Discrete Mathematics 344 (2021) 112386

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

Multicovering hypergraphs

Anand Babu *, Sundar Vishwanathan

Department of Computer Science & Engineering, IIT Bombay, India

ARTICLE INFO

Article history:
Received 9 July 2019
Received in revised form 8 March 2021
Accepted 9 March 2021
Available online 26 March 2021

Keywords:
Hypergraph
Graham–Pollak
Multicovering

ABSTRACT

Let $f_r(n, p)$ represent the minimum number of complete r-partite r-graphs required to cover every edge of the complete r-uniform hypergraph on n vertices at least once and at most p times.

Graham–Pollak theorem states that $f_2(n, 1) = n - 1$. Upper and lower bounds were known for $r = 2$ and a general p. In this note we give bounds for $f_r(n, p)$ for general r and p.

1. Introduction

An r-uniform hypergraph H (also referred to as an r-graph) is said to be r-partite if its vertex set $V(H)$ can be partitioned into sets $V_1, V_2, \ldots, V_r$, so that every edge in the edge set $E(H)$ of H intersect V_i in one vertex. The complete r-uniform hypergraph with n vertices has an edge set consisting of all r-sized subsets of $[n]$.

Let $f_r(n)$ be the minimum number of complete r-partite r-graphs needed to partition the edge set of the complete r-uniform hypergraph on n vertices. The problem of determining $f_r(n)$ for $r > 2$ was proposed by Aharoni and Linial [1]. For $r = 2$, $f_2(n)$ is the minimum number of bipartite subgraphs required to partition the edge set of the complete graph on n vertices. Graham and Pollak ([10,11] see also [3] and [9]) proved that at least $n - 1$ bipartite graphs are required to partition the edge set of the complete graph K_n. Since the edges of the complete graph K_n can be partitioned into $n - 1$ disjoint bipartite graphs, this shows that $f_2(n) = n - 1$. Other proofs were found by Tverberg [17], Peck [15] and Vishwanathan [18,19].

Alon [1] showed that $f_3(n) = n - 2$ and showed that there exist positive constants $c_1(r)$ and $c_2(r)$ such that $c_1(r) \cdot n^{\lfloor \frac{r}{2} \rfloor} \leq f_r(n) \leq c_2(r) \cdot n^{\lfloor \frac{r}{2} \rfloor}$, for fixed $r \geq 4$. Cioabă, Kündgen and Verstraëte [6] improved Alon's bounds in the lower order terms. In a breakthrough result Leader, Milićević and Tan [13] made asymptotic improvements on $c_2(r)$. See also [13,14] and [4].

Let $[p] = \{1, 2, \ldots, p\}$. An *$r$-partite p-multicover* of a complete r-uniform hypergraph $K_n^{(r)}$ is a collection of complete r-partite r-graphs such that every hyperedge of $K_n^{(r)}$ is contained in l of the r-partite r-graphs for some $l \in [p]$. In other words, every edge of the complete r-uniform hypergraph appears at least once and at most p-times in the collection. The minimum size of such a covering is called the *r-partite p-multicovering number* and is denoted by $f_r(n, p)$.

The problem of *bipartite p-multicovering* of the complete graph K_n on n vertices was first studied by Alon [2]. For a list L if each edge occurs in some l r-partite r-graphs for some $l \in L$, then the collection of r-graphs is called an L-covering. Cioabă and Tait [7] investigated *bipartite covering* for a general list L. Note that $f_2(n, \{1\})$, the biclique partition number of K_n is the same as $f_2(n)$. For fixed $p \geq 2$, Alon [2] showed that $(1+o(1))(p!/2^p)^{1/p} n^{1/p} \leq f_2(n, p) \leq (1+o(1))pn^{1/p}$. Huang and Sudakov [12] improved Alon's lower bound to $(1+o(1))(p!/2^{p-1})^{1/p} n^{1/p} \leq f_2(n, p)$. For a fixed natural number λ, $f_2(n, \{\lambda\})$

* Corresponding author.
E-mail address: anandb@cse.iitb.ac.in (A. Babu).

https://doi.org/10.1016/j.disc.2021.112386

was studied by De Caen, Gregory, and Pritikin [5]. For list L of all odd numbers, $f_2(n, L)$ was studied by Radhakrishnan, Sen and Vishwanathan [16]. Cioabă and Tait [7] provided lower bound for *bipartite* L-covering number for any list and constructive upper bounds for $f_2(n, L)$ for several L.

In Section 2, the following lower bound of $f_r(n, p)$ is proved. For even $r > 2$,

$$f_r(n,p) \geq \frac{n^{\frac{r}{2p}} \cdot p}{e^{\frac{3r-2}{2p}+2-r} \cdot r^{(r+1)(1-\frac{1}{2p})-\frac{1}{2}}}$$

and for odd $r > 2$,

$$f_r(n,p) \geq \frac{n^{\frac{r-1}{2p}} p}{e^{\frac{3r-5}{2p}+3-r} \cdot (r-1)^{r(1-\frac{1}{2p})-\frac{1}{2}}}(1-o(1))$$

In Section 3, we achieve the following upper bound for $f_r(n, p)$ for $r > 2$.

$$f_r(n,p) \leq \frac{n^{\frac{r}{p}\lfloor \frac{r}{2} \rfloor} \cdot pe^{(\frac{r}{p}+1)\lfloor \frac{r}{2} \rfloor}}{2^{\lfloor \frac{r}{2} \rfloor} r^{(\frac{r}{p}-2)\lfloor \frac{r}{2} \rfloor}}$$

2. Lower bound

In this section, we obtain lower bounds for $f_r(n, p)$, the r-partite p-multicovering number for the complete r-graph $K_n^{(r)}$. The proof is a considerable generalization of the proof for the biclique covering of the complete graph K_n.

2.1. Preliminaries

Let r be even. Consider an r-uniform hypergraph H. Define the adjacency matrix of H, A_H as an $\binom{n}{r/2} \times \binom{n}{r/2}$ matrix, with rows and columns indexed by $r/2$ sized subsets of $[n]$, as follows:

$$A_H(e_1, e_2) = \begin{cases} 1, & e_1 \cup e_2 \in E(H) \\ 0, & \text{otherwise.} \end{cases}$$

When H is the complete r-uniform hypergraph the matrix A can also be defined as follows:

$$A(e_1, e_2) = \begin{cases} 1, & e_1 \cap e_2 = \phi \\ 0, & \text{otherwise.} \end{cases}$$

For even r, the *Kneser graph*, $KN(n, r/2)$, is the graph whose vertex set is $\binom{[n]}{r/2}$. Two vertices are adjacent if and only if they correspond to disjoint subsets. The adjacency matrix of the complete r-uniform hypergraph can be viewed as the adjacency matrix of the *Kneser graph*, $KN(n, r/2)$.

Lemma 1 ([8]). *The eigenvalues of the adjacency matrix of Kneser graph, $KN(n, r/2)$ are the integers $(-1)^i\binom{n-r/2-i}{r/2-i}$, for $i = 0, 1, \ldots, r/2$.*

Proof. Refer Theorem 9.4.3. (Page 200). □

Lemma 2. *Let A be the adjacency matrix of a Kneser graph, $KN(n.r/2)$, then rank(A) $= \binom{n}{r/2}$.*

Proof. Using Lemma 1, since all the eigenvalues of *Kneser graph*$(n, r/2)$ are non-zero, the adjacency matrix A has full rank. □

Theorem 1. *The r-partite p-multicovering number of the complete r-graph $K_n^{(r)}$, $f_r(n, p) \geq n^{r/2p} \cdot \frac{1}{(2r)^{r/2+1/2}}(1-o(1))$, for even $r > 2$.*

Proof. As in the graph case, we associate a matrix A with the complete r-uniform hypergraph and matrices N_i with each complete r-partite r-uniform hypergraph. Then write A as a sum of the N_is. The bound on the number of N_is follows by showing that the rank of A is large while the rank of each N_i is small. The adjacency matrix for complete r-partite r-graphs was used by Alon [1] and also by S.M. Cioabă, A. Kündgen and J. Verstraëte [6] for obtaining lower bounds for $f_r(n)$. It also uses a similar proof idea Huang and Sudakov [12] used for $r = 2$.

Suppose the edges of the complete r-uniform hypergraph on n vertices are covered by d complete r-partite r-graphs, $U_i \equiv (U_i^1, U_i^2, \ldots, U_i^r)$ for $1 \leq i \leq d$. Here U_i^j are the parts of the complete r-partite r-graph. The edges of the hypergraph U_i are obtained by taking one vertex from each part. such that every r-hyperedge is covered at least once and at most p times.

For each i, $1 \leq i \leq d$ and each $L \in \binom{[r]}{r/2}$, define a matrix $M(U_i, L)$ whose rows and columns are indexed by $\frac{r}{2}$ sized subsets as follows:

For $e_1, e_2 \in \binom{[n]}{\frac{r}{2}}$,

$$M(e_1, e_2) = \begin{cases} 1, & \text{if } e_1 \in \bigodot_{l\in L} U_i^l \text{ and } e_2 \in \bigodot_{l\in[r]-L} U_i^l \\ 0, & \text{otherwise.} \end{cases}$$

Here $\bigodot_{l\in L} U_i^l = \{e \in \binom{[n]}{\frac{r}{2}} : |e \cap U_i^l| = 1, \text{forl} \in \text{L}\}$.

Note that the adjacency matrix of the complete r-partite r-graph U_i denoted by $N(U_i)$ is equal to $\sum_L M(U_i, L)$. It is easy to see that the rank of $M(U_i, L)$ is one and hence by the sub-additivity of ranks we have, $Rank[N(U_i)] \leq \binom{r}{\frac{r}{2}}$. Note that $N(e_1, e_2)$ is 1 iff e_1 concatenated with e_2 is an edge in U_i.

For a non-empty set $S \subset [d]$ of indices, with size at most p, let H_S denote the hypergraph with the edge set consisting of all edges present in each of the hypergraphs U_i, for $i \in S$. $H_S = \cap_{i\in S} U_i$.

Let $N(H_S)$ denote the adjacency matrix of H_S. We show below that $N(H_S)$ can be written as a sum of $(r!)^{s-1}$ matrices each corresponding to complete r-partite r-graphs.

Fix a set $S = \{i_1, i_2, \ldots, i_s\}$. Fix permutations $\sigma_1, \sigma_2, \ldots, \sigma_{s-1}$ of $[r]$. One can now define a complete r-partite r-graph with parts $X_1, X_2, \ldots, X_r$ as follows:

$$X_j = U_{i_1}^j \cap U_{i_2}^{\sigma_1(j)} \cap \cdots \cap U_{i_s}^{\sigma_{s-1}(j)}, \; 1 \leq j \leq r$$

Let the adjacency matrix of this complete r-partite r-graph be denoted by $H_S(\sigma_1, \sigma_2, \ldots, \sigma_{s-1})$. Then $N(H_S) = \sum_{\sigma_1,\sigma_2,\ldots,\sigma_{s-1}} H_S(\sigma_1, \sigma_2, \ldots, \sigma_{s-1})$

Therefore, by the principle of inclusion–exclusion we have the adjacency matrix, $A = \sum_{S\subset[d],0<|S|\leq p} (-1)^{|S|} N(H_S)$. Since there are $s-1$ permutations of $\{1, 2, \ldots, r\}$ and $\binom{d}{s}$ choices of picking S, for $|S| = s$, H_S is the disjoint union of at most $(r!)^{s-1}$ r-partite r-graphs for fixed S. Hence A can be written as a disjoint union of at most $\sum_{s=1}^{p} (r!)^{s-1}\binom{d}{s}$ r-partite r-graphs. By sub-additivity of ranks, we have

$$Rank(A) \leq \binom{r}{\frac{r}{2}} \cdot \left[\sum_{s=1}^{p} (r!)^{s-1} \binom{d}{s}\right] \tag{1}$$

Combining equation (1) and Lemma 2, we have:

$$\begin{aligned} \binom{n}{r/2} &\leq \binom{r}{\frac{r}{2}} \cdot \left[\sum_{s=1}^{p} (r!)^{s-1} \binom{d}{s}\right] \\ &\leq \binom{r}{\frac{r}{2}} \cdot \left[(r!)^{p-1} \sum_{s=1}^{p} \binom{d}{s}\right] \end{aligned} \tag{2}$$

Using the bounds $(\frac{n}{x})^x \leq \binom{n}{x} \leq (\frac{en}{x})^x$ for positive integers n, x with $1 \leq x \leq n$, Eq. (2) becomes:

$$\begin{aligned} \left(\frac{n}{r/2}\right)^{r/2} &\leq \left(\frac{er}{r/2}\right)^{r/2} \cdot \left[(r!)^{p-1} \sum_{s=1}^{p} \frac{d^s}{s!}\right] \\ \left(\frac{n}{r/2}\right)^{r/2} &\leq \left(\frac{er}{r/2}\right)^{r/2} \cdot \left[(r!)^{p-1} \sum_{s=1}^{p} \left(\frac{p^s}{s!} \cdot \frac{d^s}{p^s}\right)\right] \end{aligned} \tag{3}$$

Since $e^x = \sum_{i=0}^{\infty} \frac{x^i}{i!}$ and using $x! \leq ex^{x+1/2}e^{-x}$ for positive integer x, Eq. (3) becomes:

$$\begin{aligned} \left(\frac{n}{r/2}\right)^{r/2} &\leq \left(\frac{er}{r/2}\right)^{r/2} \cdot \left[(r!)^{p-1} \left(e^p \cdot \frac{d^p}{p^p}\right)\right] \\ \left(\frac{n}{r/2}\right)^{r/2} &\leq \left(\frac{er}{r/2}\right)^{r/2} \cdot \left[(e \cdot r^{r+1/2} e^{-r})^{p-1} \left(e^p \cdot \frac{d^p}{p^p}\right)\right] \\ n^{r/2} &\leq e^{r/2+p-1-rp+r+p} \cdot r^{r/2+rp-r+p/2-1/2} \frac{d^p}{p^p} \\ d &\geq \frac{n^{\frac{r}{2p}} \cdot p}{e^{\frac{3r-2}{2p}+2-r} \cdot r^{(r+1)(1-\frac{1}{2p})-\frac{1}{2}}} \text{ for even } r > 2 \quad \square \end{aligned}$$

Since for odd r, we have $f_r(n,p) \geq f_{r-1}(n-1,p)$, we have $f_r(n,p) \geq \frac{(n-1)^{\frac{r-1}{2p}} p}{e^{\frac{3r-5}{2p}+3-r} \cdot (r-1)^{r(1-\frac{1}{2p})-\frac{1}{2}}} \geq \frac{n^{\frac{r-1}{2p}} p}{e^{\frac{3r-5}{2p}+3-r} \cdot (r-1)^{r(1-\frac{1}{2p})-\frac{1}{2}}}(1-o(1))$.

3. Upper bound

In this section, we obtain upper bounds for $f_r(n,p)$, the r-partite p-multicovering number for the complete r-uniform hypergraph $K_n^{(r)}$. We construct a covering of the complete r-graph using $p \cdot r^r n^{\frac{r^2}{2p}}$ complete r-partite r-graphs, so that each edge is covered at least once and at most p times.

When $p = 1$, a simple construction (see [13]) using $\binom{n}{\lfloor r/2 \rfloor}$ complete r-partite r-graphs is as follows. For odd r, define a complete r-partite r-graph corresponding to each subset of $[n]$ of size $\lfloor \frac{r}{2} \rfloor$. The elements of the subsets form parts themselves and elements not in the subset between two elements in the subset form the other $\lceil \frac{r}{2} \rceil$ parts.

A natural way to get an r-partite p-multicovering is to consider cross products. The elements of the base set are $[n] \times [n] \times \cdots \times [n]$($p$ times). For each co-ordinate i, we have a family of $\binom{n}{\lfloor r/2 \rfloor}$ complete r-partite r-graphs mimicking the one dimensional construction. Here the set size is n^p and the number of complete r-partite r-graphs is $p \cdot \binom{n}{r/2}$. It is seen that no element is covered more than p times, however, there are r-subsets of the universe that are not covered by the family.

We show below that one can find a large enough subset of $[n]^p$ for which the family described above covers every r-set. The key property we need of the elements of the universe is that for every subset of size r, the cover can be determined by some co-ordinate. That is, for at least one co-ordinate all r-values must be distinct.

This motivates the following definition. Consider vectors which are ordered p-tuples each of whose co-ordinates takes values from $\{1, 2, \ldots, n\}$. A set of ordered p-tuple vectors is defined to be *r-split* if for every collection of r vectors in the set, there is at least one index where they all differ.

Lemma 3. *There exists a set of r-split vectors in $[n]^p$ of size at least $\frac{2^{\frac{p}{r}} r^{1-\frac{2p}{r}}}{e} n^{\frac{p}{r}}$.*

Proof. Consider a set S of m vectors, obtained by picking each of them uniformly and independently at random from the set of all p-tuple vectors. Let A_i be the event that the ith subset of r vectors does not differ at any index. The probability that all elements at jth index of a fixed set of r vectors are different for $1 \leq j \leq p$, is $\frac{\binom{n}{r} \cdot r!}{n^r} = (1-\frac{1}{n}) \cdot (1-\frac{2}{n}) \cdots (1-\frac{r-1}{n}) \geq (1-\frac{r \cdot (r-1)}{2n}) \geq (1-\frac{r^2}{2n})$. Therefore, the probability that all elements at jth index are not different is $1-\frac{\binom{n}{r} \cdot r!}{n^r} \leq \frac{r^2}{2n}$. Thus we have,

$$Pr[A_i] \leq \left[1 - \frac{\binom{n}{r} \cdot r!}{n^r}\right]^p \leq \left(\frac{r^2}{2n}\right)^p$$

Since there are $\binom{m}{r}$ subsets of size r in S, the probability that at least one of the events A_i occurs is at most $\binom{m}{r}(\frac{r^2}{2n})^p$.

For $m = \frac{2^{\frac{p}{r}} r^{1-\frac{2p}{r}}}{e} \cdot n^{\frac{p}{r}}$, we have $\binom{m}{r}(\frac{r^2}{2n})^p < 1$. Thus with positive probability, no event A_i occurs and there is a set of r-split p-tuple vectors of size at least $\frac{2^{\frac{p}{r}} \cdot r^{1-\frac{2p}{r}}}{e} \cdot n^{\frac{p}{r}}$. $\square$

Theorem 2. *The r-partite multicovering number of the complete r-graph, $f_r(n,p) \leq p \cdot r^r \cdot n^{r^2/2p}$, for $r > 2$.*

Proof. Suppose a p-tuple from a set of r-split vectors as $\langle v_1, v_2, \ldots, v_p \rangle$. Consider the following family of complete r-partite r-graphs, $H[i, a_1, \ldots, a_{\lfloor \frac{r}{2} \rfloor}]$, where $i \in \{1, 2, \ldots, p\}$ and $a_1 < a_2 < \cdots < a_{\lfloor \frac{r}{2} \rfloor}$ where $a_1, a_2, \ldots, a_{\lfloor \frac{r}{2} \rfloor} \in \{1, 2, \ldots, n\}$. Each part of $H[i, a_1, \ldots, a_{\lfloor \frac{r}{2} \rfloor}]$ is defined as follows:

For odd r, part 1 contains those vectors whose $v_i < a_1$, part $2j$ contains those whose $v_i = a_j$ and part $2j+1$ contains those whose $a_j < v_i < a_{j+1}$ and the last part contains those whose $v_i > a_{\lfloor \frac{r}{2} \rfloor}$. Likewise for even r, part 1 contains those vectors whose $v_i < a_1$, part $2j$ contains those whose $v_i = a_j$ and part $2j+1$ contains those whose $a_j < v_i < a_{j+1}$ and the last part contains those whose $v_i = a_{\lfloor \frac{r}{2} \rfloor}$.

It is clear that the above family of at most $p \cdot \binom{n}{\lfloor r/2 \rfloor}$ complete r-partite r-graphs are sufficient to cover all r-sized subsets of the set of r-split vectors. Note that each r-sized subset of r-split vectors correspond to a r-hyperedge and the family of complete r-graphs $H[i, a_1, \ldots, a_{\lfloor \frac{r}{2} \rfloor}]$ covers each r-hyperedge at most p times and at least once.

To obtain the upper bound for $f_r(n,p)$, the number of r-split vectors used in the above construction that are analogous to number of vertices of $K_n^{(r)}$ must be n. In order to do so, we consider r-split vectors which take values from $\{1, 2, \ldots, N =$

$\frac{e^{\frac{r}{p}}n^{\frac{r}{p}}}{2r^{\frac{r}{p}-2}}\}$. Lemma 3 shows that there exists a set of r-split vectors of size at most n. Hence,

$$f_r(n,p) \le p \cdot \binom{N}{\lfloor r/2 \rfloor} \le \frac{p \cdot e^{\lfloor r/2 \rfloor} N^{\lfloor r/2 \rfloor}}{\lfloor r/2 \rfloor^{\lfloor r/2 \rfloor}}$$
$$\le \frac{p \cdot e^{\lfloor r/2 \rfloor} \cdot e^{\frac{r}{p}\lfloor \frac{r}{2} \rfloor} n^{\frac{r}{p}\lfloor \frac{r}{2} \rfloor}}{2^{\lfloor \frac{r}{2} \rfloor} r^{(\frac{r}{p}-2)\lfloor \frac{r}{2} \rfloor}}$$
$$\le \frac{p \cdot e^{(\frac{r}{p}+1)\lfloor \frac{r}{2} \rfloor} n^{\frac{r}{p}\lfloor \frac{r}{2} \rfloor}}{2^{\lfloor \frac{r}{2} \rfloor} r^{(\frac{r}{p}-2)\lfloor \frac{r}{2} \rfloor}} \quad \text{for} \quad r > 2. \quad \square$$

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

References

[1] N. Alon, Decomposition of the complete r-graph into complete r-partite r-graphs, Graphs Combin. 2 (1986) 95–100.
[2] N. Alon, Neighborly families of boxes and bipartite coverings, in: The Mathematics of Paul Erdös II, Springer, 1997, pp. 27–31.
[3] L. Babai, P. Frankl, Linear Algebra Methods in Combinatorics: With Applications to Geometry and Computer Science, University of Chicago, 1992.
[4] A. Babu, S. Vishwanathan, Bounds for the Graham-Pollak theorem for hypergraphs, Discrete Math. 342 (11) (2019) 3177–3181.
[5] D. de Caen, D.A. Gregory, D. Pritikin, Minimum biclique partitions of the complete graph and related designs, in: R. Rees (Ed.), Graphs, Matrices and Designs, 1993.
[6] S.M. Cioabă, A. Kündgen, J. Verstraëte, On decompositions of complete hypergraphs, J. Combin. Theory Ser. A 116 (7) (2009) 1232–1234.
[7] S.M. Cioabă, M. Tait, Variations on a theme of Graham and Pollak, Discrete Math. 313 (2013) 665–676.
[8] C. Godsil, Royle. G.F., Algebraic Graph Theory, Vol. 207, Springer Science & Business Media, 2013.
[9] R.L. Graham, L. Lovász, Distance matrix polynomials of trees, Adv. Math. 29 (1) (1978) 60–88.
[10] R.L. Graham, H.O. Pollak, On the addressing problem for loop switching, Bell Syst. Tech. J. 50 (8) (1971) 2495–2519.
[11] R.L. Graham, H.O. Pollak, On embedding graphs in squashed cubes, in: Graph Theory and Applications, Springer, 1972, pp. 99–110.
[12] H. Huang, B. Sudakov, A counterexample to the Alon-Saks-Seymour conjecture and related problems, Combinatorica 32 (2) (2012) 205–219.
[13] I. Leader, L. Milićević, T.S. Tan, Decomposing the complete r-graph, J. Combin. Theory Ser. A 154 (2018) 21–31.
[14] I. Leader, T.S. Tan, Improved bounds for the Graham-Pollak problem for hypergraphs, Electron. J. Combin. 25 (1) (2018) 1–4.
[15] G.W. Peck, A new proof of a theorem of Graham and Pollak, Discrete Math. 49 (3) (1984) 327–328.
[16] J. Radhakrishnan, P. Sen, S. Vishwanathan, Depth-3 arithmetic circuits for $S_n^2(x)$ and extensions of the Graham-Pollak theorem, in: International Conference on Foundations of Software Technology and Theoretical Computer Science, Springer, 2000, pp. 176–187.
[17] H. Tverberg, On the decomposition of K_n into complete bipartite graphs, J. Graph Theory 6 (4) (1982) 493–494.
[18] S. Vishwanathan, A polynomial space proof of the Graham–Pollak theorem, J. Combin. Theory Ser. A 115 (4) (2008) 674–676.
[19] S. Vishwanathan, A counting proof of the Graham–Pollak theorem, Discrete Math. 313 (6) (2013) 765–766.