{15})^k < 1$, for positive k, for the terms of asymptotic order $n^{\frac{m}{2}-1}\log n$, we have

$$\begin{aligned} f_m(n) \leq & \Bigg[2\left(\frac{14}{15}\right)^{\frac{m}{6}} \frac{n^{\frac{m}{2}}}{2^{\frac{m}{2}}(\frac{m}{2})!} + 2\left(\frac{14}{15}\right)^{\frac{m-2}{6}} \frac{n^{\frac{m}{2}}}{2^{\frac{m}{2}}(\frac{m}{2}-1)!} + 2\left(\frac{14}{15}\right)^{\frac{m+2}{6}} \frac{n^{\frac{m}{2}}}{2^{\frac{m}{2}}\cdot 2!\cdot(\frac{m}{2}-2)!} + \cdots \\ & + 2\left(\frac{14}{15}\right)^{\frac{m}{6}} \frac{n^{\frac{m}{2}}}{2^{\frac{m}{2}}\cdot(\frac{i}{2})!(\frac{m}{2}-\frac{i}{2})!} + \cdots + \left(\frac{14}{15}\right)^{\frac{m}{6}} \frac{n^{\frac{m}{2}}}{2^{\frac{m}{2}}(\frac{m}{4})!(\frac{m}{4})!} \Bigg] \\ & + \Bigg[\frac{2n^{\frac{m}{2}-1}\log(\frac{n}{2})}{2^{\frac{m}{2}-1}} + \frac{2n^{\frac{m}{2}-1}\log(\frac{n}{2})}{2^{\frac{m}{2}-1}} + \frac{2n^{\frac{m}{2}-1}\log(\frac{n}{2})}{2^{\frac{m}{2}-1}\cdot 2!} + \frac{2n^{\frac{m}{2}-1}\log(\frac{n}{2})}{2^{\frac{m}{2}-1}\cdot(\frac{m}{2}-2)!} \\ & + \frac{2n^{\frac{m}{2}-1}\log(\frac{n}{2})}{2^{\frac{m}{2}-1}\cdot 3!} + \frac{2n^{\frac{m}{2}-1}\log(\frac{n}{2})}{2^{\frac{m}{2}-1}\cdot(\frac{m}{2}-3)!} + \cdots \\ & + \frac{n^{\frac{m}{2}-1}\log(\frac{n}{2})}{2^{\frac{m}{2}-1}\cdot(\frac{m}{4})!} + \frac{n^{\frac{m}{2}-1}\log(\frac{n}{2})}{2^{\frac{m}{2}-1}\cdot(\frac{m}{4})!} \Bigg] \\ & + \frac{2n^{\frac{m}{2}-2}\log^2(\frac{n}{2})}{2^{\frac{m}{2}-2}}\frac{m}{4} \\ & + \left[2\frac{n^{\frac{m}{2}-1}}{2^{\frac{m}{2}-1}\cdot(\frac{m}{2}-1)!} + 2\frac{n^{\frac{m}{2}-1}}{2^{\frac{m}{2}-1}\cdot(\frac{m}{2}-2)!} + \cdots + 2\frac{n^{\frac{m}{2}-1}}{2^{\frac{m}{2}-1}\cdot(\frac{m}{4})!(\frac{m}{4}-1)!}\right] \end{aligned}$$

Using the identity $\sum_{i=0}^{N} \frac{1}{i!\cdot(N-i)!} = \frac{2^N}{N!}$ on the terms in the first and last square braces we get,

$$f_m(n) \leq \Bigg[\left(\frac{14}{15}\right)^{\frac{m}{6}} \frac{n^{\frac{m}{2}}}{(\frac{m}{2})!} - 2\left(\frac{14}{15}\right)^{\frac{m}{6}} \frac{n^{\frac{m}{2}}}{2^{\frac{m}{2}}(\frac{m}{2}-1)!} - 2\left(\frac{14}{15}\right)^{\frac{m}{6}} \frac{n^{\frac{m}{2}}}{2^{\frac{m}{2}}\cdot 2!\cdot(\frac{m}{2}-2)!} + 2\left(\frac{14}{15}\right)^{\frac{m-2}{6}} \frac{n^{\frac{m}{2}}}{2^{\frac{m}{2}}(\frac{m}{2}-1)!}$$
$$+ 2\left(\frac{14}{15}\right)^{\frac{m+2}{6}} \frac{n^{\frac{m}{2}}}{2^{\frac{m}{2}}\cdot 2!\cdot(\frac{m}{2}-2)!} \Bigg] + \left[\frac{2n^{\frac{m}{2}-1}\log(\frac{n}{2})}{2^{\frac{m}{2}-1}}\right]\left[1+1+\frac{1}{2!}+\cdots+\frac{1}{(\frac{m}{2}-2)!}\right]$$
$$+ \frac{2n^{\frac{m}{2}-2}\log^2(\frac{n}{2})}{2^{\frac{m}{2}-2}}\frac{m}{4} + \frac{n^{\frac{m}{2}-1}}{(\frac{m}{2}-1)!}.$$

Upon simplifying, we have

$$f_m(n) \leq \left(\frac{14}{15}\right)^{\frac{m}{6}} \frac{n^{\frac{m}{2}}}{(\frac{m}{2})!}\left[1 - \frac{m}{2^{\frac{m}{2}}}\left\{1-\left(\frac{14}{15}\right)^{-\frac{1}{3}}\right\} - \frac{m^2}{2^{\frac{m}{2}+2}}\left\{1-\left(\frac{14}{15}\right)^{\frac{1}{3}}\right\}\right] + \left[\frac{2n^{\frac{m}{2}-1}\log(\frac{n}{2})}{2^{\frac{m}{2}-1}}\right]e$$
$$+ \frac{2n^{\frac{m}{2}-2}\log^2(\frac{n}{2})}{2^{\frac{m}{2}-2}}\frac{m}{4} + \frac{n^{\frac{m}{2}-1}}{2^{\frac{m}{2}-1}}$$
$$\leq \left(\frac{14}{15}\right)^{\frac{m}{6}} \frac{n^{\frac{m}{2}}}{(\frac{m}{2})!}\left[1 - \frac{m}{2^{\frac{m}{2}}}\left\{1-\left(\frac{14}{15}\right)^{-\frac{1}{3}}\right\} - \frac{m^2}{2^{\frac{m}{2}+2}}\left\{1-\left(\frac{14}{15}\right)^{\frac{1}{3}}\right\}\right] + \Bigg[\frac{2\cdot e\cdot n^{\frac{m}{2}-1}\log n}{2^{\frac{m}{2}-1}}$$
$$- \frac{2\cdot e\cdot n^{\frac{m}{2}-1}}{2^{\frac{m}{2}-1}}\Bigg] + \frac{2n^{\frac{m}{2}-2}\log^2(\frac{n}{2})}{2^{\frac{m}{2}-2}}\frac{m}{4} + \frac{n^{\frac{m}{2}-1}}{2^{\frac{m}{2}-1}}$$
$$\leq \left(\frac{14}{15}\right)^{\frac{m}{6}} \frac{n^{\frac{m}{2}}}{(\frac{m}{2})!} + \frac{2e\cdot n^{\frac{m}{2}-1}\log n}{2^{\frac{m}{2}-1}} + \frac{mn^{\frac{m}{2}-2}\log^2(\frac{n}{2})}{2\cdot 2^{\frac{m}{2}-2}}$$
$$\leq \left(\frac{14}{15}\right)^{\frac{m}{6}} \frac{n^{\frac{m}{2}}}{(\frac{m}{2})!} + n^{\frac{m}{2}-1}\log n\left[\frac{2e}{2^{\frac{m}{2}-1}} + \frac{m\log(\frac{n}{2})}{n\cdot 2^{\frac{m}{2}-1}}\right].$$

For all values of $m \geq 8$, $\left[\frac{2e}{2^{\frac{m}{2}-1}} + \frac{m\log(\frac{n}{2})}{n\cdot 2^{\frac{m}{2}-1}}\right] \leq 1$. Therefore,

$$f_m(n) \leq \left(\frac{14}{15}\right)^{\frac{m}{6}} \frac{n^{\frac{m}{2}}}{(\frac{m}{2})!} + n^{\frac{m}{2}-1}\log n.$$

For the case that m is a multiple of 2 but not 4, an argument similar to the above can be used to show that $f_m(n) \leq (\frac{14}{15})^{\frac{m}{6}} \frac{n^{\frac{m}{2}}}{(\frac{m}{2})!} + n^{\frac{m}{2}-1}\log n$. □

We now prove the stated bound for c_{125}.

Lemma 1. *For any S and T, and even a and b and $r = a+b+1$, $\binom{S}{a}\times\binom{T}{b+1}\cup\binom{S}{a+1}\times\binom{T}{b}$ can be exactly covered using at most $\binom{|S|}{\frac{a}{2}}\cdot\binom{|T|}{\frac{b}{2}}$ blocks.*

Proof. Order the elements of S and T. Pick $\frac{a}{2}$ elements of S, say $s_{i_1}, s_{i_2}, \dots, s_{i_{\frac{a}{2}}}$, and $\frac{b}{2}$ elements of T, say $t_{j_1}, t_{j_2}, \dots, t_{j_{\frac{b}{2}}}$. We associate a block corresponding to these sets as follows:

$$\{s_1,\dots,s_{i_1-1}\}, \{s_{i_1}\}, \dots, \{s_{i_{\frac{a}{2}-1}+1},\dots,s_{i_{\frac{a}{2}}-1}\}, \{s_{i_{\frac{a}{2}}}\},$$
$$\{t_1,\dots,t_{j_1-1}\}, \{t_{j_1}\}, \dots, \{t_{j_{\frac{b}{2}-1}+1},\dots,t_{j_{\frac{b}{2}}-1}\}, \{t_{j_{\frac{b}{2}}}\}, \{s_{i_{\frac{a}{2}}+1},\dots,s_p, t_{j_{\frac{b}{2}}+1},\dots,t_q\}.$$

Here p is the index of the last remaining element after picking $s_{i_1}, s_{i_2}, \dots, s_{i_{\frac{a}{2}}}$ from S in the ordering. Likewise, q is the index of the element from T. Among these take only the blocks which have $a+b+1$ parts. Note that these form a disjoint cover. □

Lemma 2. *For large S and T, and even a and b, $\binom{S}{a}\times\binom{T}{b+1}\cup\binom{S}{a+1}\times\binom{T}{b}$ can be exactly covered using $[(\frac{14}{15})^{\frac{b}{6}} + (\frac{14}{15})^{\frac{a}{6}}](1+o(1))\binom{|S|}{\frac{a}{2}}\cdot\binom{|T|}{\frac{b}{2}}$ blocks. Here the $o(1)$ term is as $|S|$ & $|T|$ go to ∞.*

Proof. The hypergraph $\binom{S}{a}\times\binom{T}{b+1}$ can be exactly covered using $f_a(|S|)\cdot f_{b+1}(|T|)$ blocks. By Theorem 1, this is at most $(\frac{14}{15})^{\frac{a}{6}}(1+o(1))\binom{|S|}{\frac{a}{2}}\binom{|T|}{\frac{b}{2}}$ blocks. Likewise, $\binom{S}{a+1}\times\binom{T}{b}$ can be exactly covered using $(\frac{14}{15})^{\frac{b}{6}}(1+o(1))\binom{|S|}{\frac{a}{2}}\binom{|T|}{\frac{b}{2}}$ blocks. □

Lemma 3. *For any odd $r = 4d + 1$, if $\binom{\frac{n}{2}}{\lfloor \frac{r}{2} \rfloor} \times \binom{\frac{n}{2}}{\lceil \frac{r}{2} \rceil} \cup \binom{\frac{n}{2}}{\lceil \frac{r}{2} \rceil} \times \binom{\frac{n}{2}}{\lfloor \frac{r}{2} \rfloor}$ can be covered using $\alpha(1 + o(1))\binom{\frac{n}{2}}{\lfloor \frac{r}{4} \rfloor}^2$ blocks such that $\alpha < 1$, then $f_r(n) \le c_r(\alpha) \cdot (1 + o(1))\binom{n}{\lfloor \frac{r}{2} \rfloor}$ where $c_r(\alpha) < 1$.*

Proof. Recall the inequality.

$$f_r(n) \le 2 \cdot f_r\left(\frac{n}{2}\right) + 2 \cdot f_1\left(\frac{n}{2}\right) \cdot f_{r-1}\left(\frac{n}{2}\right) + \cdots + 2 \cdot f_{\frac{r-1}{2}}\left(\frac{n}{2}\right) \cdot f_{\frac{r+1}{2}}\left(\frac{n}{2}\right)$$

Pairing up two consecutive terms and using Lemma 1 for each pair we have:

$$f_r(n) \le 2(1 + o(1))\left[\sum_{i=0}^{\frac{r-5}{4}} \binom{\frac{n}{2}}{i}\binom{\frac{n}{2}}{\frac{r-1}{2} - i}\right] + f_{\lfloor \frac{r}{2} \rfloor}\left(\frac{n}{2}\right)f_{\lceil \frac{r}{2} \rceil}\left(\frac{n}{2}\right) + f_{\lceil \frac{r}{2} \rceil}\left(\frac{n}{2}\right)f_{\lfloor \frac{r}{2} \rfloor}\left(\frac{n}{2}\right)$$

Using the hypothesis and by adding and subtracting $\binom{\frac{n}{2}}{\lfloor \frac{r}{4} \rfloor}^2$ to the above equation we have:

$$\begin{aligned}
f_r(n) &\le 2(1 + o(1))\left[\sum_{i=0}^{\frac{r-5}{4}} \binom{\frac{n}{2}}{i}\binom{\frac{n}{2}}{\frac{r-1}{2} - i}\right] + \binom{\frac{n}{2}}{\lfloor \frac{r}{4} \rfloor}^2 + \alpha(1 + o(1))\binom{\frac{n}{2}}{\lfloor \frac{r}{4} \rfloor}^2 - \binom{\frac{n}{2}}{\lfloor \frac{r}{4} \rfloor}^2 \\
&\le (1 + o(1))\binom{n}{\lfloor \frac{r}{2} \rfloor} + \alpha(1 + o(1))\binom{\frac{n}{2}}{\lfloor \frac{r}{4} \rfloor}^2 - \binom{\frac{n}{2}}{\lfloor \frac{r}{4} \rfloor}^2 \\
&\le (1 + o(1))\binom{n}{\lfloor \frac{r}{2} \rfloor} - (1 - \alpha)(1 + o(1))\binom{\frac{n}{2}}{\lfloor \frac{r}{4} \rfloor}^2 \\
&\le [1 - \frac{(1 - \alpha)}{e^{\frac{r}{2}}}]\binom{n}{\lfloor \frac{r}{2} \rfloor} \quad \square
\end{aligned}$$

Lemma 4. *$f_{125}(n) \le c \cdot (1 + o(1))\binom{n}{62}$, for a constant $c < 1$.*

Proof. For $a = b = 62$ and $|S| = |T| = \frac{n}{2}$ in Lemma 2, the hypergraph $\binom{n/2}{62} \times \binom{n/2}{63} \cup \binom{n/2}{63} \times \binom{n/2}{62}$ can be exactly covered using at most $2 \cdot \left(\frac{14}{15}\right)^{\frac{62}{6}}(1 + o(1))\binom{n/2}{31}^2 \le 0.981 \cdot (1 + o(1))\binom{\frac{n}{2}}{31}^2$ blocks. The result follows from Lemma 3. $\square$

As a consequence of Lemma 4, we have $c_{125} < 1$. In fact solving the recurrence exactly for Theorem 1 using a computer program yields $c_{113} < 1$.

Declaration of competing interest

None.

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