Note

Bounds for the Graham–Pollak theorem for hypergraphs

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ABSTRACT

Let \( f_r(n) \) represent the minimum number of complete \( r \)-partite \( r \)-graphs required to partition the edge set of the complete \( r \)-uniform hypergraph on \( n \) vertices. The Graham–Pollak theorem states that \( f_2(n) = n - 1 \). An upper bound of \( (1 + o(1))(\frac{n}{r}) \) was known. Recently this was improved to \( \frac{14}{13}(1 + o(1))(\frac{n}{r}) \) for even \( r \geq 4 \). A bound of \( \left( \frac{\sqrt{r} + 1}{\sqrt{r} - 1} + o(1) \right)(1 + o(1))(\frac{n}{r}) \) was also proved recently. Let \( c_r \) be the limit of \( \frac{f_r(n)}{(\frac{n}{r})} \) as \( n \to \infty \). The smallest odd \( r \) for which \( c_r < 1 \) that was known was for \( r = 295 \). In this note we improve this to \( c_{113} < 1 \) and also give better upper bounds for \( f_r(n) \), for small values of even \( r \).

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1. Introduction

An \( r \)-uniform hypergraph \( H \) (also referred to as an \( r \)-graph) is said to be \( r \)-partite if its vertex set \( V(H) \) can be partitioned into sets \( V_1, V_2, \ldots, V_r \), so that every edge in the edge set \( E(H) \) of \( H \) consists of choosing precisely one vertex from each set \( V_i \). That is, \( E(H) \subseteq V_1 \times V_2 \times \cdots \times V_r \). Let \( f_r(n) \) be the minimum number of complete \( r \)-partite \( r \)-graphs needed to partition the edge set of the complete \( r \)-uniform hypergraph on \( n \) vertices. The problem of determining \( f_r(n) \) for \( r > 2 \) was proposed by Aharoni and Linial [1]. For \( r = 2 \), \( f_2(n) \) is the minimum number of bipartite subgraphs required to partition the edge set of the complete graph. Graham and Pollak ([5,6] see also [4]) proved that at least \( n - 1 \) bipartite graphs are required to cover the complete graph on \( n \) vertices. Other proofs were found by Tverberg [10], Peck [9] and Vishwanathan [11,12].

For a general \( r \), constructions due to Alon [1] and later Cioabă, Kündgen and Verstraëte [2] give an upper bound for \( f_r(n) \). Cioabă et al. showed that by ordering the vertices, the collection of \( r \)-graphs whose even positions are fixed partitions the edge set of the complete \( r \)-uniform hypergraph. The cardinality of the collection of \( r \)-graphs obtained is \( \binom{n}{r-1}/(r-1)! \) for odd \( r \), and \( \binom{n}{r/2}^{r-1} \) for even \( r \). The upper bound described below is from the above construction and the lower bound is obtained using the ideas from linear algebra by Alon [1].

\[
\frac{2}{(\frac{r}{2})!(\frac{r}{2})!} (1 + o(1)) \left( \frac{n}{\binom{r}{2}} \right) \leq f_r(n) \leq (1 - o(1)) \left( \frac{n}{\binom{r}{2}} \right).
\]

Alon also proved that \( f_2(n) = n - 2 \) [1]. Cioabă and Tait [3] showed that the construction is not tight in general but there was no asymptotic improvement to Alon’s upper bound. In a breakthrough paper, Leader, Miličević and Tan [7] showed that \( f_4(n) \leq \left( \frac{14}{13} \right)(1 + o(1))(\binom{n}{4}) \). Using this they observed that \( f_r(n) \leq (\frac{14}{13})(1 + o(1))(\binom{n}{r}) \) for even \( r \). Let \( c_r \) be the smallest
c such that \( f_r(n) \leq c(1 + o(1)) \left( \frac{n}{r} \right) \). Later, Leader and Tan [8] showed that for a general \( r \geq 4 \), \( c_r \leq \frac{14}{15} r^2 + o(1) \) and as a direct consequence showed that \( c_{295} < 1 \) [8]. The smallest odd \( r_0 \) for which \( c_{r_0} < 1 \) is important since this implies that \( c_r < 1 \) for all \( r > r_0 \). In this note we improve the smallest known odd \( r \), for which \( c_r < 1 \) to \( r = 113 \). We also give an improved upper bound for \( f_r(n) \) for even \( r \) and \( 8 \leq r \leq 1096 \) which is used in the above result. We show that for all even \( r \geq 6 \),

\[
f_r(n) \leq \left( \frac{14}{15} \right)^2 (1 + o(1)) \left( \frac{n}{r} \right).
\]

2. The main result

Let \( S \) and \( T \) be two disjoint sets. Let \( \binom{S}{a} \times \binom{T}{b} \) denote all subsets \( X \) of \( S \cup T \) such that \( |X \cap S| = a \) and \( |X \cap T| = b \).

A set \( I^r \) of complete \( r \)-partite \( r \)-graphs over \( S \cup T \) is said to exactly cover a hypergraph \( F \), if the hypergraphs in \( I^r \) are edge-disjoint and the union of the edges of the hypergraphs in \( I^r \) is \( F \). A complete \( r \)-partite \( r \)-graph is also referred to as a block.

So \( f_r(n) \) is the minimum number of complete \( r \)-partite \( r \)-graphs required to exactly cover the edge set of the complete \( r \)-uniform hypergraph on \( n \) vertices.

**Theorem 1.** For even \( r \geq 6 \), \( f_r(n) \leq \left( \frac{14}{15} \right)^2 (1 + o(1)) \left( \frac{n}{r} \right) \).

(Here the \( o(1) \) term is as \( n \to \infty \) with \( r \) fixed.)

**Proof.** We show that for even \( m \geq 8 \), and \( n \geq m \),

\[
f_m(n) \leq \left( \frac{14}{15} \right)^2 \frac{n^m}{(m/2)!} + n^{m-1} \log n.
\]

The proof is by induction on \( m \) and \( n \).

We use the following known bounds: \( f_2(n) \leq n - 1 \), \( f_3(n) \leq n - 2 \).

By dividing the set \( [n] \) into two parts of size \( \frac{n}{2} \) each, we get the following recurrence for \( f_m(n) \).

\[
f_m(n) \leq 2 \cdot f_m \left( \frac{n}{2} \right) + 2 \cdot f_{m-1} \left( \frac{n}{2} \right) + 2 \cdot f_{m-2} \left( \frac{n}{2} \right) f_2 \left( \frac{n}{2} \right) + 2 \cdot f_{m-3} \left( \frac{n}{2} \right) f_3 \left( \frac{n}{2} \right) + \cdots + 2 \cdot f_m \left( \frac{n}{2} \right) f_2 \left( \frac{n}{2} \right) + \left( f_m \left( \frac{n}{2} \right) \right)^2.
\]

The bound \( f_6(n) \leq \left( \frac{14}{15} \right)^2 n^3 + n \log n \) follows from [7]. We prove that \( f_6(n) \leq \left( \frac{14}{15} \right)^3 n^2 + n^2 \log n \) first. The base case for \( f_6(n) \) holds since \( f_6(6) = 1 \). Assume it is true for all values less than \( n \).

\[
f_6(n) \leq 2f_6 \left( \frac{n}{2} \right) + 2f_5 \left( \frac{n}{2} \right) + 2f_5 \left( \frac{n}{2} \right) f_4 \left( \frac{n}{2} \right) + f_4 \left( \frac{n}{2} \right) f_3 \left( \frac{n}{2} \right)
\]

\[
\leq 2 \left[ \left( \frac{14}{15} \right)^2 n^3 \cdot 8 \cdot 3! + \frac{n^2 \log(2)}{4} \right] + 2 \cdot \frac{n^2}{4 \cdot 2!} + 2 \cdot \left( \frac{14}{15} \right)^2 \left( \frac{n^2}{4 \cdot 2!} + \frac{n \log(2)}{2} \right) + \frac{n^2}{4}
\]

\[
\leq \left( \frac{14}{15} \right)^3 n^3 + n^2 \log n
\]

Now we prove that \( f_m(n) \leq \left( \frac{14}{15} \right)^2 \frac{n^m}{(m/2)!} + n^{m-1} \log n \). The base case for \( f_m(n) \) holds since \( f_m(m) = 1 \). Assume it is true for all values less than \( n \). as stated earlier. Suppose \( m \) is a multiple of 4. By rearranging the terms in (1) according to even and odd indices we have,

\[
f_m(n) \leq 2 \cdot f_m \left( \frac{n}{2} \right) + 2 \cdot f_{m-2} \left( \frac{n}{2} \right) f_2 \left( \frac{n}{2} \right) + \cdots + \left( f_m \left( \frac{n}{2} \right) \right)^2
\]

\[
+ 2 \cdot f_{m-1} \left( \frac{n}{2} \right) + 2 \cdot f_{m-3} \left( \frac{n}{2} \right) f_3 \left( \frac{n}{2} \right) + \cdots + 2 \cdot f_{m-1} \left( \frac{n}{2} \right) f_2 \left( \frac{n}{2} \right) + \left( f_{m-1} \left( \frac{n}{2} \right) \right)^2
\]
Substituting for \( f_1\left(\frac{m}{2}\right) \), \( f_4\left(\frac{n}{2}\right) \) and \( f_6\left(\frac{n}{2}\right) \) and using the inductive hypothesis for all \( i \geq 8 \) and \( f_i\left(\frac{n}{2}\right) = \left(\frac{n}{2}\right)^{i-1} \) for all odd \( i \geq 3 \), we get

\[
f_m(n) \leq 2 \left[ \binom{14}{15} \frac{n^{m^2}}{2^m (\frac{m}{2})!} + \frac{n^{m^2-1} \log(\frac{n}{2})}{2^{m^2-1}} \right] + 2 \left[ \binom{14}{15} \frac{n^{m^2}}{2^m (\frac{m}{2})!} + \frac{n^{m^2-1} \log(\frac{n}{2})}{2^{m^2-1}} \right] + \cdots
\]

\[
+ 2 \left[ \binom{14}{15} \frac{n^{m^2}}{2^m (\frac{m}{2})!} + \frac{n^{m^2-1} \log(\frac{n}{2})}{2^{m^2-1}} \right] + \cdots
\]

Grouping terms according to their asymptotic behavior of \( n \) and since there are at most \( \frac{m}{4} \) terms, each contributing \( \frac{2^{m-n^2} \log(\frac{n}{2})}{2^{m^2-1}} \), we have

\[
f_m(n) \leq 2 \left[ \binom{14}{15} \frac{n^{m^2}}{2^m (\frac{m}{2})!} + \frac{n^{m^2-1} \log(\frac{n}{2})}{2^{m^2-1}} \right] + 2 \left[ \binom{14}{15} \frac{n^{m^2}}{2^m (\frac{m}{2})!} + \frac{n^{m^2-1} \log(\frac{n}{2})}{2^{m^2-1}} \right] + \cdots
\]

Since \( \left(\frac{14}{15}\right)^k < 1 \), for positive \( k \), for the terms of asymptotic order \( n^{m^2-1} \log n \), we have

\[
f_m(n) \leq 2 \left[ \binom{14}{15} \frac{n^{m^2}}{2^m (\frac{m}{2})!} + \frac{n^{m^2-1} \log(\frac{n}{2})}{2^{m^2-1}} \right] + 2 \left[ \binom{14}{15} \frac{n^{m^2}}{2^m (\frac{m}{2})!} + \frac{n^{m^2-1} \log(\frac{n}{2})}{2^{m^2-1}} \right] + \cdots
\]
Using the identity $\sum_{i=0}^{N} \frac{1}{i!(N-i)!} = \frac{2^N}{N!}$ on the terms in the first and last square braces we get,

$$f_m(n) \leq \left(\frac{14}{15}\right)^m \frac{n^m}{m!} \left[ 1 - \frac{m}{2\pi} \log \left(\frac{n}{2\pi-1}\right) \right] + \left[ \frac{2m^{2-1} \log^2 \left(\frac{n}{2\pi-1}\right)}{2^{2-1}} \right] \left[ 1 - \frac{1}{\left(\frac{14}{15}\right)^{\frac{1}{2}} + \frac{1}{\left(\frac{14}{15}\right)^{\frac{1}{3}}} + \cdots + \frac{1}{\left(\frac{14}{15}\right)^{\frac{1}{6}}} \right]$$

Upon simplifying, we have

$$f_m(n) \leq \left(\frac{14}{15}\right)^m \frac{n^m}{m!} \left[ 1 - \frac{m}{2\pi} \log \left(\frac{n}{2\pi-1}\right) \right] + \left[ \frac{2m^{2-1} \log^2 \left(\frac{n}{2\pi-1}\right)}{2^{2-1}} \right] \left[ 1 - \frac{1}{\left(\frac{14}{15}\right)^{\frac{1}{2}} + \frac{1}{\left(\frac{14}{15}\right)^{\frac{1}{3}}} + \cdots + \frac{1}{\left(\frac{14}{15}\right)^{\frac{1}{6}}} \right]$$

For all values of $m \geq 8$, $\left[ \frac{2m^{2-1} \log^2 \left(\frac{n}{2\pi-1}\right)}{n \cdot 2^{2-1}} \right]$ $\leq 1$. Therefore,

$$f_m(n) \leq \left(\frac{14}{15}\right)^m \frac{n^m}{m!} + n^{\frac{m}{2} - 1} \log n.$$

For the case that $m$ is a multiple of 2 but not 4, an argument similar to the above can be used to show that $f_m(n) \leq \left(\frac{14}{15}\right)^m \frac{n^m}{m!} + n^{\frac{m}{2} - 1} \log n$. □

We now prove the stated bound for $c_{125}$.

**Lemma 1.** For any $S$ and $T$, and even $a$ and $b$ and $r = a + b + 1$, $\binom{a}{b} \times (\binom{T}{a+b+1}) \cup (\binom{S}{a+b+1}) \times \binom{T}{b}$ can be exactly covered using at most $\left(\frac{14}{15}\right)^r \binom{S}{a+b+1} \cdot \binom{T}{b}$ blocks.

**Proof.** Order the elements of $S$ and $T$. Pick $\frac{a}{2}$ elements of $S$, say $s_1, s_2, \ldots, s_{\frac{a}{2}}$, and $\frac{b}{2}$ elements of $T$, say $t_1, t_2, \ldots, t_{\frac{b}{2}}$. We associate a block corresponding to these sets as follows:

$$\{s_1, s_2, s_3, s_4, \ldots, s_{\frac{a}{2}}, t_1, t_2, t_3, t_4, \ldots, t_{\frac{b}{2}}\}.$$ 

Here $p$ is the index of the last remaining element after picking $s_1, s_2, \ldots, s_{\frac{a}{2}}$ from $S$ in the ordering. Likewise, $q$ is the index of the element from $T$. Among these take only the blocks which have $a+b+1$ parts. Note that these form a disjoint cover. □

**Lemma 2.** For large $S$ and $T$, and even $a$ and $b$, $\binom{S}{a} \times (\binom{T}{b+1}) \cup (\binom{S}{a+b+1}) \times \binom{T}{b}$ can be exactly covered using $(\left(\frac{14}{15}\right)^{\frac{a}{2}} + (\left(\frac{14}{15}\right)^{\frac{b}{2}}) \cdot (1 + o(1)) \cdot \binom{S}{a+b+1} \cdot \binom{T}{b}$ blocks. Here the $o(1)$ term is as $|S|$ and $|T|$ go to $\infty$.

**Proof.** The hypergraph $\binom{S}{a} \times (\binom{T}{b+1})$ can be exactly covered using $f_a(|S|) \cdot f_{b+1}(|T|)$ blocks. By Theorem 1, this is at most $(\left(\frac{14}{15}\right)^{\frac{a}{2}}(1 + o(1)) \cdot \binom{S}{a+b+1} \cdot \binom{T}{b}$ blocks. Likewise, $(\binom{S}{a+b+1}) \times \binom{T}{b}$ can be exactly covered using $(\left(\frac{14}{15}\right)^{\frac{b}{2}}(1 + o(1)) \cdot \binom{S}{a+b+1} \cdot \binom{T}{b}$ blocks. □
Lemma 3. For any odd \( r = 4d + 1 \), if \( \left( \frac{n}{12} \right) \times \left( \frac{n}{12} \right) \cup \left( \frac{n}{12} \right) \times \left( \frac{n}{12} \right) \) can be covered using \( \alpha(1 + o(1)) \left( \frac{n}{4} \right)^2 \) blocks such that \( \alpha < 1 \), then \( f_r(n) = c_r(\alpha) \cdot (1 + o(1)) \left( \frac{n}{12} \right)^2 \) where \( c_r(\alpha) < 1 \).

Proof. Recall the inequality.

\[
f_r(n) \leq 2 \cdot f_r\left( \frac{n}{2} \right) + 2 \cdot f_r\left( \frac{n}{2} \right) \cdot f_{r-1}\left( \frac{n}{2} \right) + \cdots + 2 \cdot f_{r-1}\left( \frac{n}{2} \right) \cdot f_{r+1}\left( \frac{n}{2} \right)
\]

Pairing up two consecutive terms and using Lemma 1 for each pair we have:

\[
f_r(n) \leq 2(1 + o(1)) \left[ \sum_{i=0}^{\frac{r-1}{2}} \left( \frac{n}{2} \right) \left( \frac{n}{2} \right) + f_{r+1}\left( \frac{n}{2} \right) + f_{r+1}\left( \frac{n}{2} \right) f_{r+1}\left( \frac{n}{2} \right) f_{r+1}\left( \frac{n}{2} \right) \right]
\]

Using the hypothesis and by adding and subtracting \( \left( \frac{n}{12} \right)^2 \) to the above equation we have:

\[
f_r(n) \leq 2(1 + o(1)) \left[ \sum_{i=0}^{\frac{r-1}{2}} \left( \frac{n}{2} \right) \left( \frac{n}{2} \right) + \left( \frac{n}{2} \right)^2 + \alpha(1 + o(1)) \left( \frac{n}{4} \right)^2 - \left( \frac{n}{4} \right)^2 \right]
\]

\[
\leq (1 + o(1)) \left( \frac{n}{4} \right) + \alpha(1 + o(1)) \left( \frac{n}{4} \right)^2 - \left( \frac{n}{4} \right)^2
\]

\[
\leq (1 + o(1)) \left( \frac{n}{4} \right) - (1 - \alpha)(1 + o(1)) \left( \frac{n}{4} \right)^2
\]

\[
\leq 1 - \left( 1 - \frac{1 - \alpha}{e^2} \right) \left( \frac{n}{4} \right)
\]

Lemma 4. \( f_{125}(n) \leq c \cdot (1 + o(1)) \left( \frac{n}{63} \right)^2 \), for a constant \( c < 1 \).

Proof. For \( a = b = 62 \) and \( |S| = |T| = \frac{n}{2} \) in Lemma 2, the hypergraph \( \left( \frac{n}{62} \right) \cup \left( \frac{n}{62} \right) \times \left( \frac{n}{62} \right) \times \left( \frac{n}{62} \right) \) can be exactly covered using at most \( 2 \cdot \left( \frac{14}{15} \right)^{\frac{62}{n}} \left( 1 + o(1) \right) \left( \frac{n}{31} \right)^2 \) blocks. The result follows from Lemma 3.

As a consequence of Lemma 4, we have \( c_{125} < 1 \). In fact solving the recurrence exactly for Theorem 1 using a computer program yields \( c_{113} < 1 \).

Declaration of competing interest

None.

References

[1] N. Alon, Decomposition of the complete \( r \)-graph into complete \( r \)-partite \( r \)-graphs, Graphs Combin. 2 (1) (1986) 95–100.