Even cycle problem for directed graphs

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Overview

I will discuss the following in this presentaion:

- Problem Description
- Terminology
- The central proof
  - this will comprise of various parts–viz parts (1) - (18)
- Proofs of lemmas used
- Conclusion

The problem

Definition

The even cycle problem is "Does a given directed graph D contain an even cycle?"

Why is the problem hard?

- Harder than the 'undirected' case
- Harder than the 'odd' case

Terminology

- Digraphs, etc.

Terminology

- Splitting and subdivision

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- Strongly k-connected digraph

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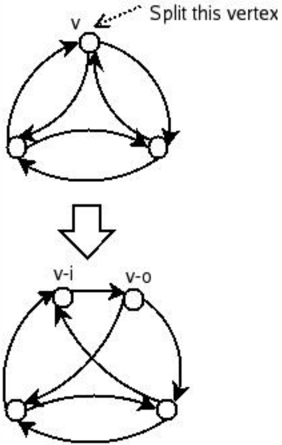
Terminology

- Splitting and subdivision
- Strongly k-connected digraph
- Initial and terminal components
- Weak k-double cycle



A weak odd double cycle obtained from 3-double cycle

Characterization of the problem

Definition

A digraph D is even, if and only if *every* subdivision of D contains a cycle of even length.

Characterization on the basis of even digraphs

- Equivalence of even-length and even-total-weight based definitions
- Characterization
  - A digraph is even if and only if it contains a weak-odd-double cyle

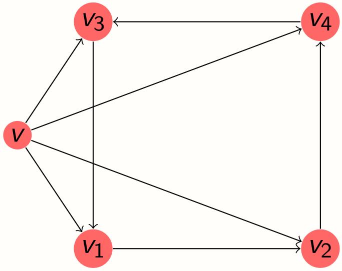
Lemmas used in the proof

We use the following four lemmas in the proof

- Lemma 1
  If we contract an arc such that either its initial vertex has outdegree one or its terminal vertex has in-degree one, then the resulting digraph contains a weak k-double cycle if and only if the original one does
- Lemma 2
  If the digraph obtained by terminal-component-reduction of a digraph contains a weak 3-double cycle, then original graph also contains one.

Lemmas contd..

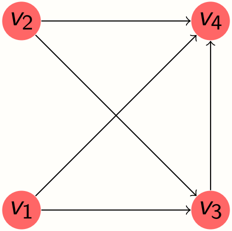
Lemma 3
If a strongly 2-connected digraph contains a dominating/dominated cycle then it contains a weak 3-double cycle



Lemmas contd..

Lemma 4

If a strongly 2-connected digraph contains vertices v_1, v_2, v_3, v_4 and the arcs v_1v_3, v_1v_4, v_2v_3, v_2v_4 and v_3v_4. Then D contains a weak 3-double cycle.



Outline of the proof

Theorem

If a strong digraph has minimum outdegree at least 3, except possibly for three vertices and, if we remove any vertex all the remaining vertices are still reachable from a vertex v_1 of outdegree 2, Then D contains a weak 3-double cycle.

The proof proceeds as follows:

Outline of the proof

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ii. Using the lemmas (1)-(4), obtain smaller graph G than D

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i. Assume that the theorem is false–D be minimal counterexample
ii. Using the lemmas (1)-(4), obtain smaller graph G than D
iii. Prove G to be a counterexample, contradicting minimality of D

Parts 1 and 2

1 D is strongly 2-connected

- Assume its not. D'' be a terminal component reduction of D
- Prove D'' is a counterexample smaller than D
  - D'' has minimum outdegree 2
  - Some vertex of D'' plays role of v_1
  - D'' contains no weak 3-double cycle

2 v_1 has outdegree 2 in D

Again, what if this were false :

- Remove an arc comming to it, say from z
- Now we get a smaller graph satisfying the conditions with $v_2 = z$
- Any v_i can play the role of v_1 ; so contradiction

Part 3

3 Delete v_1u_2, contract v_1u_1 ; gets digraph with minimum outdegree 2

Reasons why this might go wrong?

i. Outdegree of u_1 in D was 2 and it dominated v_1

ii. Some vertex z_1 of outdegree 2 in D dominated both u_1 and v_1 in D

   Or, if we flip roles of u_1 and u_2,

iii. Outdegree of u_2 in D was 2 and it dominated v_1

iv. Some vertex z_2 of outdegree 2 in D dominated both u_2 and v_1 in D

So, with v_1u_1 contracted we get D_1 and v_1u_2 contracted we get D_2

Part 3 Contd..

D_1 cannot be strongly 2-connected because

- It has at most three vertices of outdegree 2
- It does not contain a weak 3-double cycle
- It is smaller than D, can't be a counterexample

So, $D_1 - z_1$ not strong, find D_1', terminal component reduction of D_1 at z_1

Terminal component is H_1 and all other vertices in set I_1

Parts 4 and 5

4 Where do u_2, u'_1 lie?

- $u_2 \epsilon I_1$ and $u'_1 \epsilon H_1 \cup \{z_1\}$
  - If v_1 (or u'_1) lies in I_1, $D - z_1$ will fail to be strong
  - And if u_2 does not lie in I_1, $D - z_1$ or $D - u_1$ fail to be strong

5 D'_1 is strongly 2-connected

To prove this

- Prove if any vertex removed, all others can reach z_1 AND,
- z_1 can reach all others
- Removal of z_1 itself is trivial

Parts 6

6 D_1' has precisely four vertices of outdegree 2

- At least four, because otherwise D_1' becomes smaller counterexample
- Who else is candidate other than z_1, v_2 and v_3?
  - a vertex of outdegree 3 which dominates both v_1 and u_1 in D
  - u_1 if it has outdegree 3 and dominates v_1 in D
- But only one such candidate is possible

Some Implications

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i. We get $v_2, v_3 \epsilon H_1$ So, $u_2 \neq v_2, v_3$ as $u_2 \epsilon I_1$

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i. We get $v_2, v_3 \epsilon H_1$ So, $u_2 \neq v_2, v_3$ as $u_2 \epsilon I_1$

ii. So outdegree of u_2 in D is 3 (Similarly for u_1)

Parts 8 and 9

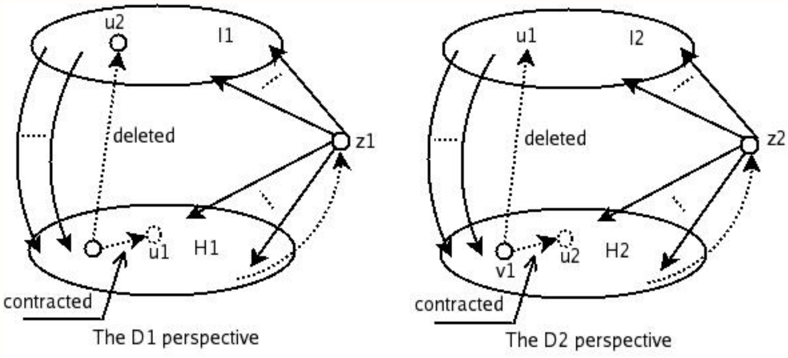
8 Some vertex of $I_1 \cup \{z_1\}$ dominates v_1 in D

- Either u_2 dominates v_1 or some vertex of outdegree 3 dominates v_1 and u_2
- As $u_2 \epsilon I_1$, vertex dominating it is not in H_1
- In any case, the dominating vertex is from I_1 or it is z_1

9 Either $z_1 \neq u_1'$ or $z_2 \neq u_2'$

- If $z_1 = u_1'$, every path from v_2 to u_2 in $D - v_1$ contains u_1
- Likewise, if $z_2 = u_2'$, every path from v_2 to u_1 in $D - v_1$ contains u_2
- But D strongly 2-connected, so $D - v_1$ has a v_2-$\{u_1, u_2\}$ dipath; contradiction

An Intuition for parts (10)-(12)



Part 10

10 If $z_2 = u'_2$ or $z_2 \epsilon V(I_1) - \{u_2\}$, then $z_1 \epsilon V(H_2)$

- Less the boundary conditions, it says that if z_2 is in I_1, then z_1 is in H_2
- Any $v_2 - z_1$ dipath in $D - v_1$ cannot contain any vertex from I_1 other than u_2 (because v_2 is in H_1), in particular z_2
- This is true even if $z_2 = u'_2$
- But as v_2 is in H_2, a terminal component, z_1 is also in H_2

Part 11

11 If $z_2 = u_2'$ or $z_2 \epsilon (V(H_1) - \{u_1'\}) \cup \{z_1\}$, then $I_1 - u_2 \subseteq H_2$

1. Case $z_2 = u_2'$
   - $z_2 = u_2'$ So by (10), z_1 lies in H_2
   - A z_1-I_1 dipath in $D - u_2$ is present in $D_2 - z_2$ also,because it avoids v_1, u_1
   - The start vertex of this path-z_1 is in H_2, a terminal component
   - So all possible endpoints (read all of I_1) also lie in H_2
2. Case $z_2 \epsilon (V(H_1)$ or is $\{z_1\}$
   - As $z_2 \neq u_2'$, u_2' lies in H_2
   - A u_2-I_1 dipath in $D - z_1$ is present in $D_2 - z_2$ also,because it avoids z_2, which is in H_1
   - The start vertex of this path-u_2' is in H_2, a terminal component
   - So all possible endpoints (read all of I_2) also lie in H_2

Part 12

12 If $z_2 \epsilon V(I_1) - \{u_2\}$, then
$(V(I_1) - \{u_2, z_2\}) \cup \{z_1, u_2'\} \subseteq V(H_2)$

- Simply said, if z_2 is in I_1, then all of I_1, z_1 and u_2 are contained in H_2
- z_1 is in H_2 and u_2' is in H_2 as before
- All $\{z_1, u_2\}$-I_1 *shortest* dipaths in $D - z_2$ are present in $D_2 - z_2$ also
- These start in H_2, a terminal component of $D_2 - z_2$ so also end in H_2
- So all the endpoints(read all of I_1), z_1 and u_2' lie in H_2

Parts 13

13 At most one vertex from $I_1 \cup \{z_1\}$ dominates u_1 in D

An Intuition

u_1 is in I_2 and H_2 contains almost all of I_1. And not many arcs from H_2 to I_2. So only possibilities (who dominate u_1) are z_1, z_2 and u_2

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- z_2 is in I_1 : Apply (12) to get $I_1 \subseteq H_2$ and $z_1, u_2' \in H_2$

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- z_2 is in I_1 : Apply (12) to get $I_1 \subseteq H_2$ and $z_1, u_2' \epsilon H_2$
- z_2 is in H_2 : By (11) we get $I_1 - u_2 \subseteq H_2$

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- z_2 is in I_1 : Apply (12) to get $I_1 \subseteq H_2$ and $z_1, u_2' \epsilon H_2$
- z_2 is in H_2 : By (11) we get $I_1 - u_2 \subseteq H_2$
- $z_2 = z_1$: As $z_2 \neq u_2'$, u_2' lies in H_2

Part 14

Obtain G and G'

- G obtained from the subdigraph of D induced by $I_1 \cup \{r, v_1, z_1\}$ by adding rv_1 and rz_1
- Outdegree of v_1 here is 1. Contract $v_1 u_2$ into u_2' to get G'
- This proves the following fact

14 G' doesn't contain a weak 3-double cycle

Part 15

15 G' has minimum outdegree at least 2

- Obtained from I_1 so a vertex looses outdegree only if has arcs to H_1
- Outdegree of z_1 in D (i.e.2) indicates number of such vertices

Who else can loose their outdegree in G'?

- A vertex dominating v_1, u_1 and u_2 in D can have outdegree 1 in G'; but by lemma 4, that's impossible
- u_2, if it dominates both v_1 and u_1; but by lemma 3 this is impossible

Parts 16 and 17

16 r in G' plays the role played by v_1 in D

- z_1 and v_1 have direct arcs from r. So removal of any vertex doesn't disconnect them
- For all other vertices : $D - u_2$ has paths from r to I_1; these paths are present here

17 G' is strong

- Any vertex in G' is reachable from r, by 16
- $D - u_1$ has a path to r from any vertex and outdegree of v_1 in G is one
- So, any dipath to r in $D - u_1$ from $I_1 \cup \{z_1\}$ is in G'
- Hence, any vertex in G' can reach r. And thus, G' is strong

Part 18 and Conclusion

18 G' has at most three vertices of outdegree 2

- Almost all vertices of I_1 have the same outdegree in G as in D i.e. ≥ 3
- So only r and v_1 in can have outdegree less than 3
- While forming G' from G, u_2' or a vertex dominateding both v_1 and u_2 loose outdegree
- Only one such vertex is possible, as seen above

Conclusion

From parts (13)-(18), we conlcude that we have got a smaller counterexample to the theorem. So we get a complete contradiction. Hence the proof of this theorem.

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Proving Lemma 1

Lemma 1

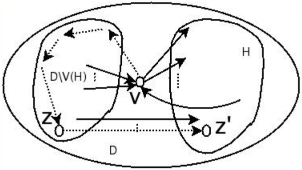
Let xy be an arc of D such that either $d^+(x, D) = 1$ or $d^-(y, D)$ $= 1$.D' be obtained from D by contracting xy into a vertex z. Then D' contains a weak k-double cycle iff D does.

- Any cycle in the original graph represents a subdivision of a cycle in the new graph
- If any cycle in the new graph is a weak k-double cycle, then so is its subdivision
- Conversely, any weak k-double cycle in the original graph is transformed to one in the new graph

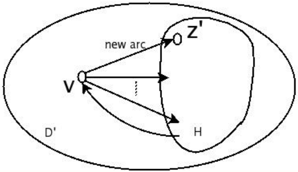
Proving Lemma 2

Lemma 2

D' be the H-reduction of D at v. If D' has a weak k-double cycle, then so does D (D-v not strong and H the terminal component)



- A weak k-double cycle in D' has an arc vz' means D has an arc to z' from some vertex outside H, say z
- P is a dipath from v to z
- Replace vz' by the dipath P, to get a weak k-double cycle in D

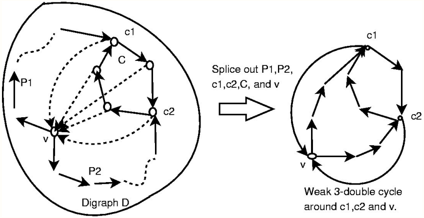


Lemma 3

Lemma 3

D strongly 2-connected. If D has a dicycle which dominates/is dominated by v, then D contains a weak 3-double cycle.

- Lets say C is a cycle whose vertices all dominate v
- There are two independent $v - C$dipaths, say P_1 and P_2
- The dicycle C, dipaths P_1 and P_2, and two arcs from C to v form a weak 3-double cycle



Lemma 4

Lemma 4

Let v_1, v_2, v_3, v_4 be vertices in a strongly 2-connected digraph D such that D contains the arcs $v_1v_3,\ v_1v_4,\ v_2v_3,\ v_2v_4$ and v_3v_4. Then D contains a weak 3-double cycle.

Two cases come out here

- P_1 and P_2 be two dipaths from v_4 to v_1 and v_2, resp.
- v_3 lies on one of the dipaths P_1 or P_2
  - P_1 gets partitioned into two dipaths–R_1(from v_4 to v_3) and R_2
  - P_3 be a $V(R_1) \cup V(P_2) - V(R_2)$ dipath in $D - v_3$
  - $P_1 \cup P_2 \cup P_3 \cup \{v_1v_3,\ v_3v_4,\ v_1v_4\}$ contains a weak 3-double cycle
- v_3 does not lie on P_1 or P_2
  - D-v_4 has a $v_3 - V(P_1) \cup V(P_2)$ dipath P_3
  - Lets say P_3 intersects P_1
  - Now $P_1 \cup P_2 \cup P_3 \cup \{v_1v_3,\ v_1v_4,\ v_2v_3,\ v_2v_4\}$ is a weak 3-double cycle