Even Cycle Problem For Directed Graphs

Seminar report

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by

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Abstract

The paper we discuss here answers the question : "Does there exist a natural number k such that any strongly k-connected digraph has an even length dicycle?" This is an attempt at proving sufficient conditions (in terms of connectivity and number of paths between any two vertices) for the existence of an even length dicycle in a graph.
We here try to assimilate and explain the proof given by Thomassen [5] in this regard.

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Chapter 1

Introduction

1.1 The problem

1.1.1 The even cycle problem

The even cycle problem is stated as follows :
"Does a directed graph D contain an even cycle?"
Here, even cycle means a cycle which comprises of an even number of edges. This problem has come up in various connection and has been studied for long. Recently (i.e. late 1990's), a polynomial time algorithm for this problem has been discovered [2]. A discussion on the recent developments can be found in a survey paper on pfaffian orientations of graphs by Robin Thomas [3].

1.1.2 Why is it hard?

In this section we first note that the same problem in case of undirected as well as in case of odd length cycle (this in case of directed also) is solvable easily in polynomial time. So why is it that *even* cycle problem for *directed* graphs only is so hard? We found some explainations as below, from the references indicated.

Harder than the 'Odd length' case

In case of odd length, existence of an odd closed walk [1] implies the existence of an odd cycle. Once you find an odd closed walk (if its not already a cycle), find a vertex that is repeated, forming a cycle around itself. Splice out this cycle. Now one of the two cycles has to be of add length, for, sum of their lengths is an odd number.(And, an odd number can't be expressed as a sum of two even numbers). There are efficient algorithms for finding an odd/even closed walk. So, effectively we have efficient algorithm for the odd cycle problem. To the contrary, an even number *can* be expressed as a sum of two odd numbers. Hence the same systematic procedure is not applicable here. A more detailed version of this discussion and an efficient algorithm for finding odd/even closed walk in a digraph can be found in a masters thesis by Michael Brundage. [1]

Harder than the 'Undirected' case

An important result in the directed graphs is that, large connectivity requires large minimum degree, but large minimum degree does not imply large connectivity. When we are in the undirected domain, life is much more easier. Because there, large minimum degree immediately implies large connectivity. So, the sufficiency conditions for the existence of even cycles can be readily expressed in terms of minimum degree of the graph. But in directed domain, life becomes suddenly difficult.
In fact Thomassen [4] proved that for each positive integer n, there are digraphs G and D on n vertices which do not contain even cycles even if :

- D_n is strong and each vertex of D_n has outdegree n

- Each vertex of G_n has in-degree n and outdegree at least n

The terms used here to express the assertion are taken from the master's thesis of Michael Brundage [1].

[1] In a closed walk, vertices can be repeated, whereas in cycle they can't.

1.2 Terminology

1.2.1 Preliminaries

In a digraph we distinguish between two endpoints of an arc(an edge is called an arc in the directed case) and if the arc is from vertex u to vertex v , we say that u dominates v.We also distinguish between *in*-degree and *out*-degree of a vertex, which we define as number of arcs entering the vertex and leaving the vertex, respectively.
A $u - v$ dipath is a directed path from vertex u to vertex v and for sets of vertices A and B, an $A - B$ dipath is a $x - y$ dipath such that $x \epsilon A$, $y \epsilon B$ and no other vertex on the path belongs either to A or to B.

Splitting and subdividing

Splitting a vertex v means replacing it by two, one of which is designated for *incomming* arcs and the other is designated for *outgoing* arcs. That is, all arcs entering vertex v will now enter, say v_i and all arcs leaving v will now leave, say v_o. Subdividing an arc uv means replacing that arc with a $u - v$ dipath by introducing one or more *new* vertices.
If A is any subset of the vertex set of a digraph D, we can remove all those vertices from D (along with arcs incident with them, of course) that are not in A and obtain a subgraph of D. This is called subdigraph *induced* by D.

1.2.2 Definitions

Strongly k-connected graph

A strong digraph is one in which every vertex can be reached from every other vertex. And a graph is called *strongly* $k - $ *connected* if it remains strong after removal of any set of vertices of size less than k.
This implies that there be two disjoint paths between every two vertices of a strongly k-connected digraph.

Initial and Terminal components

A strong component of a digraph D is a maximal strong subdigrph. That is, if a graph is not already strong, then we can find its subdigraph which is strong. The maximal such subdigraph is a strong component of a graph.

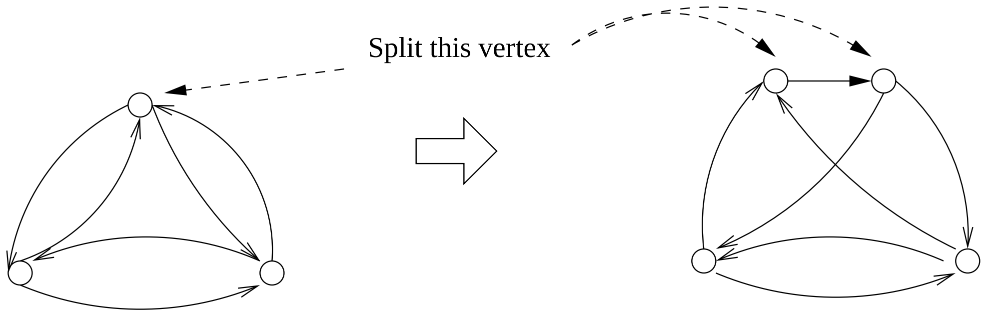
We define two special types of strong components here. Initial component is a strong component such that there is no edge leaving the component (there are edges only comming in). Similarly, a terminal component is a strong component such that there is no edge comming into the component (there are edges only going out).
Reduction at an initial/terminal component :
If H is an initial component of a digraph D, then H-reduction of D at a vertex v is a graph obtained from the subdigraph induced by the vertices in H, and v, by adding all the arcs vz for all vertices z in H to which there is an arc in D from a vertex that is not in H. (By definition of initial components, there can only be edges of this type). In short we look at the rest of the graph excluding H as only one vertex and represent all its connections to H by arcs from a single vertex - viz. z.

Similarly if H is a terminal component then H-reduction of D at a vertex v is a graph obtained from the subdigraph induced by the vertices in H, and v, by adding all the arcs zv for all vertices z in H which have an edge going out of H (to a vertex in $D - V(H)$). Here also, we are looking at the rest of the graph excluding H as a single vertex and represent the connections from H to this set as edges to a single vertex - viz. z.

Weak-odd double cycle

A double cycle is a graph whose arcs form two cycles–one in each direction. That means, a double cycle is obtained from a cycle by adding arcs parallel to the original arcs, but in the opposite direction.

We obtain a *weak double cycle* from a double cycle by splitting vertices and/or subdividing arcs. And, a weak double cycle which is obtained from a double cycle on odd number of vertices is a *weak odd double cycle*. Note that, a weak odd double cycle may itself have an even number of vertices; the term ‘odd’ refers to the number of vertices in the original double cycle from which this weak cycle is obtained (possibly, by splitting vertices, hence changing parity of number of vertices). An example of a weak odd double cycle obtained from a double cycle on 3 vertices is shown below. Observe that the weak odd double cycle shown here has 4 vertices.



A 3−double cycle (Double cycle on 3 vertices)

A weak−odd−double cycle Obtained from a 3−double cycle (3:odd)

Even Digraph

A digraph D is even, if and only if *every* subdivision of D contains a cycle of even length. If, instead of looking at the subdivisions of D, we try to assign binary weights to the arcs of D, then saying that *every* such assignment has a cycle of even *total weight* is same as saying that *every* subdivision of the graph contains a cycle of even *length*. Because a particular subdivision of the graph corresponds to a particular assignment of weights. If the subdivision divides an arc into even number of arcs, we assign a zero to that arc in the original graph. And, if the subdivision divides an arc into odd number of arcs, we assign one to that arc in the original graph. Thus, the weight assignmet corresponds to the parity of the length of any cycle in a particular assignment. A subdivision corresponding to a particular assignment can be similarly formed. It is basically possible because we only want to distinguish between odd and even lengths, which can be done using weights zero and one.

Note that the concept of even digraphs is one more way to characterize the even cycle problem. How? An even digraph already contains an even cycle, because every subdivision of it contains an even cycle. In particular the subdivision in which no vertex/arc is split/subdivided has an even cycle.

Chapter 2

The Theorem

2.1 Proofs of Important Lemmas

Characterization of the problem

A digraph is even if and only if it contains a weak-3-double cycle. This characterization was given by the same author Carsten Thomassen, which is used in this paper to prove the desired result.
Here basically we use the result that : A weak odd double cycle has an odd number of dicycles and every arc is in an even number of dicycles. So, every weak odd double cycle is even.
It is easy to see that this is true. An odd double cycle has odd number of dicycles - 2 spanning cycles in each direction and a cycle around each adjacent pair of vertices, of which only an odd number can be there. So this totals to odd number plus 2 which is always odd. And these cycles are preserved over any number of splittings/subdivisions. So the first assertion is proved. For the second, see that each arc in a double cycle is in exactly two cycles-one spanning cycle and one smaller cycle with its neighbour. Over any number of splittings/subdivisions, this will not change,for the original arcs. For the new arcs, they *have* to be a part of both the spanning cycles and they have be a part of both the smaller cycles. Hence they are in 4 dicycles. So this is proved.

But how is a graph with above property even?
This is simple to see. Suppose we assign weights to the arcs. Then let $w(C_i)$ be the total weight of cycle C_i. The sum of all $C_i's$ must be even because each arc weight is counted in an even number of times. But number of $C_i's$

itself is odd. So sum of an odd number of odd numbers can never be even. So some $w(C_i)$ must be even. That is there is an even-total-weight cycle. So the digraph is even [6].

Lemma 2.1.1 *Let xy be an arc of D such that either* $d^+(x, D) = 1$ *or* $d^-(y, D) = 1$. [1] *Let D' be obtained from D by contracting xy into a vertex z. Then D' contains a weak k-double cycle iff D does.*

This lemma states that, if we contract an arc such that either its initial vertex has outdegree one or its terminal vertex has in-degree one, then the resulting digraph contains a weak k-double cycle if and only if the original one is.
*[Proof]*Contraction of an arc xy means replacing the vertices x and y by a single vertex z such that all arcs entering or leaving x or y now enter or leave (respectively) z. Observe that for the arc we are contracting, either every path through its initial vertex passes *only through* the terminal vertex or, every path to the terminal vertex comes *only via* the initial vertex.
Intuitively, we can convince ourselves that the lemma indeed is true. Because, given the above situation, the process of contracting an edge looks much like the reversal of the process of splitting a vertex (with the difference that some extra outgoing arcs–in the former case, and some extra incoming arcs–in the latter case, are added). And splitting is a way of forming subdivisions of a digraph. So, any cycle in the original graph represents a subdivision of a cycle in the new graph. So if any cycle in the new graph is a weak k-double cycle, then so is its subdivision–in the original graph. (Remember how weak double cycles are defined)

Conversely, any weak k-double cycle in the original graph is transformed to a weak k-double cycle in the new graph. The original weak k-double cycle must be something obtained from a k-double cycle by at least one subdivision/splitting. So, we are reversing this process by contracting an edge. This gives us a cycle which in a sense is *stronger* than the former weak double cycle. So, at most it can be a proper double cycle, which is also a special case weak k-double cycle.

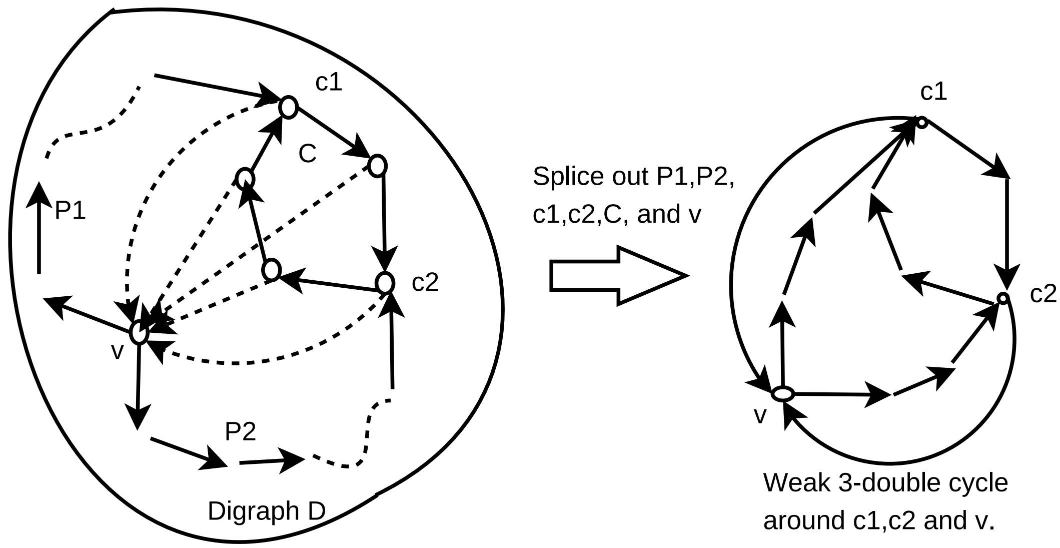
[1] $d^+(x, D)$ means outdegree of x in D and $d^-(x, D)$ means in-degree of x in D

Lemma 2.1.2 *Let D be a strong digraph such that D-v is not strong. Let H be a terminal component of D-v. Let D' be the H-reduction of D at v. If D' has a weak k-double cycle, then so does D.*

This lemma states that, if we obtain a terminal component-reduction of a digraph at a vertex, and this newly-obtained digraph contains a weak k-double cycle, then original graph also contains one. In other words, if the original digraph *does not* contain a weak k-double cycle, then the new one also cannot contain a weak k-double cycle(Contrapositive of the lemma).
[Proof] Let M' be a weak k-double cycle in D'. If it has an arc vz' then that means D has an arc to z' from some vertex outside H, say z. If P is a dipath from v to z. If we replace vz' in M' by the dipath P, M' is transformed into a weak k-double cycle in D. Observe that if one more such arc, say y'(and vertex y corresponding to it), exists, then we will walk backwards from y towards v and stop where we cut P. This subdipath is the replacement for the arc vy' if it is present in the weak k-double cycle. Thus any weak k-double cycle in D' can be transformed into one in D. Hence proof.

Lemma 2.1.3 *Let v be a vertex in a strongly 2-connected digraph D. If D-v contains a dicycle whose vertices all are dominated by v (in D) , or a dicycle whose vertices all dominate v (in D) then D contains a weak 3-double cycle.*

[Proof] Here, we use the Menger's theorem to prove the lemma. In simple terms, Menger's theorem states that, if a digraph is (strongly) k-connected then there are at most k independent (pairwise internally disjoint) paths between any two vertices in the digraph. In particular, there are 2 independent paths between any two vertices in a strongly 2-connected digraph.
Consider a dicycle C whose vertices all dominate v. By Menger's theorem, there are two independent paths, say P_1 and P_2, between v and vertices of C. That is to say, if we take any two different vertices, say, c_1 and c_2 on the dicycle C, then the two $v - V(C)$ dipaths–$v - c_1$ and $v - c_2$ have nothing in common except v. Now consider the dicycle C, the two dipaths P_1 and P_2, and the arcs in D from c_1 and c_2 to v(any vertex on the dicycle dominates v). The union of all these actually forms a weak 3-double cycle with v, c_1 and c_2 being the 'three main' vertices. This situation is pictured as follows:



Thus, D contains a weak 3-double cycle. The case where v *dominates* all the vertices of C is proved similarly using two $V(C)$ - v paths. Hence the lemma is proved.

Lemma 2.1.4 *Let v_1,v_2,v_3,v_4 be vertices in a strongly 2-connected digraph D such that D contains the arcs v_1v_3, v_1v_4, v_2v_3, v_2v_4 and v_3v_4. Then D contains a weak 3-double cycle.*

[Proof] We again use the Menger's theorem here. As D is strongly 2-connected, there should be two arc-disjoint paths between any two vertices. Lets say P_1 and P_2 be two dipaths from v_4 to v_1 and v_2, respectively. The only common vertex to these graphs is v_4 itself. Now there are two possibilities for the remaining vertex v_3 : either it lies on one of the dipaths P_1 or P_2 or it does not. We consider both cases and prove the lemmas in both the cases.

Assume that v_3 does not lie either on P_1 or on P_2. Now, observe that, one dipath from v_3 to $V(P_1)$ is the arc v_3v_4. By Menger's theorem, there has to be one more such path, say P_3. We can safely assume that P_3 intersects P_1. Then we can form a v_3v_4 dipath as follows: we can leave v_3 along P_3, reach P_1 somewhere and then follow P_1 from there to v_1.

Thus, we have three vertices now - namely, v_1, v_3 and v_4. The dipath formed by concatenating P_1, the arc v_1v_3, and arc v_3v_4 is a dicycle in one direction–$v_4v_1v_3v_4$. The dipath formed by concatenating the arc v_1v_4, P_2

(from v_4 to v_2) and arc v_2v_3 and then the path from v_3 back to v_1 as discussed above is a dicycle in another direction–$v_1v_4v_3v_1$. Thus we have found a weak 3-double cycle.

Consider the second case. Here $v_3 \epsilon P_1$. So, P_1 now gets partitioned into two dipaths–R_1(from v_4 to v_3) and R_2(from v_3 to v_1). Also, one way for the vertices in $V(R_1) \cup V(P_2)$ to reach to those in $V(R_2)$ is through v_3. So, there has to be one more path not involving $v-3$. Call this path P_3. Now we can form a v_4 v_1 dipath (not passing through v_3) by leaving $v-4$ along P_1, then leave P_1(or if P_3 is from a vertex on P_2, we leave v_4 along P_2 and then leave P_2) along P_3, before v_3 is reached and reach on R_2 part of P_1 and then follow rest of R_2 to reach v_1. We have necessarily bypassed the vertex v_3.

We again form a weak 3-double cycle around vertices v_1, v_3 and v_4 as follows: The cycle formed by concatenating arcs v_1v_3 and v_3v_4 with the v_4v_1 dipath as formed above is a dicycle in one direction–$v_1v_3v_4v_1$. Similarly the cycle obtained by concatenating arc v_1v_4, dipath R_1 and dipath R_2 is a dicycle in the other direction–$v_1v_4v_3v_1$.
Hence the proof of this lemma.

2.2 Proof of main result

Theorem 1 *Let D be a strong digraph such that each vertex has outdegree at least 2. Let v_1, v_2,v_3 be vertices such that all other vertices of D have outdegree at least 3. Assume further that if we remove any vertex other than v_1, all the remaining vertices are still reachable from v_1. Then D contains a weak 3-double cycle. In particular, D is even.*

Outline of proof
The proof proceeds as follows:

- First assume the theorem to be false and let D be a minimal counterexample to the theorem. That is, a counterexample with as few vertices and arcs as possible subject to the conditions in the theorem statement.
- Using lemmas from previous chapter obtain graphs *smaller* than D that retain certain properties of D(like evenness or non-evenness).

- Now construct a graph G, from these intermediate small graphs, such that G is in some sense *smaller* than D and still is a counterexample to the theorem.
- Lastly, G is a contradiction to the minimality of D. And hence the proof.

The proof proceeds in various parts. So we present the explanation of the proof in various sections that follow. All the related parts of the original proof have been clubbed up under one section.

2.2.1 Proving properties of graph D

(1) D is strongly 2-connected
We prove this property by contradiction. Suppose that D was not strongly 2-connected. That means there is some vertex u such that $D-u$ is not strong. Let D' be a terminal component of $D-u$. We have assumed that if we remove any vertex, *all* other vertices are still reachable from v_1. So v_1 can't be in the terminal component D'.(By definition, any vertex in a terminal component can have arcs to vetices from the same component only.) Note that although v_1 cannot be in the terminal component, u can itself be equal to v_1, which is fine.
Obtain the D'-reduction of D at u – call it D''. Now we will prove that D'' is a smaller counterexample to the theorem. For this, we need to prove:

- D'' has minimum outdegree at least two.
- Some vertex of D'' plays the role of v_1.
- D'' contains no weak 3-double cycle.

To prove the first assertion, we observe that the outdegree of all vertices in D'' is same as their outdegree in D, except for u. (Because the only change we made was removing u and we restore that while forming the reduction. All other outgoing arcs have to be within the terminal component only) So, $d^+(x, D'') = d^+(x, D'') \geq 2 \ \forall \ x \ \epsilon \ D'$. We also need to prove that $d^+(u, D'') \geq 2$, because u might have 'lost' some arcs to vertices not in the terminal component. For that, lets assume that outdegree of u is less than 2, say 1. This means that there is at most one arc from the rest of the graph to the terminal component D'(While forming the reduction, we add arcs from

u into D' to account for this connectivity). So, if we remove the only vertex from D' to which u has an arc, there is no way for v_1 (which is outside the terminal component) to reach any vertex from D'. But this is not true. Hence outdegree of u should also be at least 2.
Now that u has outdegree at least two in D'', it can play the roll of v_1 in D''. This proves the second assertion. And, by lemma 2.1.2, as D does not have a weak 3-double cycle(assumption in the proof), D'' also doesnot have one. This proves the third assertion.

We have proved that D'' is a smaller counterexample to the theorem so it contradicts the minimality of D. We conclude that the assumption that we started off with is false. Hence D is strongly 2-connected.

(2) Outdegree of v_1 (in D) is 2
To prove this, note that if all vertices had outdegree at least 3, we could take a vertex say z dominating v_1, remove the arc zv_1 and get a graph smaller than D with $v_2 = z$(as D is strongly 2-connected, any of the $v_i's$ can play the role of v_1); and this would again contradict the minimality of D. Consequently, some v_i has to have outdegree 2 and it will then play the role of v_1. So if $d^+(v_1, D) = 2$, let u_1 and u_2 be two vertices dominated by v_1.

(3) If we delete the arc v_1u_2 and contract v_1u_1, then the resulting digraph has minimum outdegree at least 2
To prove that this indeed is the case, we will investigate the possible situations where this is *not* the case. So if after deleting v_1u_2 and contracting v_1u_1, the minimum outdegree is not 2, it has to be the case that either:

i. Outdegree of u_1 in D was 2 and one of the two vertices it dominated was $v - 1$ and due to contraction v_1u_1, the new vertex formed now has outdegree 1. OR

ii. Some vertex z_1 of outdegree 2 in D dominated both u_1 and v_1 in D. And because u_1 and v_1 are now the same vertex, z_1 has now only one outgoing arc.

   As the roles of u_1 and u_2 can be interchanged (v_1u_1 deleted and v_1u_2 contracted), we also have the other two symmetric possibilities where the above statement might be violated. So we also have either:

iii. Outdegree of u_2 in D was 2 and one of the two vertices it dominated was $v-1$ and due to contraction of v_1u_2, the new vertex formed now has outdegree 1. OR

iv. Some vertex z_2 of outdegree 2 in D dominated both u_2 and v_1 in D. And because u_2 and v_1 are now the same vertex, z_1 has now only one outgoing arc.

Now we systematically investigate and prove that the statement is true in all the four scenarios. Let's see first what happens if (ii) or (iv) are true. If (ii) is true and z_1 is equal to u_2, then the situation is something like this : Here there is a cycle whose vertices all dominate a vertex. And by lemma 2.1.3, such a graph should contain a weak 3-double cycle, which, by assumption is false. So, if (ii) is true, then z_1 can't be same as u_2. Symmetrically, if (iv) is true, then z_3 can't be same as u_1. So, if (ii) ever holds, we can choose the notation such that $z_1,\ v_1,\ u_1$ (or $z_1,\ v_1,\ u_1$) play the roles of $v_1,\ u_1,\ u_2$ (that is above statement is true if we contract z_1v_1 and delete z_1u_1). Note that this is possible because here z_1 can't be u_2. And also because the following : there are at most three vertices of outdegree 2 in D and if (ii) holds, two of them are v_1 and z_1. The third can either be u_2 (if (iii) is true) or z_2 (if (iv) holds). Understand that (i) and (ii) become true/false independent of (iii)/(iv).

And if $z_1,\ v_1,\ u_1$ cannot play the roles of $v_1,\ u_1,\ u_2$ that means due to contraction of z_1v_1, some vertex of outdegree 2 that dominated z_1 as well as v_1 lost its outdegree. And the only vertex remaining which can be of outdegree two is u_2. So (iii) holds. But given this situation, $z_1,\ u_1,\ v_1$ can play the roles of $v_1,\ u_1,\ u_2$. That is instead of contracting z_1v_1, we will contract z_1u_1. This will work because now there is no vertex of outdegree 2 that can dominate both z_1 and u_1.

We now consider the case when (i) or (iii) hold. So if (i) and (iii) hold, v_1 dominates and is dominated by both u_1 and u_2. Now the scenario looks like that of a cycle with all dominating/dominated vertices. So if there is an arc between u_1 and u_2 then lemma 2.1.3 will be applicable. And will imply that the graph contains a weak 3-double cycle, which is again false. So, there is no arc between u_1 and u_2.So if y be the vertex ($\neq v_1$) dominated by u_1. Then $u_1,\ y,\ v_1$ can play the role of $v_1,\ u_1,\ u_2$.

2.2.2 Obtaining D_1 and D_2

We saw in the previous section that a digraph obtained from D by contracting arc v_1u_1 and deleting arc v_1u_2 has minimum outdegree at least 2. We now call this digraph D_1 and call the new vertex formed (due to contraction) as u_1'. Similarly the digraph obtained by interchanging roles of u_1 and u_2 (i.e. by contracting v_1u_2 and deleting v_1u_1) is called D_2 and the new vertex here is called u_2'. We note here that all the statements about D_1 that we prove in next section are also true for D_2.

As D_1 is obtained from D by contraction of an arc whose initial vertex had outdegree 1 (before we contracted v_1u_1 we deleted v_1u_2, making outdegree of v_1 1), lemma 2.1.1 applies here. So, as D doesn't contain a weak 3-double cycle, D_1 also doesn't contain a weak 3-double cycle. Next we claim that there are at most three vertices of outdegree 2. To see this, note that when we contract v_1u_1, we are loosing a vertex of outdegree 2. And if at all any new vertex of outdegree 2 is created anew, there can exist only one such vertex. Why? Because of the following: New vertex of outdegree 2 may be produced either because u_1 had out-degree 3 and dominated v_1 or because some other vertex w of outdegree 3 dominated both u_1 and v_1. But if u_1 dominated v_1 then existence of w creates the scenario of lemma 2.1.3 (the cycle $v_1u_1v_1$ with all vertices dominated by w) and implies that D_1 contains a weak 3-double cycle. And the existence of 2 such w's creates the scenario of lemma 2.1.4 (two vertices dominating a pair of vertices–u_1 and v_1, which have an arc between them) and again implies that D_1 contains a weak 3-double cycle. But as D_1 doesnot contain a weak 3-double cycle, either u_1 dominates v_1 or some other w (only one) dominates both u_1 and v_1. Therefore, if D_1 happens to be strongly 2-connected, it will be a smaller counterexample to the theorem, contradicting the minimality of D and so D_1 is not strongly 2-connected.

2.2.3 Obtaining and proving properties of D_1' and D_2'

Since D_1 is not strongly 2-connected, we can find a vertex z_1 such that $D_1 - z_1$ is not strong. We now choose a z_1 such that the terminal component H_1 of $D_1 - z_1$ is relatively minimal. That is if we choose any other such vertex z' then the terminal component obtained is either bigger than this or is equal to this (i.e. H_1). Call the set of vertices of D_1 *o*ther than z_1 that are not in the terminal component H_1 as I_1. We now investigate which vertices of

$u'_1, u_2,$ *and* z_1 lie in which parts of the graph D_1. (A similar analysis can be done about D_2 also, as noted in the previous section).

(4) $u_2 \in I_1$ **and** $u'_1 \in H_1 \cup \{z_1\}$
Where can u'_1 lie in D_1? Observe the situation here. As $D_1 - z_1$ is not strong, and H_1 is the terminal component created by removing z_1, we know that in D_1, the only way for the vertices in H_1 to reach to other vertices (namely those in I_1) is through z_1 (which got cut away by the removal of z_1). But D_1 has been obtained from D by contraction of v_1u_1. If we restore that arc back, we get back D (of course we have to add the deleted arc v_1u_2). But this will not make any difference to the locality of v_1 and u_1. They will lie only where u'_1 lies in D_1. And so, if v_1 happens to be in I_1, then even after restoring to D the only way out for vertices in H_1 is through z_1 only. But there has to be one more because D is strongly 2-connected. For this reason, v_1 must be in H_1 so that u_2 is in I_1 and when we add that arc back to get D, we form one more way out for the vertices in H_1 (other than through z_1). So, u'_1 lies in H_1 (or equals z_1) and u_2 lies in I_1.

(5) D'_1 **is strongly 2-connected**
We obtain the H_1-reduction of D_1 at z_1 and call it D'_1. As D_1 has outdegree at least 2, there are at least 3 vertices in D'_1. Now we need to prove that if we remove any vertex from D'_1, it still remains strong. Clearly as D'_1 is a reduction at z_1, if we remove z_1 what remains is itself a strong (terminal) component. So we prove this for all other vertices than z_1. Now if we show that if we remove any other vertex, z_1 can reach the remaining vertices and the remaining vertices can reach z_1. This proves that, the graph is still strong. Now, any vertex of D'_1 must be able to reach z_1. Because if that is not the case, we can exclude those vertices from H_1 and still form a terminal component H'_1 which in fact is smaller that H_1 and hence contradicts the minimality of H.

So what remains to be proved is that if we remove any vertex from D'_1, z_1 can still reach remaining vertices. We prove it as follows:
Here we make use of the fact that D is strongly 2-connected and hence $D - t$ has a path from z_1 to all other vertices, in particular, those in H_1 (because those are the ones in D'_1). So basically we want to prove that these paths do exist in D'_1 as well. So, in D, (as there are now two logical parts of D now, namely I_1 and H_1) this z_1-H_1 path can either come to H_1 directly or

it can come via a vertex of I_1. In the first case we are done. This path will surely be included in D_1 (This follows from definition of H-reduction). And if that path comes via I_1, then at some point the path *must* leave I_1 and enter H_1(and afterwards remain in H_1 only). Let this happen at the arc w_1w_2 (such that $w_1 \epsilon I_1$ and $w_2 \epsilon H_1$) Now as w_2 has an arc from I_1, while forming the reduction D_1', we will add an arc from z_1 to w_2. So, in D_1', take this arc from z_1 to w_2 and then follow the original path to any vertex in question. This completes the proof that D_1' is strongly 2-connected.

(6) D_1' has precisely four vertices of outdegree 2. Three of them are z_1, v_2, v_3(if one of v_2, v_3 is u_1 then we take u_1' in its place). The fourth vertex is either u_1' or a vertex of outdegree 3 that dominates both v_1 and u_1.

First, by lemma 2.1.2, D_1' does not contain a weak 3-double cycle. It also is strongly 2-connected. So if it also has at most three vertices with outdegree 2, then it will become smaller counterexample to the theorem, contradicting the minimality of D. So it must violate this condition in the theorem. So, one thing is clear that D_1' has at least four vertices of outdegree 2. Now investigate who are the candidates.

One possibility is z_1, then v_2 and v_3 are also candidates because they had outdegree 2 in D as well. Then there can be a vertex of outdegree 3 which dominates both v_1 and u_1 or u_1 itself can be of outdegree 3 and could dominate v_1 in D. But as discussed in section 2.2.2 exactly one of these two can happen and also in case the latter happens, there can be at most one such vertex. Hence the three stable candidates are z_1, v_2, v_3. (with one of them possibly equal to u_1). This reveals the fact that in D, v_2 and v_3 lie in the H_1 part because they are in D_1' and D_1' is composed of $V(H_1)$ and z_1. Possibility is that one of v_2 or v_3 are same as u_1 and also z_1 is u_1'. But these two can't happen together because there are at least 4 vertices of outdegree 2 in D_1'. So conclusion is that,

$$v_2, v_3 \ \epsilon \ H_1$$

Also note that, as $u_2 \epsilon I_1$, and v_1, v_2 are in H_1, u_2 can't be same as v_2 *or* v_3. Neither can it be the case that v_1 *or* v_2 dominate u_2. But, there are only three vertices of D which can possibly have outdegree 2. And they are v_1, v_2, v_3. Now that u_2 can't be same as v_1 is but obvious. Hence u_2 has outdegree at least 3 in D.

In section 2.2.2 we pointed out that the statements being made for D_1' have counterparts for D'_2 and they hold there. So, as from analysis of D'_1 we got that u_2 has outdegree 3 or higher in D, we get that u_1 has outdegree 3 or higher in D, from the analysis of D'_2. Essentially we want to emphasize here that $u-1$ and u_2 are different than $\{v_1, v_2, v_3\}$. And hence, although v_1, v_2 belong to $H-1$ (and u_1 also lies there) they can't be equal to u_1. So we refine the above relation as :

$$v_2, v_3 \epsilon H_1 - \{u'_1\}.$$

(7) $u_1 \epsilon I_1$ and $v_2, v_3 \epsilon H_2$-$\{u'_2\}$

This is immediate from the fact that the statements made about D'_1 above are all true for D'_2 also.

(8) Some vertex of $I_1 \cup \{z_1\}$ dominates v_1 in D

Now we consider the D'_2 counterpart of the statements about D'_1. There are precisely 4 vertices of outdegree 2 in D'_2 also. Follows the fact that either u_2 dominates v_1 or some other vertex of outdegree 3 dominates both v_1 and u_2. (Obtained by just interchanging the occurrences of u_1 and u_2 from the statement in previous section). Now we know that u_2 lies in H_1. So, a vertex dominating it cannot lie in H_1. Because, there are no other paths from H_1 to I_1 than through z_1 and the arc v_1u_2. So in both cases, a vertex from I_1 dominates v_1. Note that this vertex can also be z_1. Because there is a gap in this argument which is filled by z_1.

(9) Either $z_1 \neq u'_1$ or $z_2 \neq u'_2$

That is to say, both of the above clauses cannot be false at the same time. That is, z_1 being u'_1 and z_2 being u'_2 cannot happen at the same time. How do we prove this?

Consider $z_1 = u'_1$. What does this mean when we move back to the original graph D? If we split back z_1 (which is now u'_1) to give us v_1u_1, we get back D (after of course, replacing the deleted arc v_1u_2). So, now the two independent paths from the vertices in H_1 to those in I_1 are – one through the arc v_1u_2 and other through the vertex u_1. So, as u_2 is in I_1, and v_2 is in $H-1$, any path from v_2 to u_2 in $D - v_1$ must contain u_1.

Similarly, we can start with $z_2 = u'_2$, and can conclude from the analysis of D'_2 that, any path from v_2 to u_1 in $D - v_1$ must contain u_2. But because, as D is strongly 2-connected, $D - v_1$ should have a v_2-$\{u_1, u_2\}$ dipath. (That is either a path to u_1 which doesn't have u_2 or vice versa) This contradiction proves that, both the assertions cannot be true at the same time.

So, we can choose the notation such that, say, $z_1 \neq u_1'$ and then $z_2 = u_2'$. But once we do this, we can no longer interchange between u_1 and u_2. Hence we would like to investigate further the vertices in I_1 and H_1.

2.2.4 Investigating vertices in I_1, H_1, I_2, H_2

(10) If $z_2 = u_2'$ or $z_2 \epsilon V(I_1) - \{u_2\}$, then $z_1 \epsilon V(H_2)$
The meaning of the claim : This means to say that if z_2 lies in the vertex set I_1 , then z_1 lies in H_2. The union and set difference operations are just to handle some boundary conditions. For a first read we can skip it. So what are the boundary conditions?

Here we write $V(I_1) - \{u_2\}$ to emphasize that, while defining z_2, in D_2, there is no u_2, we have contracted it but when we talk of I_1, we know that u_2 is there in it. And hence z_2 is in I_1 but can't be equal to u_2 and u_2' is not in I_1 but still z_1 can be equal to u_2'.
[Proof] To understand why this is true, note first that v_2 belongs in H_1. And secondly, as D is strongly connected $D - v_1$ is strong. So if we consider any $v_2 - z_1$ dipath in $D - v_1$, it will not contain any vertex from I_1 other than u_2 because we know that only way from H_1 to I_1 other than through z_1 is the arc v_1u_2. So in particular, if z_2 is in I_1 (and of course is not u_2), then any $v_2 - z_2$ dipath will not contain z_2. But what if $z_2 = u_2'$? Even then any v_2z_1 dipath cannot contain z_2 because, we can split back the contracted arc v_1u_2 and get back D. But in $D - v_1$ this arc is lost. So if you now want to reach any vertex in I_1 you have to go through z_1 only.

Hence on a $v_2 - z_1$ dipath in $D - v_1$, z_2 will never occur. But now try to look at this from the D_2 perspective. We also know that v_2 lies in H_2. And similarly only way out from H_2 in $D - v_1$ is through z_2. But if no $v_2 - z_1$ path ever contains z_2 then the path will never move out of H_2. In particular, its end, z_1 will be in H_2.
Hence, if z_2 is in I_1 then z_1 will be in H_2.

(11) If $z_2 = u_2'$ or $z_2 \epsilon (V(H_1) - \{u_1'\}) \cup \{z_1\}$, then $I_1 - u_2 \subseteq H_2$
The meaning of the claim : In the previous claim we investigated the effect on z_1's location of z_2 being in I_1. Now we study the effect when z_2 lies in the other part of graph–i.e. H_2 and z_1. The claim says, again in simple terms, that if z_2 lies in $H_1 \cup \{z_1\}$ then the whole of I_2 is contained in H_2.

Once again the set differences and all are here to take care of boundary conditions. When we say $z_2 \epsilon H_1$, we emphasize that while defining z_2, we did not contract u_1. So u_1' is non-existent at this time although it is there in H_1 ; so we must rule out the possibility of $z_2 = u_1'$ while talking of z_1 being in H_1 in general. Again when we say all of I_2 is contained in H_2, we must not mean that u_2, which is otherwise an element of I_1, is there in H_2. Because basically, while obtaining H_2, we contracted v_1u_2 to get u_2' and so u_2 is non-existent at this time.

[Proof] We will prove it separately for the cases $z_2 = u_2'$ and z_2 is in H_1. Assume first that $z_2 = u_2'$. So by the previous property, we now say that z_1 lies in H_2. Now, as D is strongly 2-connected, $D - u_2$ is strong. In particular, there is a z_1-I_1 dipath. We want to prove that this path is present in $D_2 - z_2$ also. So while forming D_2, we have deleted v_1u_1 and if this path avoids u_1 and z_1, then we are done. This indeed is true. Because, v_1, u_1 are in H_1 and the only way from H_1 to I_1 other than through z_1 is the arc v_1u_2. But in $D - u_2$, we don't have that arc. So any such $z_1 - I_1$ path is there in $D_2 - z_2$ also. But in $D_2 - z_2$, H_2 is a terminal component. Recall what we started off with–z_1 is in H_2. So any path from z_1 must also end in H_2. So all those vertices to which z_1 has a path, particularly, all vertices in I_1, are in H_2. Hence H_2 contains all of I_1.

The second case is z_2 belongs in H_1 or is equal to z_1. Here as $z_2 \neq u_2'$, u_2' must lie in H_2. (It is either equal to z_2 or it has to be in H_2) Again as D is strongly 2-connected, $D - z_1$ is strong. So it has dipaths from u_2 to all other vertices of I_1. We want these paths to be present in $D_2 - z_2$ as before. For this note that because we are removing z_2 which is not u_2' from D_2. And D_2 is obtained by contracting v_1u_2 into u_2'. So all u_2-I_1 paths from $D - z_2$ come in $D_2 - z_2$ (They become u_2'-I_1 dipaths now). But H_2, where u_2' lies, is a terminal component of $D_2 - z_2$. So all these paths also end in H_2. Hence all the endpoints, namely, all the vertices of I_1 lie in H_2. This exhaustively proves the assertion.

(12) If $z_2 \epsilon V(I_1) - \{u_2\}$, then $(V(I_1) - \{u_2, z_2\}) \cup \{z_1, u_2'\} \subseteq V(H_2)$

The meaning of the claim : This claim says that if z_2 is in I_1, then again all of I_1 is contained in H_2. Once again $V(I_1) - \{u_2\}$ stands for the same purpose as discussed previously. And here, when we say I_1 is contained in H_2, we don't mean it for u_2 as before, as well as for z_2 because now z_2 also is there in I_1. Clearly z_2 is outside H_2 and u_2 is non-existent at the time of defining H_2. And this time we are making the assertion a bit stronger by saying that z_1 and u_2' are also in H_1.

*[Proof]*To prove this we do the folowing. We proceed on the similar lines. If z_2 is in I_1, we already know that z_1 is in H_2 and u_2' is in H_2 as before,

because it is not equal to z_2. So we will try to find paths from these to all other vertices in I_1. So, as D is strongly 2-connected, $D - z_2$ is strong. So it has paths from $\{z_1, u_2\}$ to all other vertices of I_1. We argue that these dipaths come in $D_1 - z_2$ also. To see this, as usual, observe that, z_2 is not u_2' and contracting $v_1 u_2$ does not affect paths passing *through* u_2. They just become u_2' paths now. Also, to make sure that these paths remain within I_1 only, we here consider only *shortest* paths, so a path from u_2 cannot go to H_1 since then it has to come back only through z_1 and in that case, we will consider the shorter path through z_1 only. The same argument applies to paths from z_1 also. Hence all these dipaths avoid the vulnerable v_1 and u_1.

So, we have proved that there are $\{z_2, u_2'\}$ - I_1 paths in $D_2 - z_2$. But both these starting points lie in the terminal component of this graph–H_2. So they all must end in H_2 itself. So all there endpoints (namely all of I_2) are in H_2. This proves the assertion.

(13) There is at most one vertex in $I_1 \cup \{z_1\}$ that in D dominates u_1
We know that u_1 is in I_2 and H_2 contains almost all of I_1. And there are not many arcs from H_2 to I_2. So intuitively, there are not many vertices from I_1 either that in D dominate u_1. We formally prove this.
[Proof] The only possibilities here are z_1, z_2 and u_2. We prove that exactly one of these three possibilities hold at any time. We do this by exhaustively considering the places where z_2 can lie as follows :

- $z_2 = u_2'$: In this case, by (10) and (11), z_1 lies in H_2 and so we don't care about it. So, only possibility is z_2 $(= u_2')$

- z_2 is in I_1 : Apply (12) to get $I_1 \subseteq H_2$ and $z_1, u_2' \epsilon H_2$. So none of these two are a concern. Only possible choice here is z_2.

- z_2 is in H_2 : Here by (11) we get $I_1 - u_2 \subseteq H_2$. So nobody else than u_2 is from I_1.

- $z_2 = z_1$: So here as $z_2 \neq u_2'$, u_2' lies in H_2. Once again we are left with only one choice–z_2 $(=z_1)$.

This proves that at most one vertex from $I_1 \cup \{z_1\}$ can dominate u_1 in D.

We now form the graphs G and G' in the next section and prove that G' is a counterexample to the theorem.

2.2.5 Obtaining G and G'

Let G be the digraph obtained from the subdigraph of D induced by $I_1 \cup \{r, v_1, z_1\}$ by adding the arcs rv_1 and rz_1, if they are not present already. As D is strongly 2-connected, D contains two independent $r - \{v_1, z_1\}$ paths (that is, they have nothing common except r). If we look at these paths as subdivisions of the arcs rv_1 and rv_2, then these dipaths together with the subdigraph from which we formed G is a subdivision of G. Hence G does not contain a weak 3-double cycle. Observe that v_1 has outdegree 1 in G (because, it can only dominate u_2, which is from I_1 in G). So we can contract v_1u_2 into u'_2 to get a new digraph called G'. So G' also does not contain a weak 3-double cycle.
We now prove that G' satisfies the conditions of the theorem and hence it is a smaller counterexample than D, contradicting the minimality of D. This will complete our proof.

(14) All vertices of G' have outdegree at least 2 in G'
Why is this true?

We reason when this might go wrong. It will only happen when some vertex of I_1 has an arc outside I_1, that is to H_1.(other possibility is arc to z_1 but z_1 is already included in G) So how many of such vertices can be there in I_1? And which are they? Recall how we formed the H_1 reduction at z_1. We added arcs from z_1 to the vertices of H_1 which had an incoming arc from some vertex of I_1. So, these arcs from z_1 to H_1 in D'_1 represent the arcs from I_1 to H_1. We have already proved in (6) that number of such arcs is 2(outdegree of z_1 is 2) So, at most 2 vertices from I_1 dominate some vertex from H_1. And these vertices from H_1 which have arcs from I_1 in D are the vertices dominated by z_1 in D'_1. So, one of them is u'_1 and the other is r. And we have already included these two vertices in our digraph G. Note that although we have not included u'_1 we have its representative from D viz. v_1. So all these vertices preserve their degrees in D. And in D, we know that the minimum outdegree is at least 2. Consequently, minimum outdegree in G' also is 2.
So is there any other reason why any vertex would loose its outdegree while comming from D to G'? The answer is yes. If a vertex dominates v_1, u_1 and u_2 in G will have outdegree 1 in G'. But we know that this situation calls for the application of lemma 2.1.4 and so implies that D contains a weak 3-double cycle, which we know is false. So this is impossible. What else? I a

u_2 dominates both v_1 and u_1 (and because we are contracting v_1u_2) now u_2' may have an outdegree equal to 1 in G'. But already we know that this also is impossible, because it calls for the application of lemma 2.1.3 and implies that D contains a weak 3-double cycle, which is a contradiction.

(15) If we remove any vertex from G', the remaining vertices are still reachable from r (r plays the role of v_1).
Here, r in G' has direct arcs to u_2' (because we added rv_1 in G) and z_1 so if we remove any vertex, $v-1$ and z_1 are already reachable from r. What about all others? We observe that D is strongly 2-connected and hence $D-\textit{any vertex}$ has a path from r. And also in particular, $D-u_2$ has a path from r to any vertex. We easily get this dipath in G' for the vertices we are interested in–those in I_1. Hence r in G' can play the role of v_1 in D.

(16) G' is strong
We have already proved that any vertex in G' is reachable from r. Now if we prove that any vertex can reach r in G', we are done. For this, first observe that $D-u_1$ has a path to r from any vertex because D is strongly 2-connected. But any dipath in $D-u_1$ from $I_1 \cup \{z_1\}$ is in G' also because outdegree of v_1 in G is one (it only dominates u_2). So we conclude that G' is indeed strong.

(17) G' has at most three vertices of outdegree
Lastly we investigate the vertices of outdegree 2 in G'. Here, r has out-degree 2 in G (and so in G'). We have already asserted that all other vertices that belong to I_1 have the same outdegree in G as in D which is ≥ 3. So the only other vertex in G of outdegree 2 is obviously v_1. While forming G' from G, we may create a new vertex of outdegree 2 which is either u_2' or a vertex that dominated both v_1 and u_2. But again by application of lemmas 3 and 4 and forming a contradiction, we can prove that only one of these two can happen. Thus, there are at most three vertices in G' which have outdegree 2 in G'.

2.3 conclusion

From the above three assertions, we conclude that G' satisfies the conditions in the theorem and still it doesnot contain a weak 3-double cycle. That is it is a counterexample to the theorem. But this graph is certainly smaller than

D (it is obtained from a part of D, viz. I_1). So, it contradicts the minimality of D. This contradiction completes the proof.

Note that we obtained such kind of contradictions a number of times during the proof, but the proof completed here only. Because every time we obtained a contradiction, there was always an escape from this contradiction– such as proving that the digraph is strongly 2-connected or something else. But only last time we had no escape, a complete contradiction. Hence the proof concluded there.

Bibliography

[1] Michael Brundage. From the even cycle miystery to the l-matrix problem and beyond, 1996.

[2] Neil Robertson, P. D. Seymour, and Robin Thomas. Permanents, pfaffian orientations, and even directed circuits. *Mathematics Annals*, 1999.

[3] Robin Thomas. A survey of pfaffian orientations of graphs. In *Peoceedings of the International Congress of Mathematicians,Madrid,Spain*, 2006.

[4] Carsten Thomassen. Even cycles in directed graphs. *Europion Journal of combinatorics*, 1985.

[5] Carsten Thomassen. The even cycle problem for directed graphs. *Journal of the American Mathematical Society*, 5(2), April 1992.

[6] Carsten Thomassen. The even cycle problem for planar digraphs. *Journal of algorithms*, 1993.

[7] Douglas West. *Introduction to Graph Theory*. Pearson Education, University of Illions-Urbana, second edition, 2001.