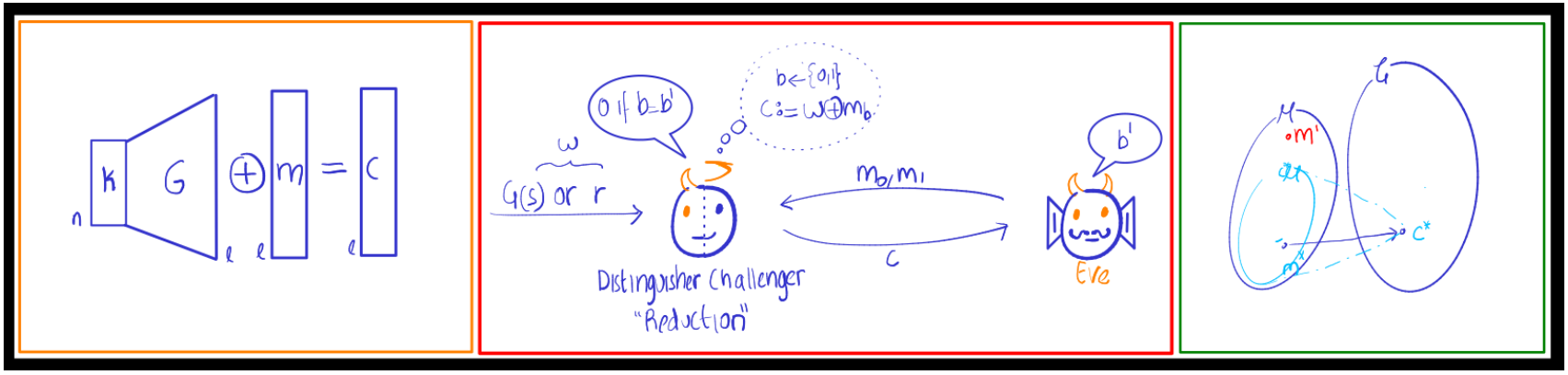


CS783: Theoretical Foundations of Cryptography

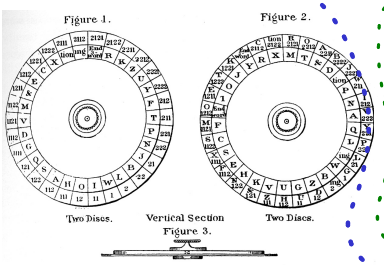
Lecture 3 (30/Jul/24)

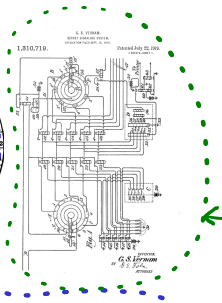
Instructor: Chethan Kamath

Recall from Last Lecture...

MODULE I
(Shared keys)

For a large part of history





Some historical ciphers

Perfect secrecy and one-time pad

Ego quoque sum conscius secreti

ooga

Past

~0

~1930s

Birth of information theory & CS

Foundations of modern cryptography

Credit for images: Wikipedia

Recall from Last Lecture...

General *template*:

1. Identify the task — secret communication with shared keys
2. Come up with precise threat model M (a.k.a security model) — Perfect secrecy against eavesdroppers
   - Adversary/Attack: What are the adversary's capabilities?
   - Security Goal: What does it mean to be secure?
3. Construct a scheme Π — One-time pad
4. Formally prove that Π in secure in model M

Plan for This Lecture...

of longer messages

General *template*:

secret communication with shared keys

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secret communication with shared keys

of longer messages

Computational

~~Perfect~~ secrecy against eavesdroppers

Computational

One-time pad

Plan for This Lecture...

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secret communication with shared keys

of longer messages

Computational

~~Perfect~~ secrecy against eavesdroppers

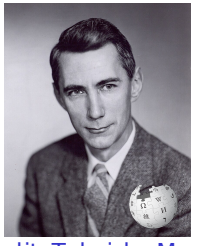
Computational

One-time pad

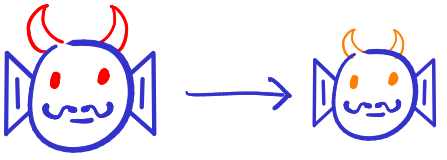
under certain assumption

↳ First security reduction!

Plan for This Lecture...



Credit: Tekniska Museet

1. Limitations of Perfect Secrecy: Shannon's Impossibility
2. Bypassing Shannon's Impossibility 
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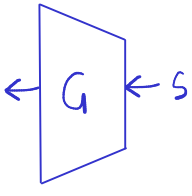

Plan for This Lecture

1. Limitations of Perfect Secrecy: Shannon's Impossibility
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3. Pseudo-Random Generators (PRGs) and Computational OTP

Shannon's Impossibility

Theorem 1 (Shannon'49)

Let $\Pi = (\mathsf{Gen}, \mathsf{Enc}, \mathsf{Dec})$ *be any perfectly-secret encryption scheme with message space* $\mathcal{M}$ *and key-space* $\mathcal{K}$*. Then* $|\mathcal{K}| \geq |\mathcal{M}|$*.*

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Proof Sketch.  Idea: proof by contradiction.

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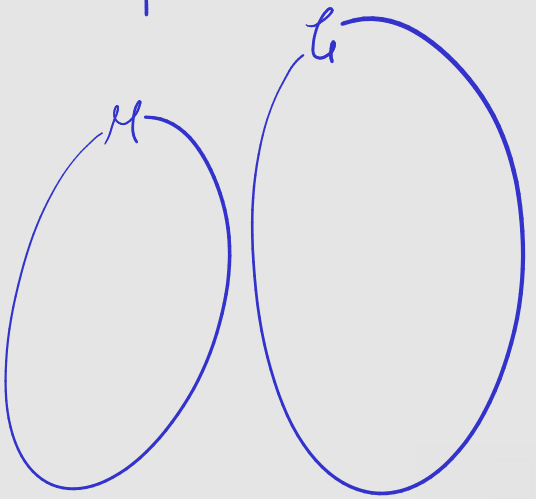

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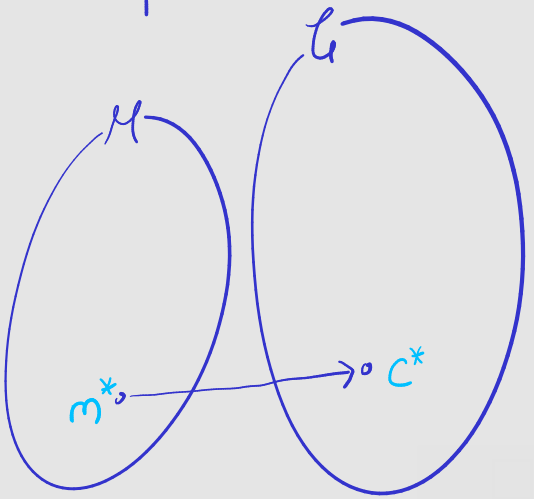

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Consider set $M_{c^*} \subseteq M$ defined as

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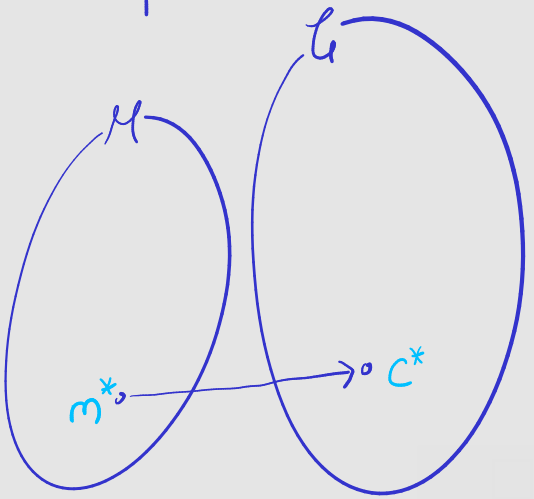

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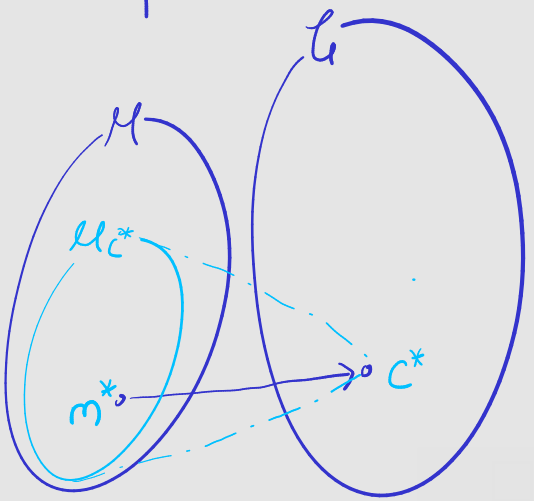

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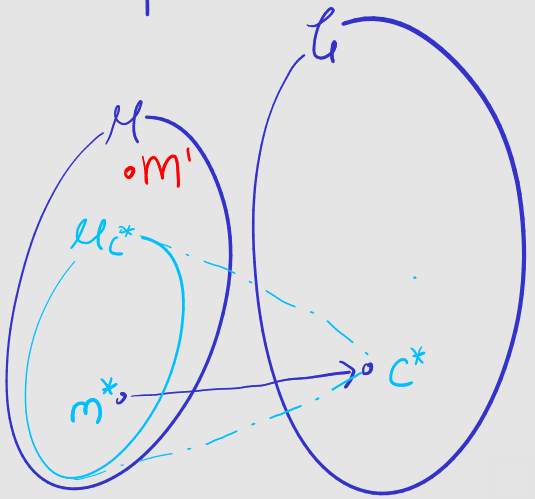
$\{m \in \mathcal{M} : \exists k \in \mathcal{K} \text{ s.t. } \mathsf{Dec}(k, c^*) = m\}$ ← ? Why?

Since $|M_{c^*}| \leq |\mathcal{K}| < |\mathcal{M}|$,

$\exists m' \in \mathcal{M} \setminus M_{c^*}$: c^* never decrypts to m'

$\Rightarrow (1/2)$



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(2/2)

Consider (m^*, m') and $\mathsf{Eve}_{c^*}(c) := \begin{cases} \text{'left'} & \text{if } c = c^* \\ \text{'right'} & \text{otherwise} \end{cases}$

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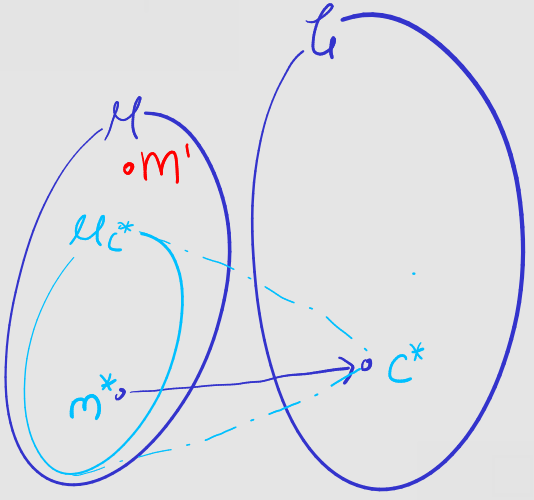

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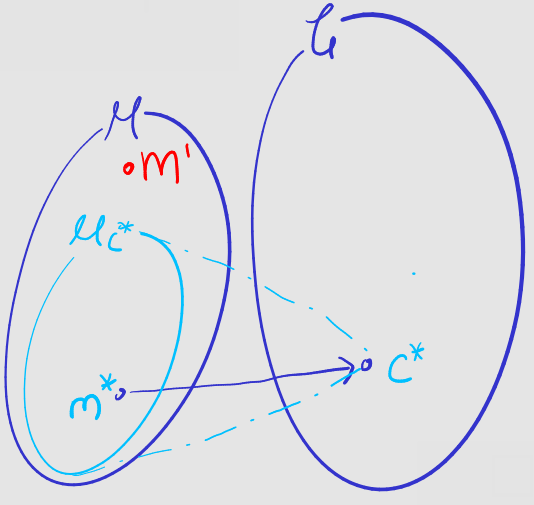

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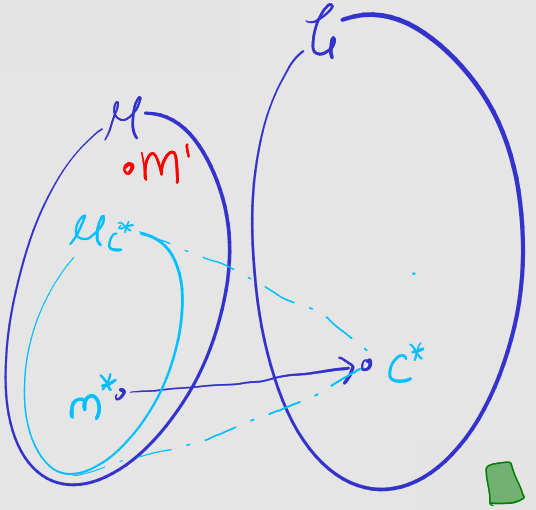
$\Rightarrow$ Π is not perfectly secure ↯



What Do We Do in Face of Shannon's Impossibility?

Defintion 1 (SKE Imitation Game)

An SKE $\Pi = (\mathsf{Gen}, \mathsf{Enc}, \mathsf{Dec})$ *is perfectly-secret if for every eavesdropper Eve and pair of messages* $(m_0, m_1) \in \mathcal{M}$*:*

$$\Pr_{\substack{k \leftarrow \mathsf{Gen} \\ c \leftarrow \mathsf{Enc}(k, m_0)}} [\mathsf{Eve}(c) = 0] - \Pr_{\substack{k \leftarrow \mathsf{Gen} \\ c \leftarrow \mathsf{Enc}(k, m_1)}} [\mathsf{Eve}(c) = 0] = 0$$

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  1. Restrict to *computationally*-bounded Eve
  2. Allow "slack": Eve may distinguish, but with "very small" prob.

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- Turns out both compromises necessary!

Plan for This Lecture

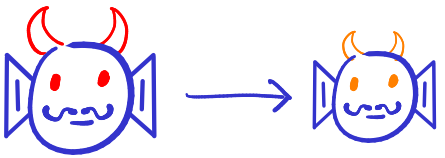
1 Limitations of Perfect Secrecy: Shannon's Impossibility

2 Bypassing Shannon's Impossibility

3 Pseudo-Random Generators (PRGs) and Computational OTP

First Compromise: Computationally Bound Eve...

- Restrict to *probabilistic polynomial-time* (PPT) Eves:
  randomised Eve that runs in time $p(n)$, for some polynomial p



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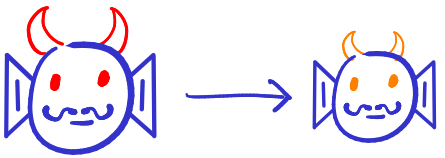

- Why PPT?
  - "Captures" efficient computation
  - The exact model of computation (Turing Machines, Random Access Machine) doesn't matter
    - Church-Turing thesis: all models of computation are polynomially equivalent
  - Polynomials have nice closure properties
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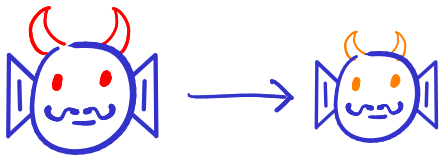

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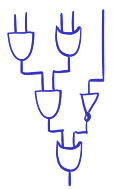
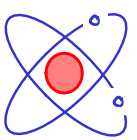
  - Randomness allowed since it is allowed for honest algorithms
- Some stronger models for Eve:



  - Polynomial-sized family of circuits: allows "non-uniform" advice
  - Quantum polynomial-time algorithms (Lecture 10)



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Candidate Defintion 1

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Exercise 1

Show that Shannon's impossibility extends to Candidate Defintion 1.

- *Hint: use similar strategy to Theorem 1.*

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- *Take-away: Eve can distinguish with a "very small" probability*

Second Compromise: Allow "Slack"...

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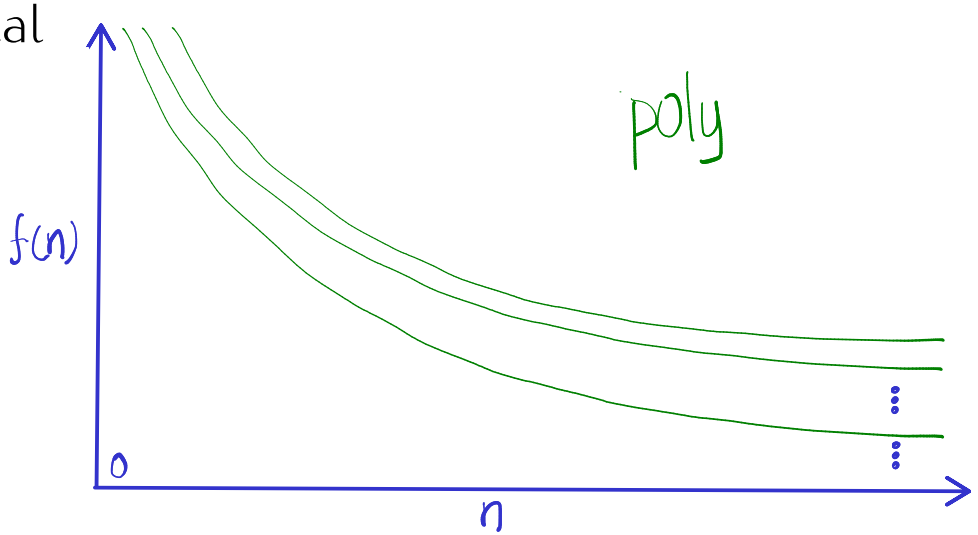

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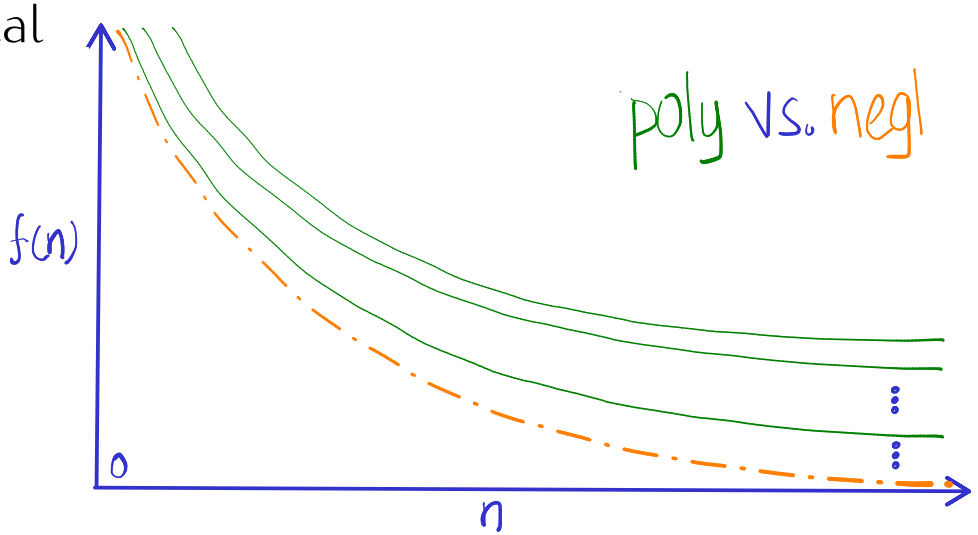

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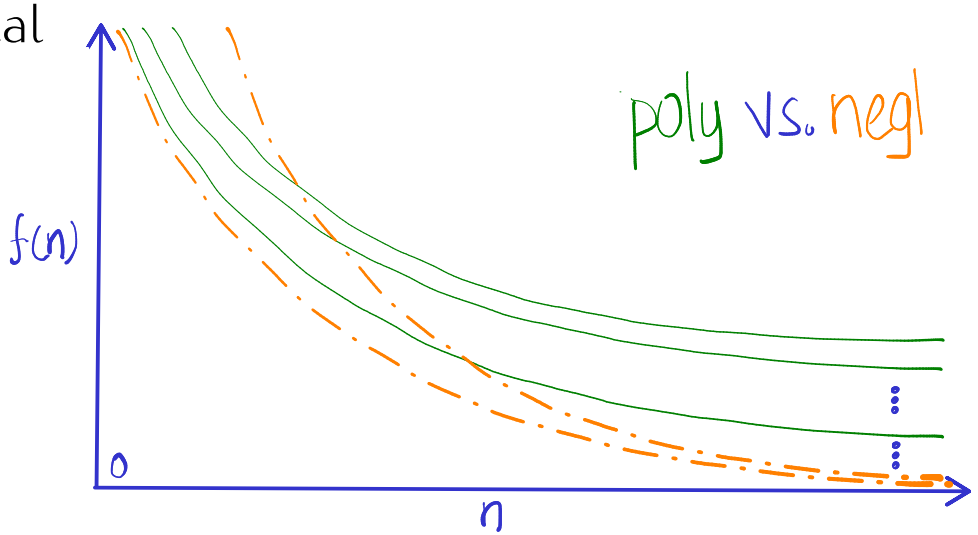

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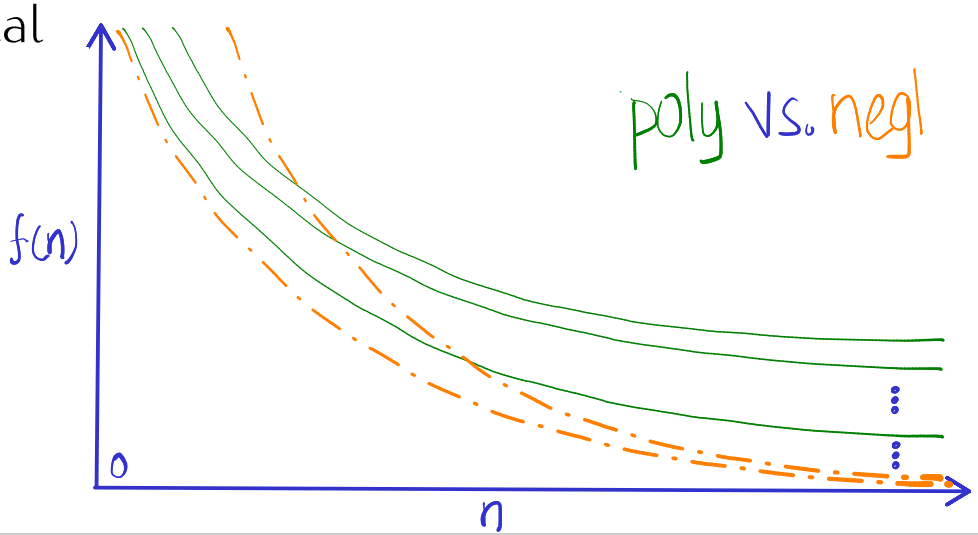

Defintion 2

A function $f : \mathbb{N} \to \mathbb{R}^+$ *is negligible if for every polynomial* p *and sufficiently large* n, $f(n) < 1/p(n)$ *holds.*

Second Compromise: Allow "Slack"...

- Okay if Eve distinguishes with "very small" probability
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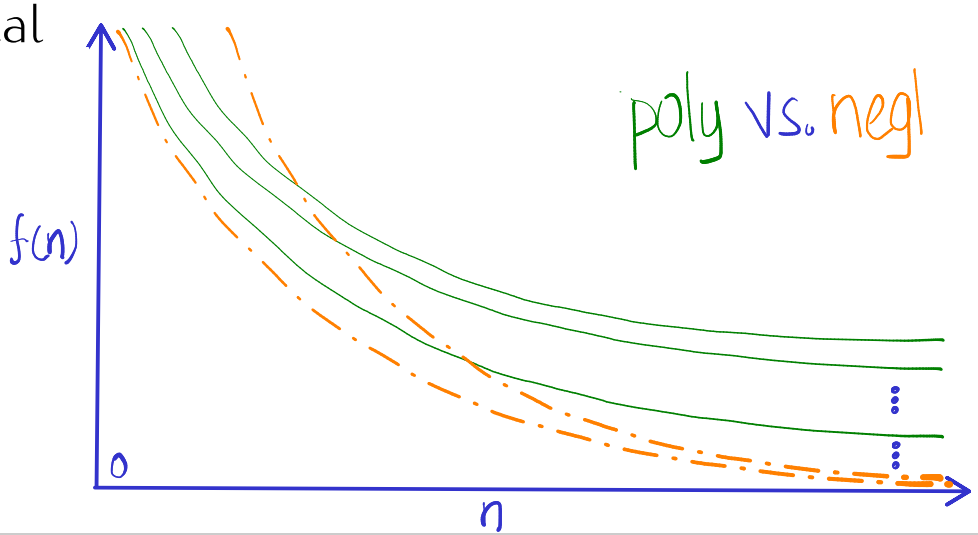

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- Why negligible? Like PPT, it behaves nicely.

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Negligible or not?

1. $f_1(n) := 1/314159n^{314159}$
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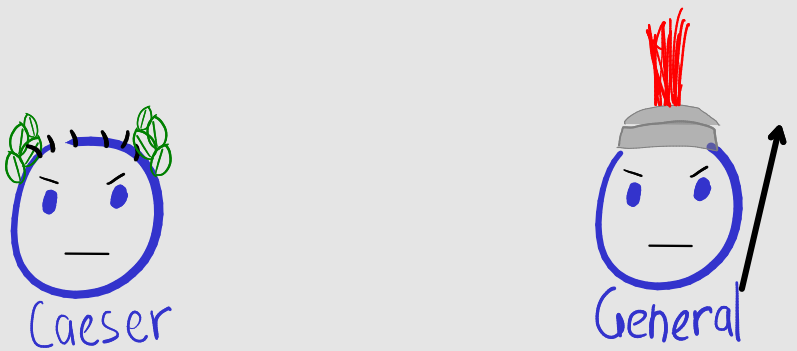
- To show that $f(n)$ is *non-negligible*, show that there exists a polynomial p such that $f(n) > 1/p(n)$ for *infinitely often* ns.

Incorporating Security Parameter into SKE Definition

Definition 2 (Shared/Symmetric-Key Encryption (SKE))

An SKE Π is a triple of efficient algorithms (**Gen**, **Enc**, **Dec**) with the following syntax:



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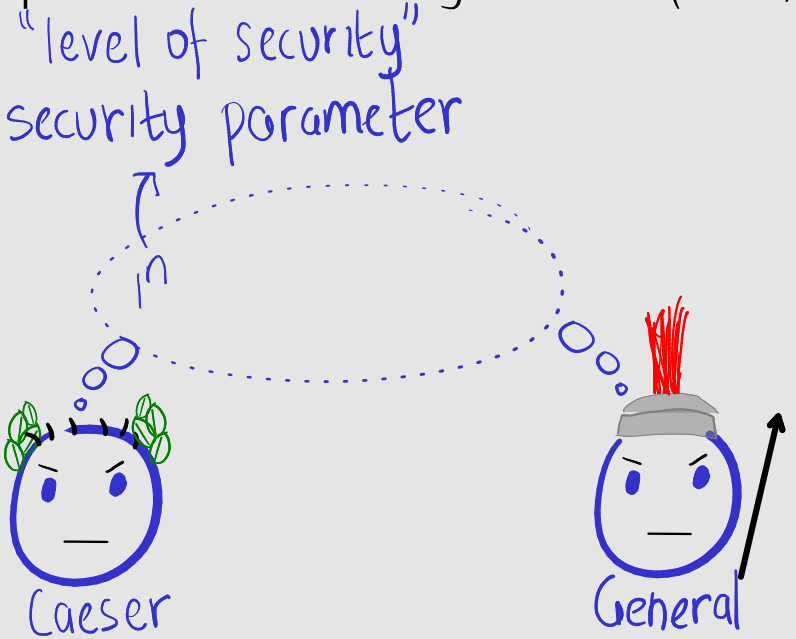

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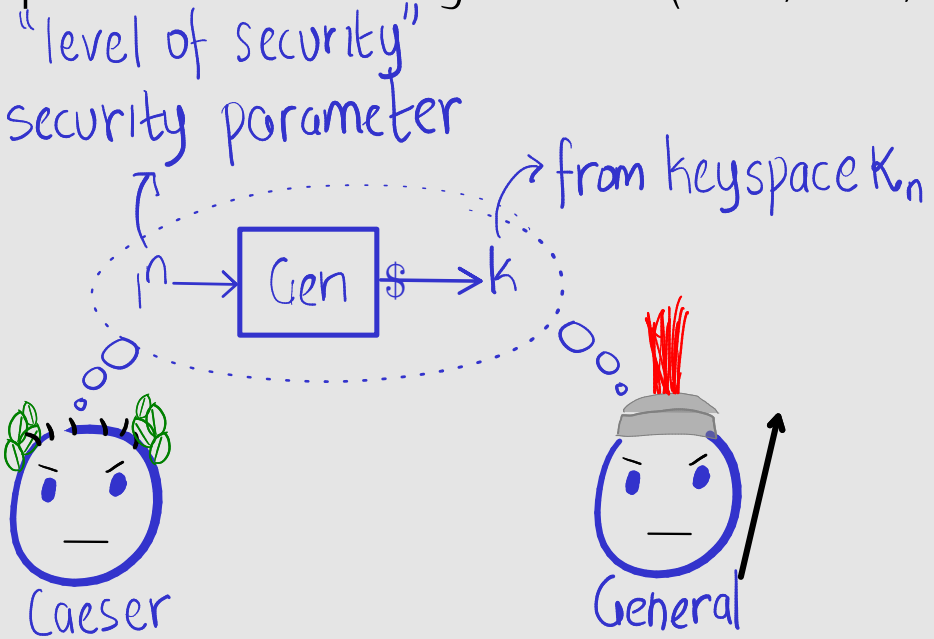

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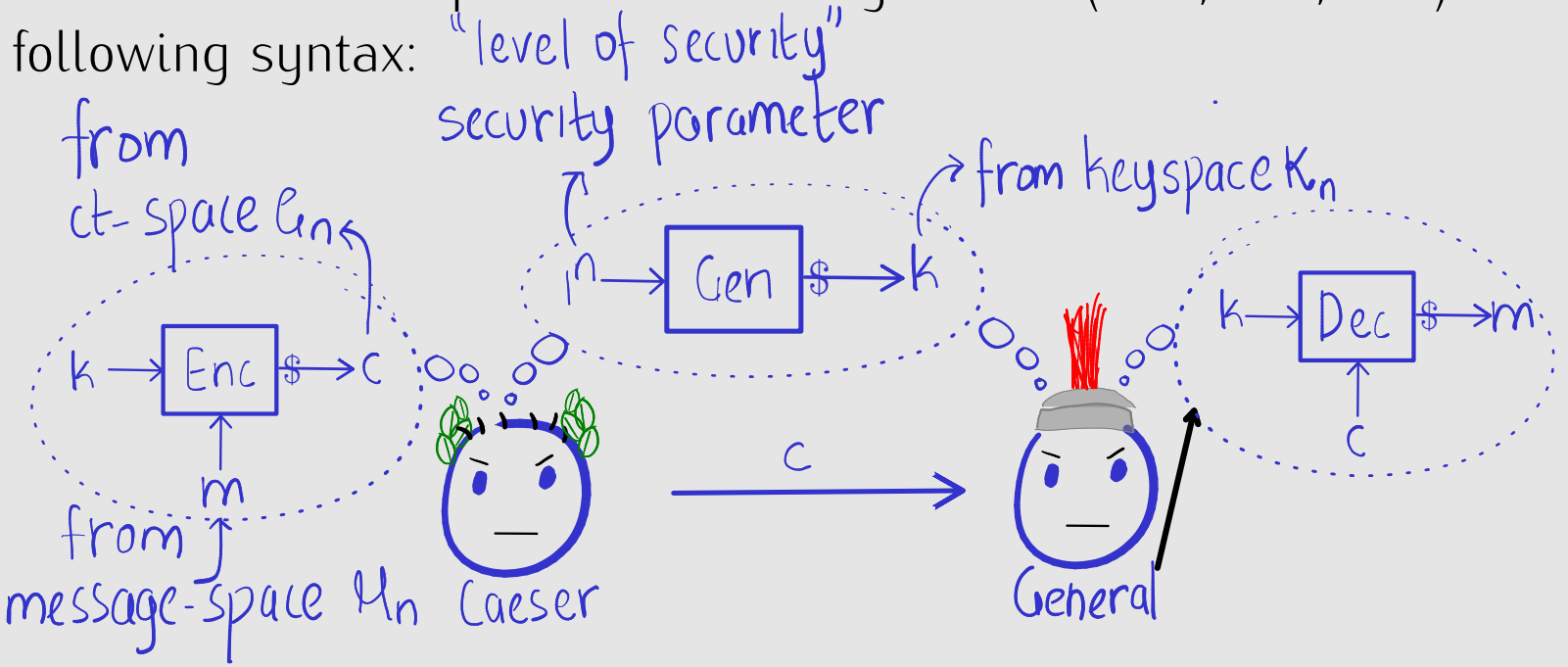

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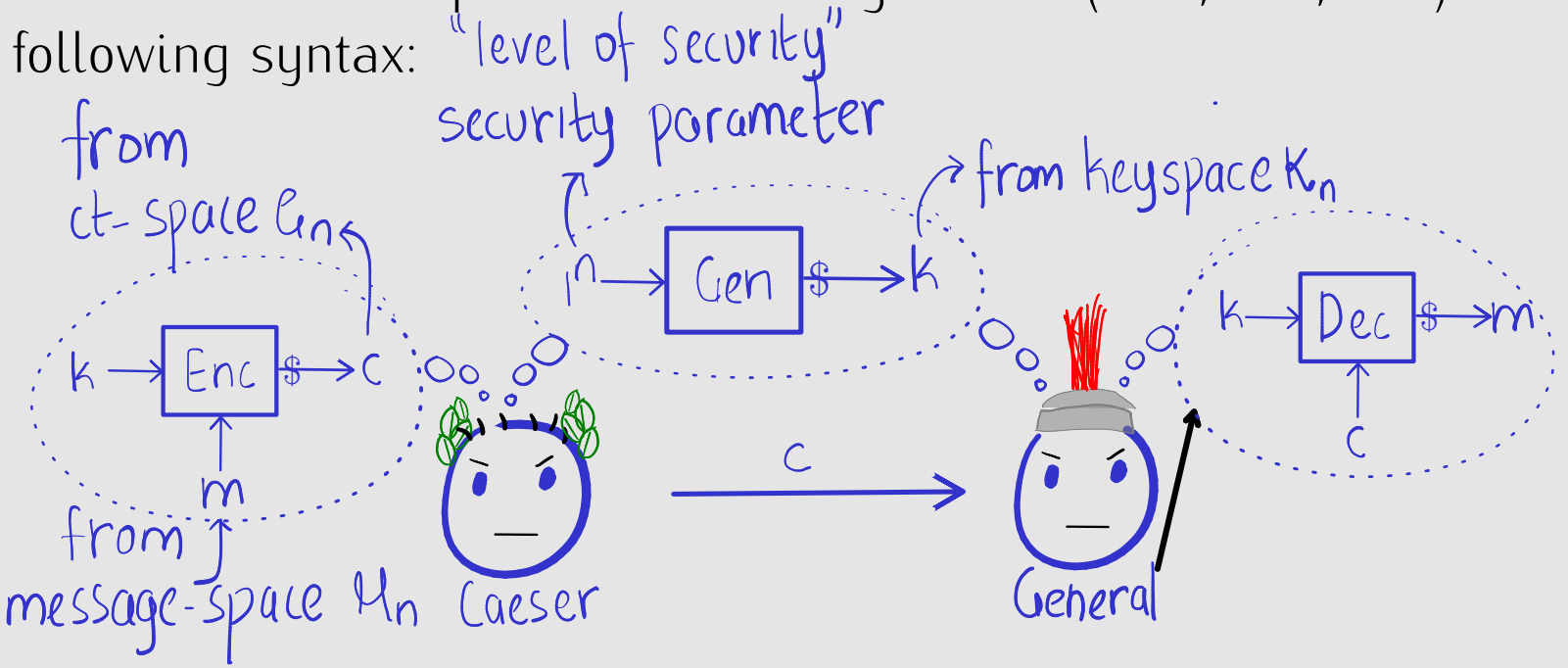
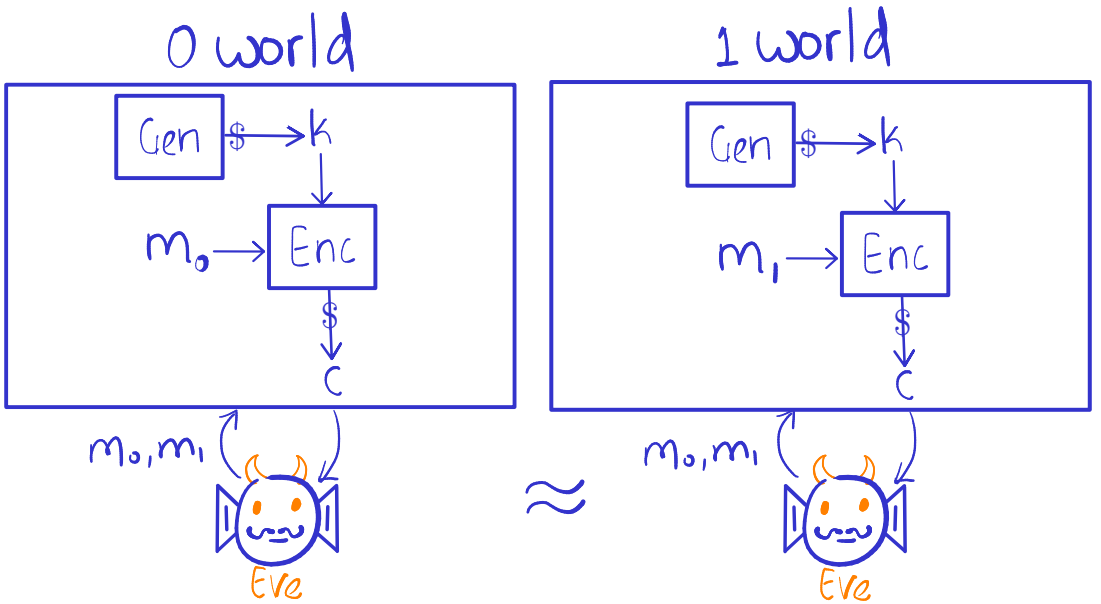


- Correctness of decryption: for every $n \in \mathbb{N}$, message $m \in \mathcal{M}_n$,

$$\Pr_{k \leftarrow \mathsf{Gen}(1^n), c \leftarrow \mathsf{Enc}(k,m)} [\mathsf{Dec}(k, c) = m] = 1$$

Let's Finally Define Computational Secrecy...



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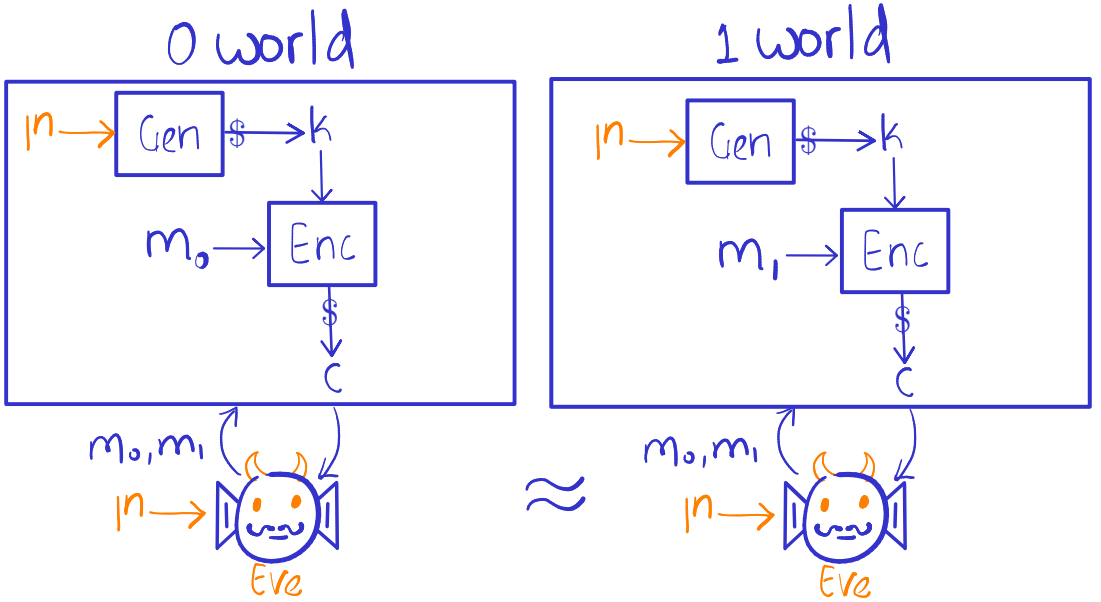

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Defintion 3 (SKE Imitation Game for PPT Eves)

An SKE $\Pi = (\mathsf{Gen}, \mathsf{Enc}, \mathsf{Dec})$ *is computationally-secret if for every PPT eavesdropper Eve*

$$\delta(n) := \left| \Pr_{\substack{(m_0,m_1)\leftarrow \mathsf{Eve}(1^n) \\ k\leftarrow \mathsf{Gen}(1^n) \\ c\leftarrow \mathsf{Enc}(k,m_0)}} [\mathsf{Eve}(c) = 0] - \Pr_{\substack{(m_0,m_1)\leftarrow \mathsf{Eve}(1^n) \\ k\leftarrow \mathsf{Gen}(1^n) \\ c\leftarrow \mathsf{Enc}(k,m_1)}} [\mathsf{Eve}(c) = 0] \right|$$

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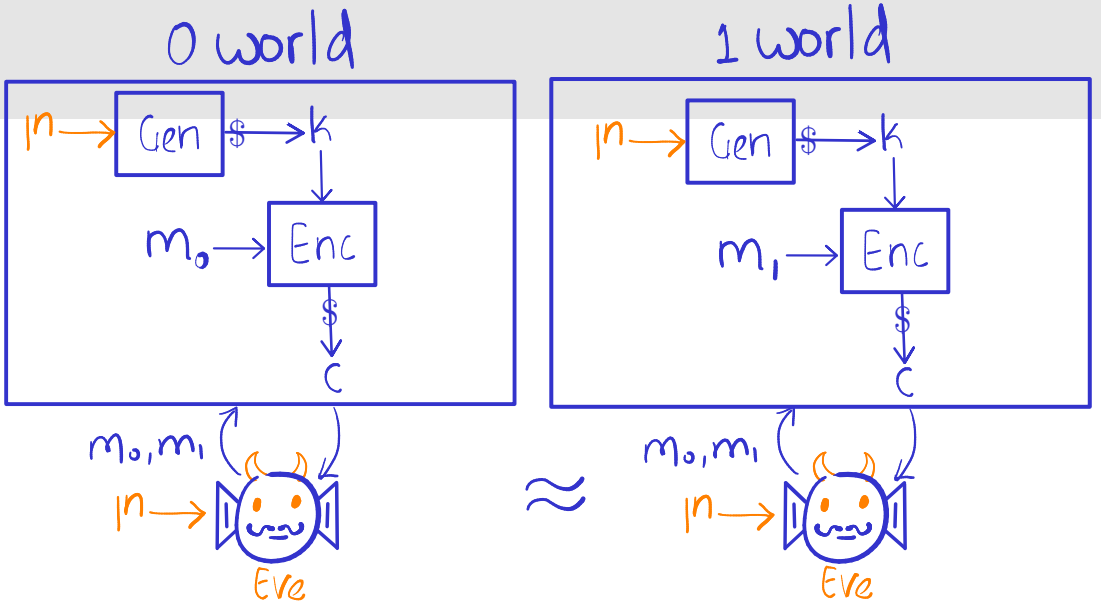

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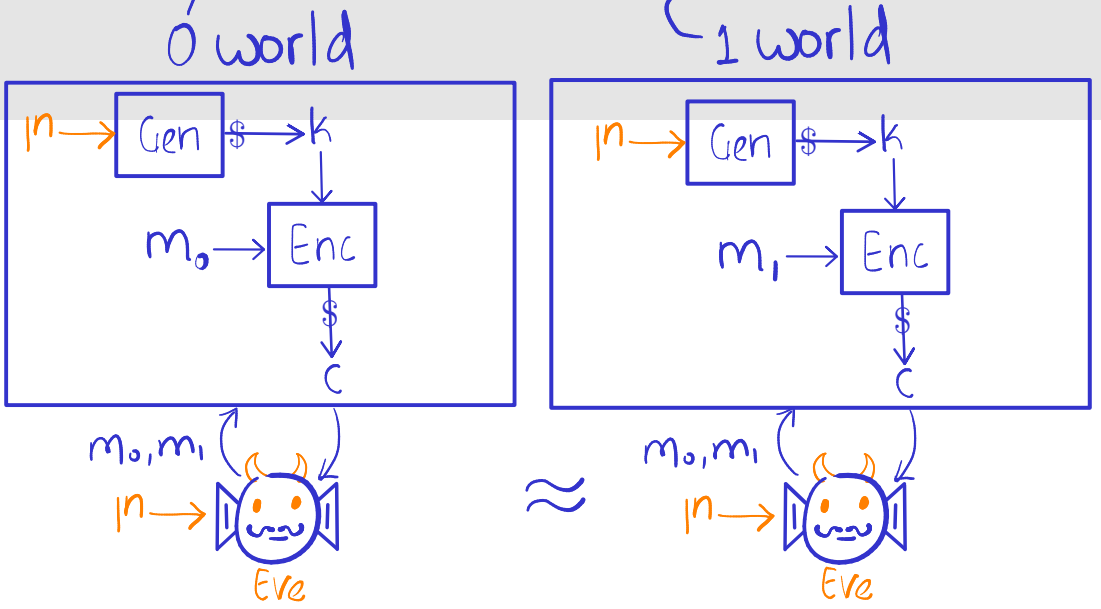

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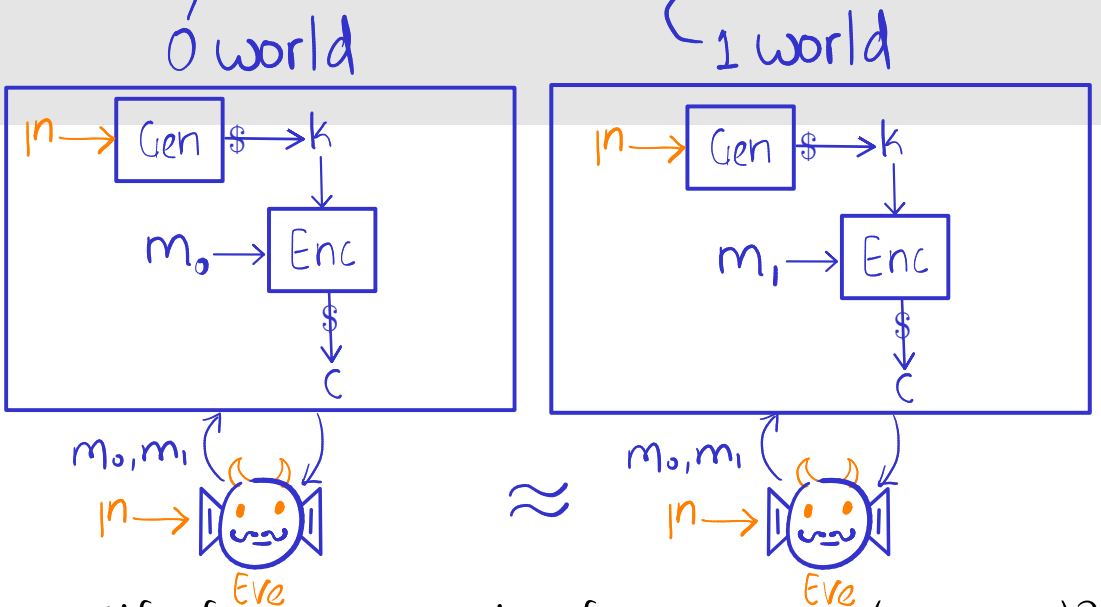

? What if we quantify for every pair of messages (m_0, m_1)?

Let's Finally Define Computational Secrecy...

A Second Way

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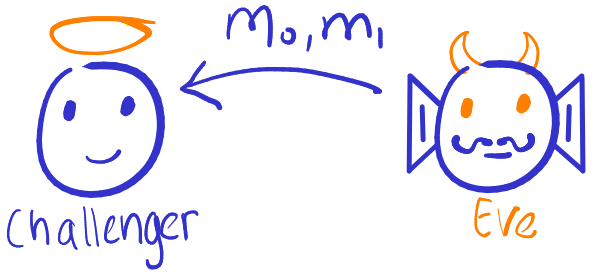

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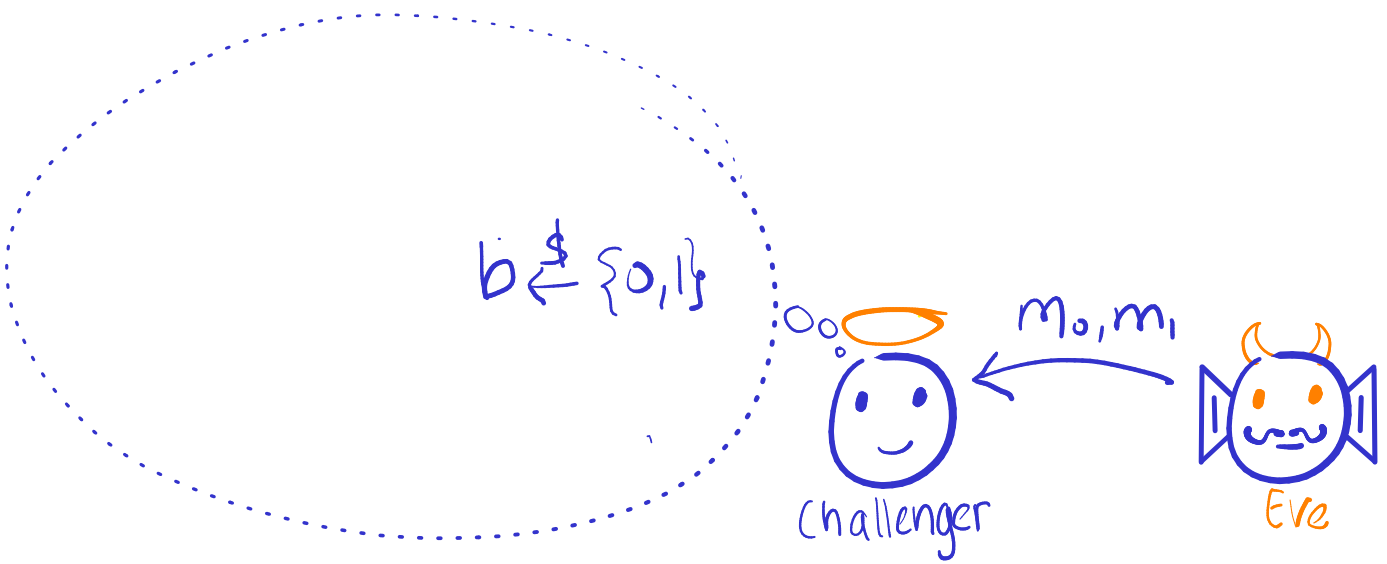

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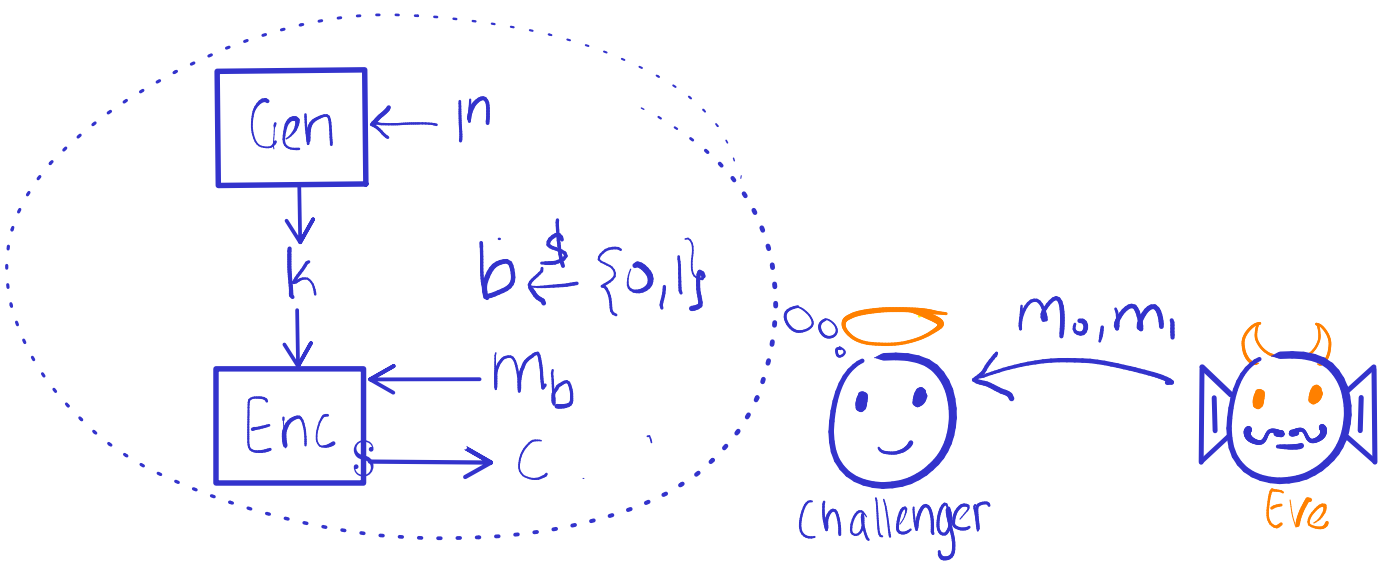

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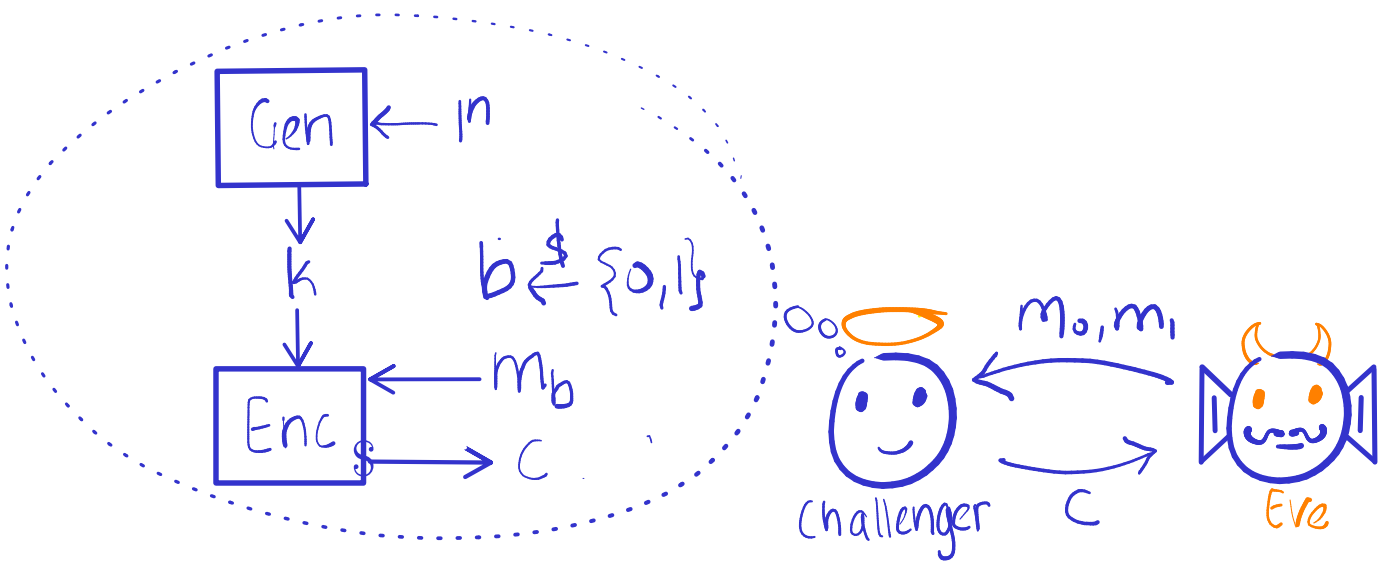

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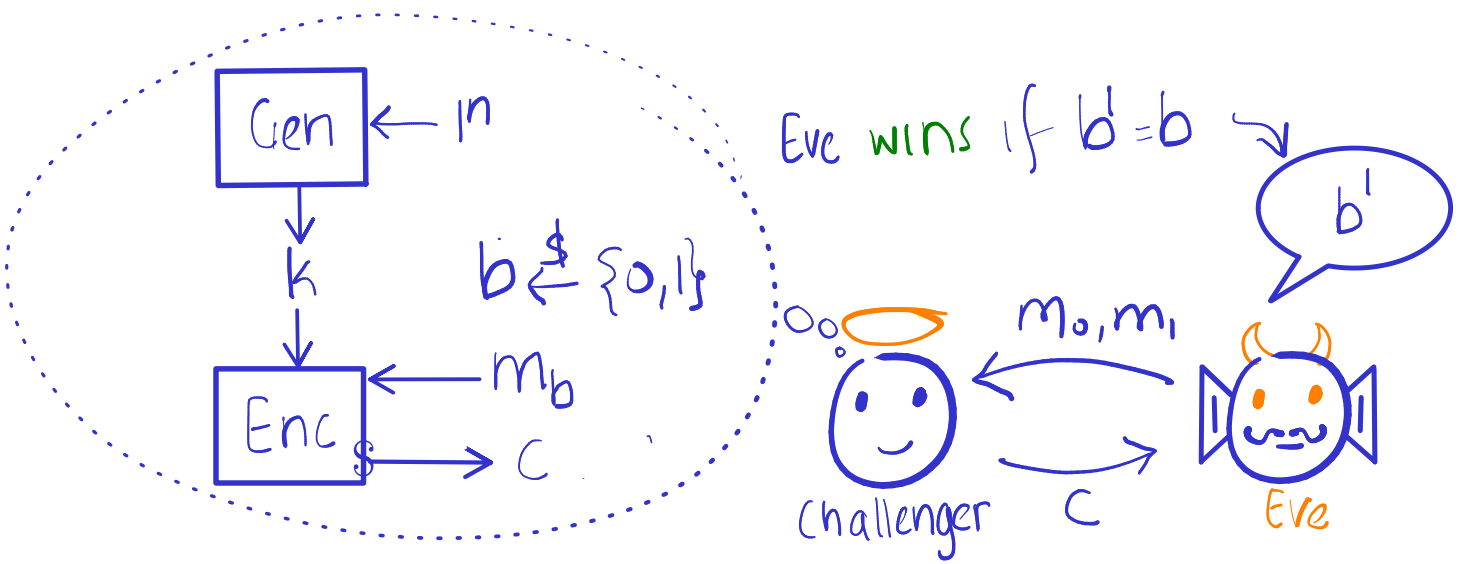


Let's Finally Define Computational Secrecy...

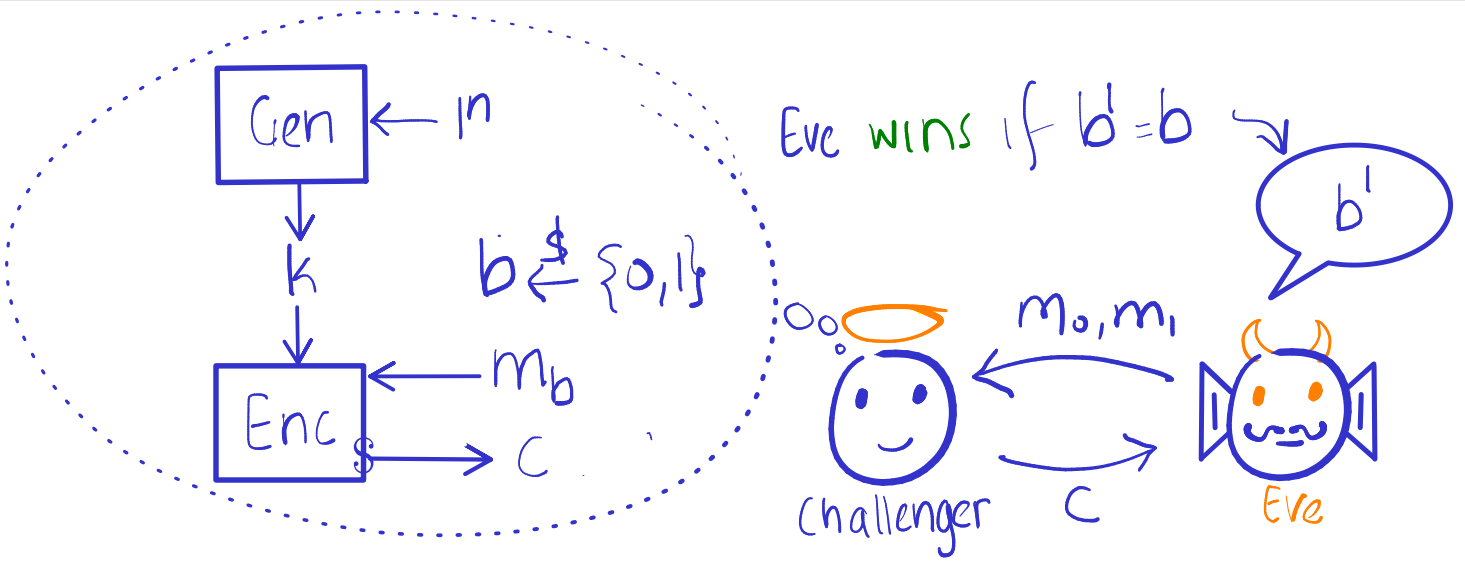
A Second Way

Defintion 4 (Adversarial Indistinguishability for PPT Eves)

An SKE $\Pi = (\mathsf{Gen}, \mathsf{Enc}, \mathsf{Dec})$ *is computationally-secret if for every PPT eavesdropper Eve*

$$\delta(n) := \Pr_{\substack{(m_0,m_1)\leftarrow \mathsf{Eve}(1^n) \\ k\leftarrow \mathsf{Gen}(1^n) \\ b\leftarrow\{0,1\} \\ c\leftarrow \mathsf{Enc}(k,m_b)}}[\mathsf{Eve}(c) = b] - \frac{1}{2}$$

is negligible.



The Two Definitions are Equivalent!

Claim 1 (Other direction exercise!)

Defintion 4 implies Defintion 3.

Proof.

Definition 4 implies for every PPT Eve the following is negligible

$$\Pr\left[\begin{array}{l} m_0, m_1 \leftarrow \mathrm{Eve}(1^n) \\ k \leftarrow \mathrm{Gen}(1^n),\ b \leftarrow \{0,1\} \\ c \leftarrow \mathrm{Enc}(k, m_b) \end{array} : \mathrm{Eve}(c) = b \right] - \frac{1}{2}$$

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Definition 4 implies for every PPT Eve the following is negligible

$$\Pr\left[\mathsf{Eve}(c)=0, b=0 \text{ or } \mathsf{Eve}(c)=1, b=1\right] - \frac{1}{2}$$

$m_0, m_1 \leftarrow \mathsf{Eve}(1^n)$
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$\Downarrow$

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$$\left| \Pr\left[\mathrm{Eve}(c)=0\right] - \Pr\left[\mathrm{Eve}(c)=0\right] \right| \text{ is negl.}$$

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? Where did I cheat?

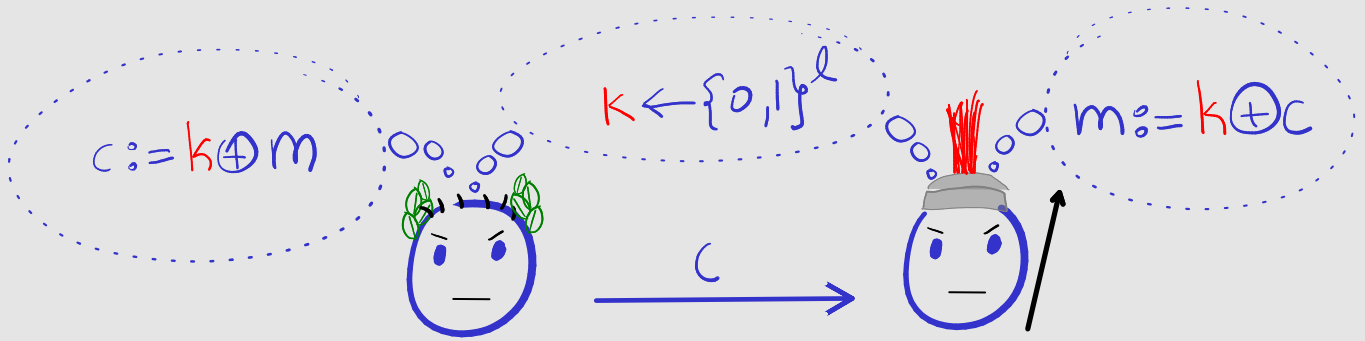
Plan for This Lecture

1. Limitations of Perfect Secrecy: Shannon's Impossibility
2. Bypassing Shannon's Impossibility
3. Pseudo-Random Generators (PRGs) and Computational OTP

↖ Our first cryptographic assumption!

Recall One-Time Pad (Vernam's Cipher)

Construction 1 (Message space $\{0,1\}^\ell$)



Pseudocode 1 (Message space $\{0,1\}^\ell$)

- *Key generation* $\mathsf{Gen}(1^\ell)$*: output* $k \leftarrow \{0,1\}^\ell$
- *Encryption* $\mathsf{Enc}(k, m)$*: output* $c := k \oplus m$
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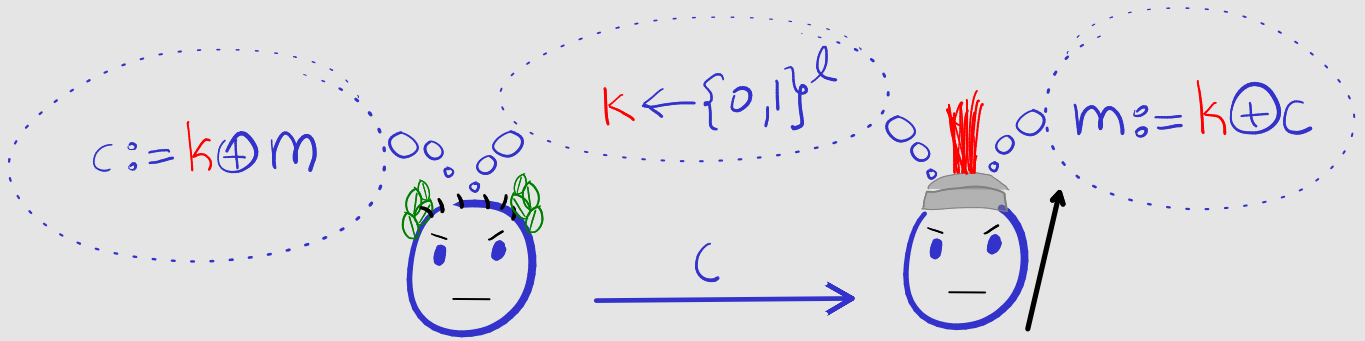

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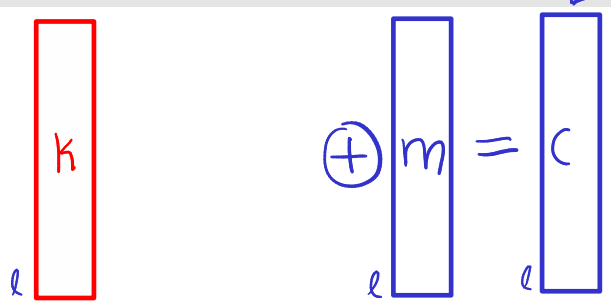

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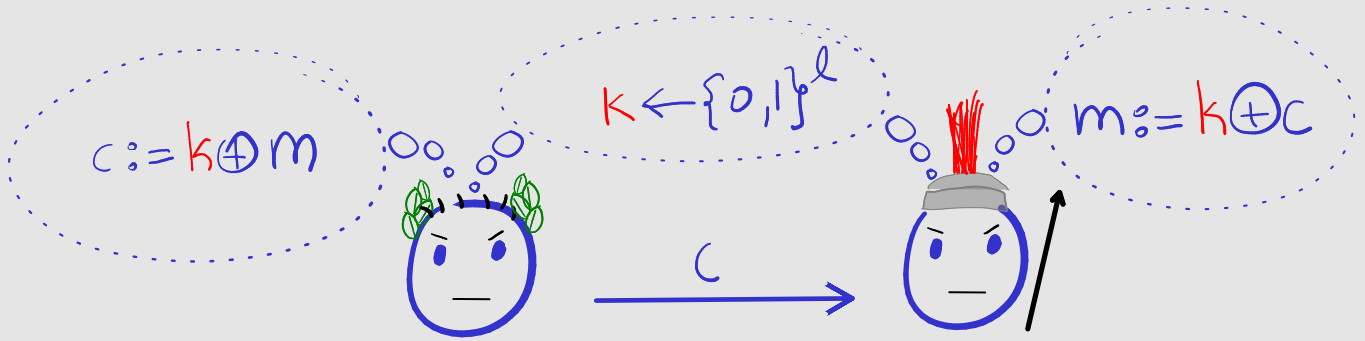

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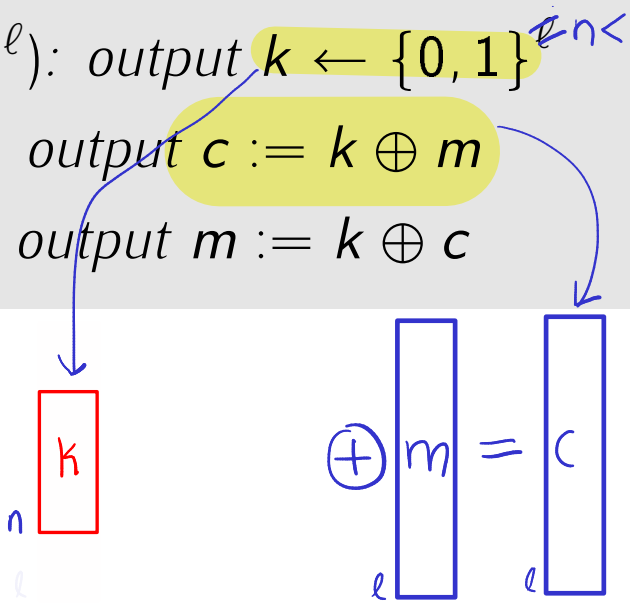
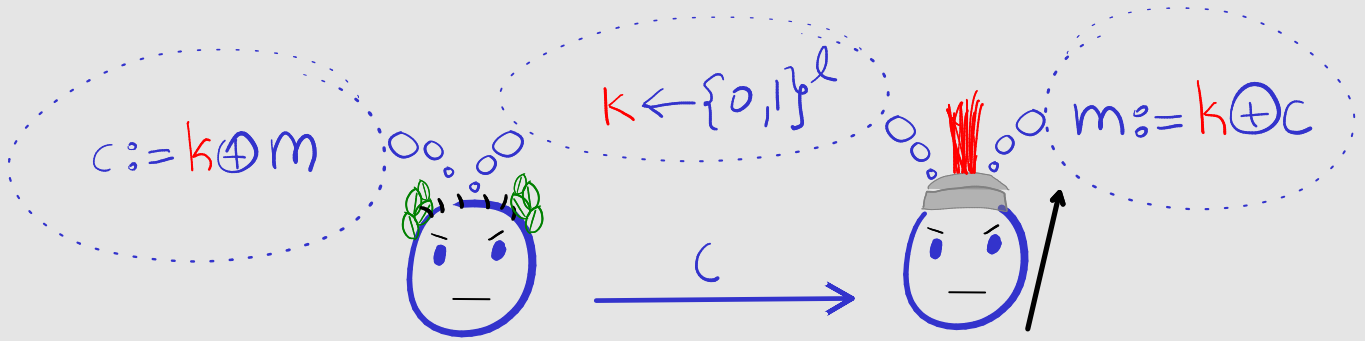


Recall One-Time Pad (Vernam's Cipher)

Construction 1 (Message space $\{0,1\}^\ell$)



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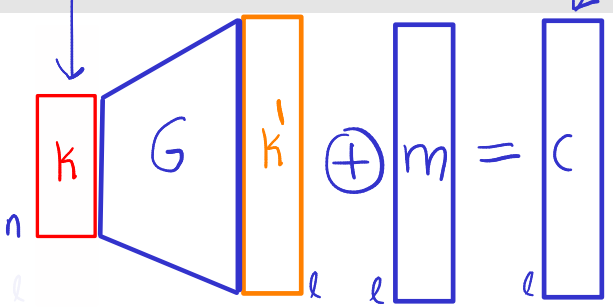

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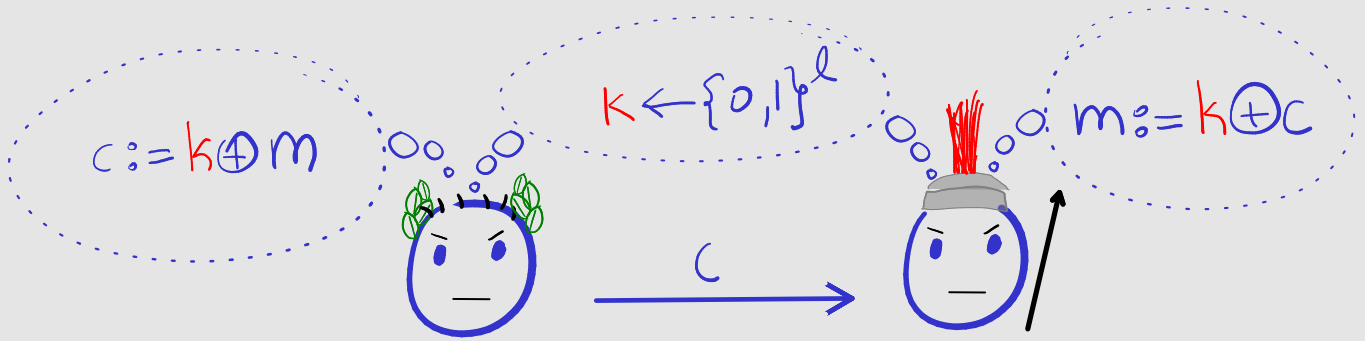

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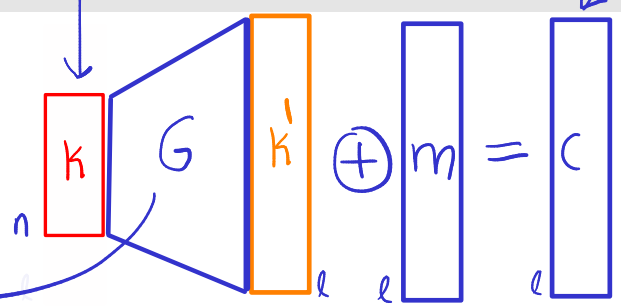
Requirement:
G expands $k \to k'$ such that k' "seems random" to



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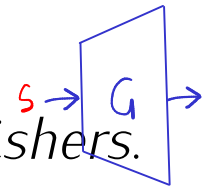

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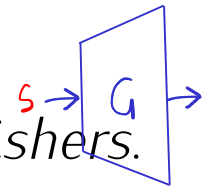

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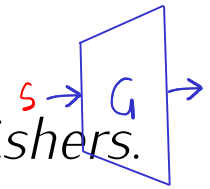

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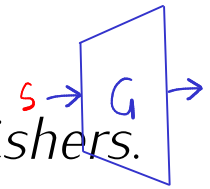

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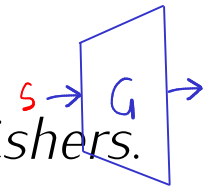

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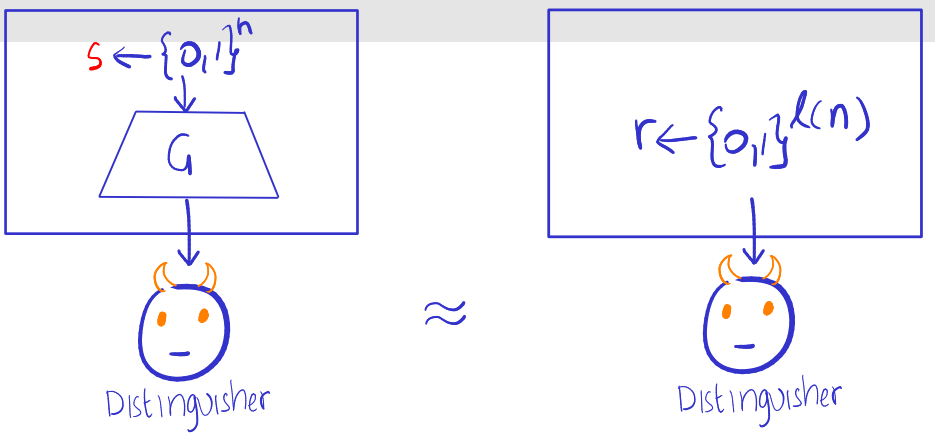

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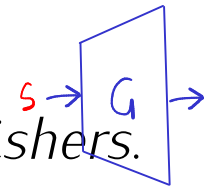

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Exercise 2

1. *Write up "adversarial indistinguishability" definition of PRG.*
2. *Show that the two definitions are equivalent.*

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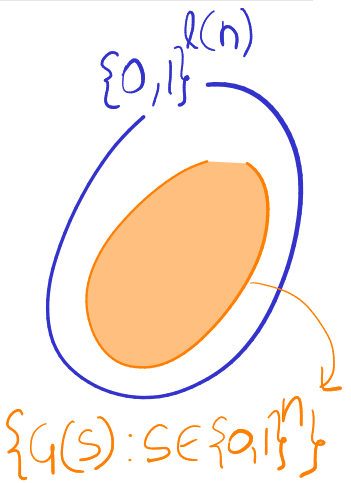

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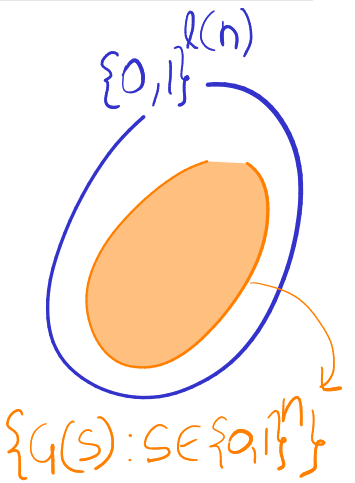

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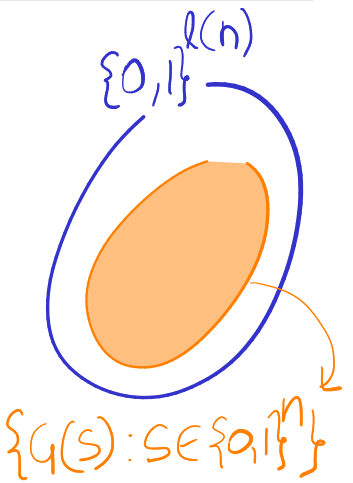
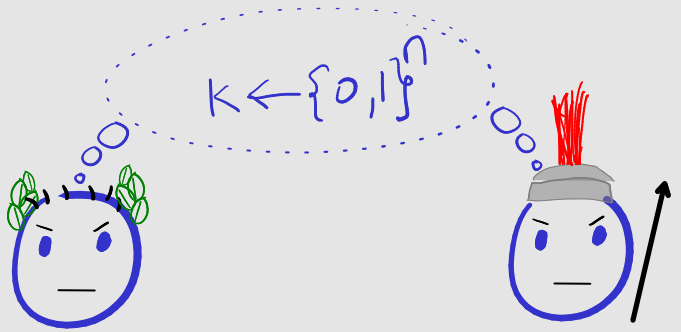


"Computational" One-Time Pad from PRG G

Construction 2 (Message space $\{0,1\}^{\ell(n)}$)



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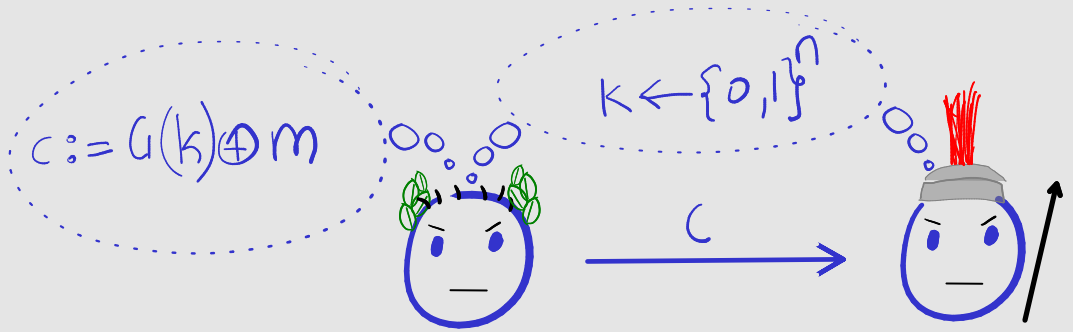

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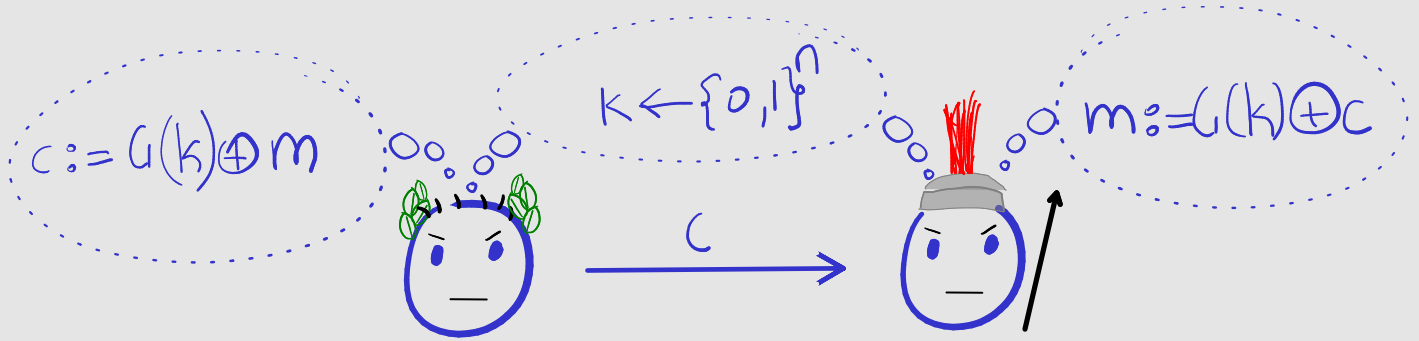

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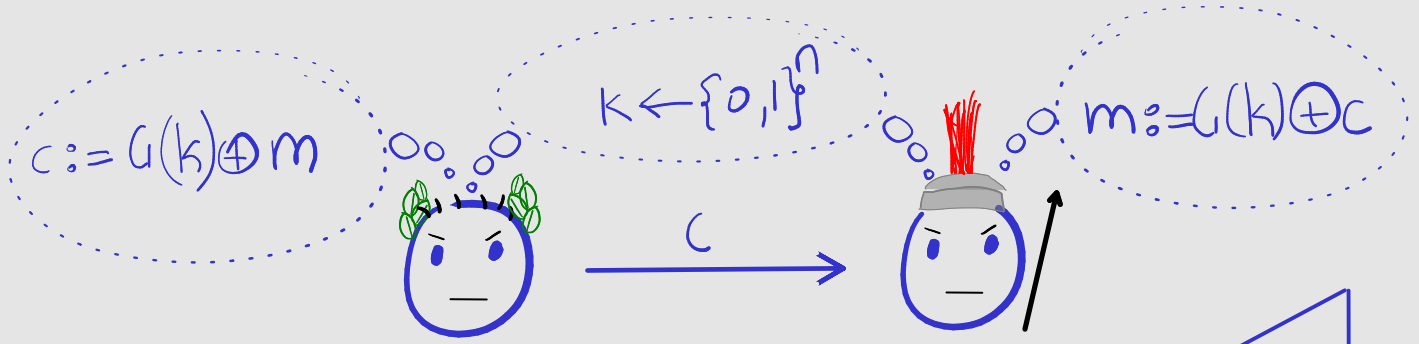

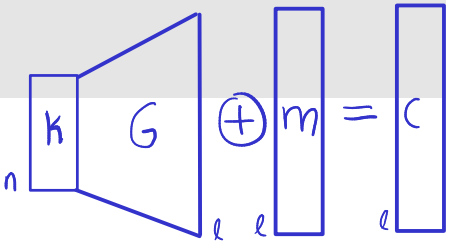

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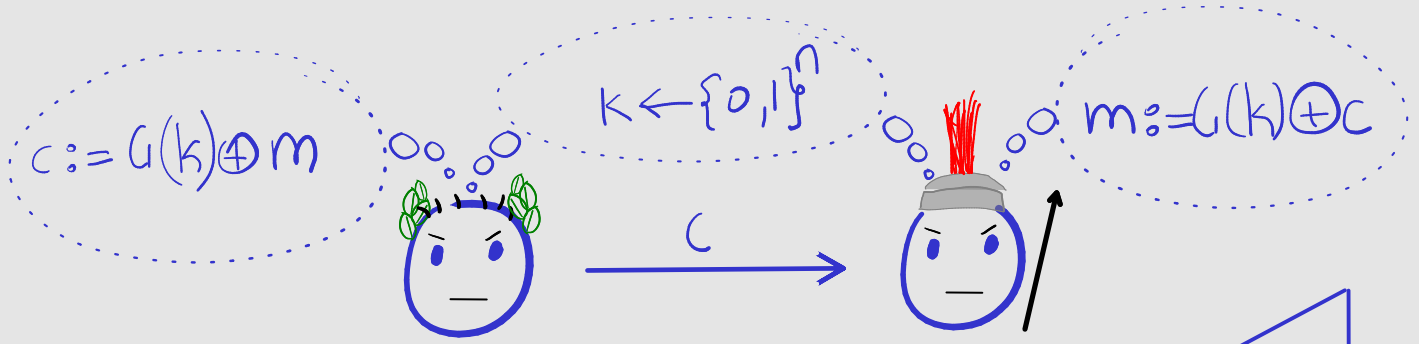

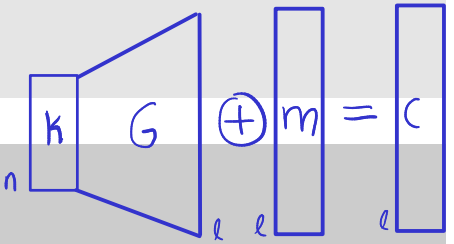

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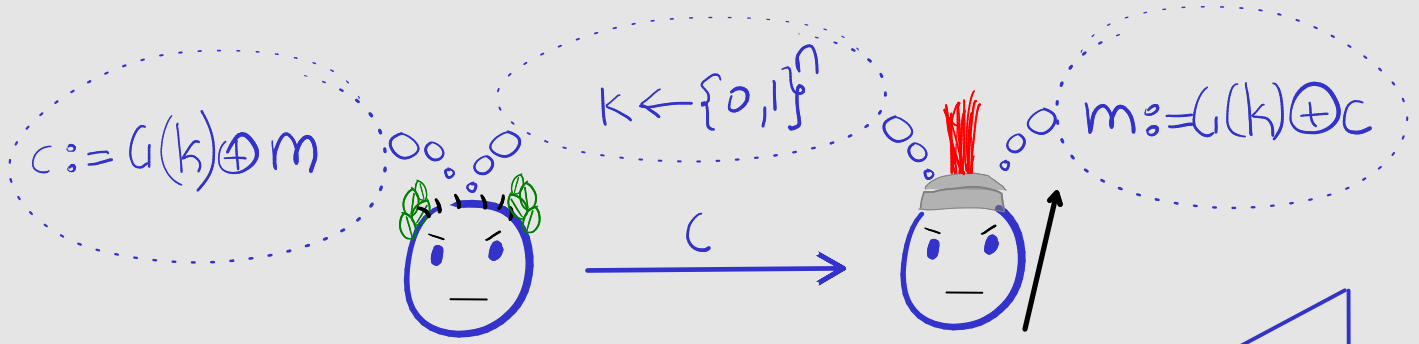

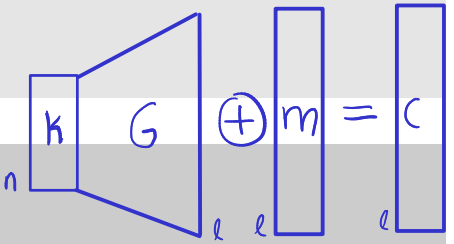

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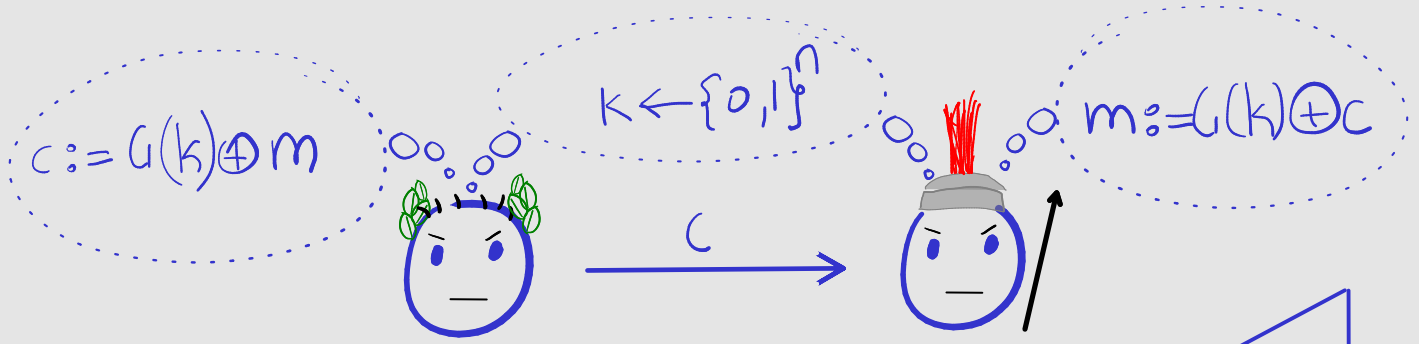

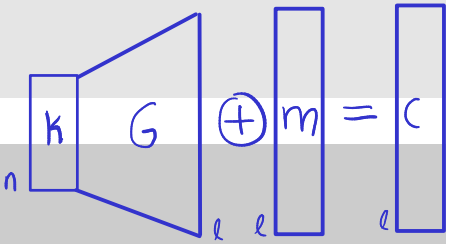

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Theorem 3

Assuming G *is a PRG, Construction 2 is computationally secret.*

Proof *by reduction.* $\exists$Eve breaking Construction 2 $\Rightarrow$ $\exists$D for G.

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Intuition: consider the following four worlds

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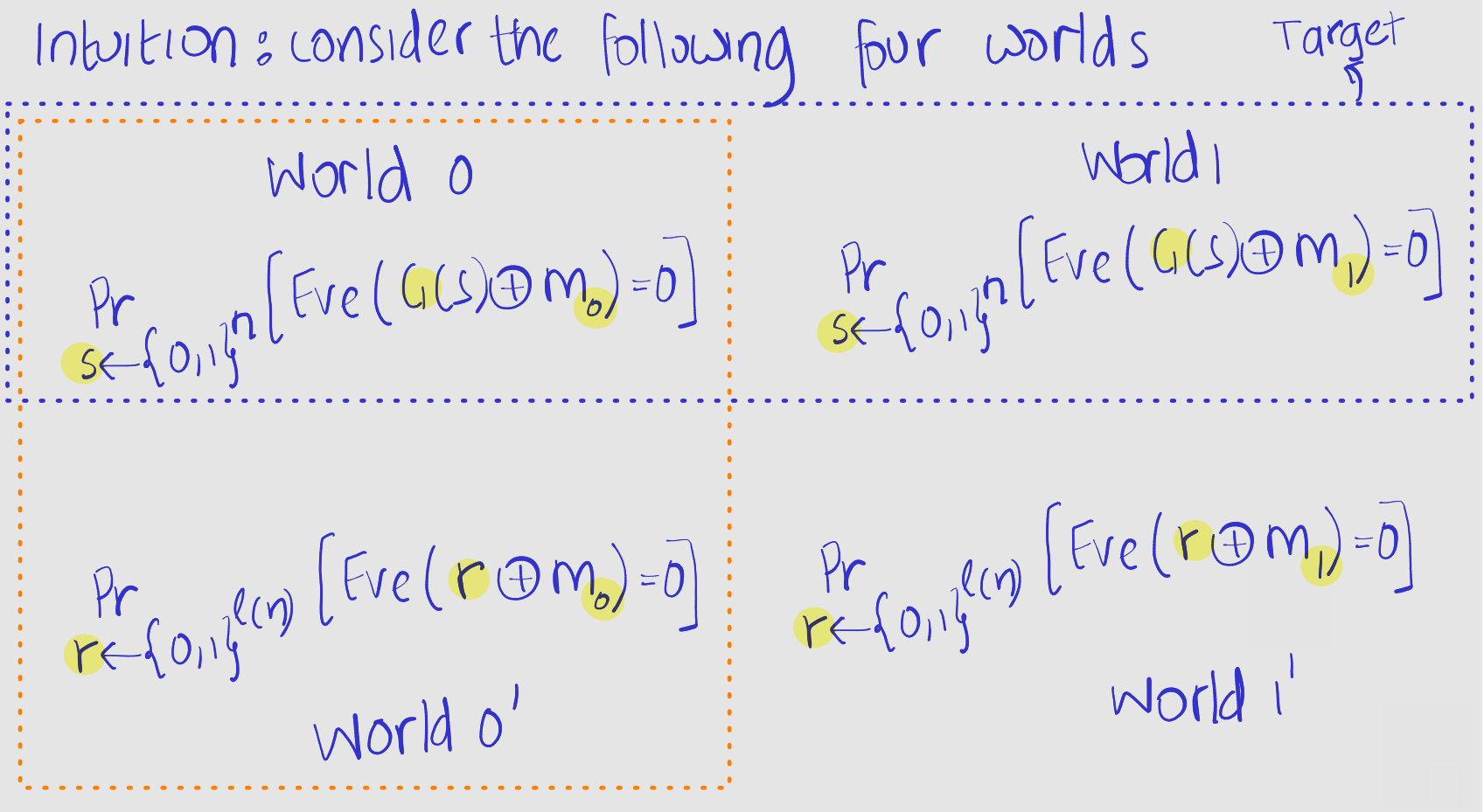

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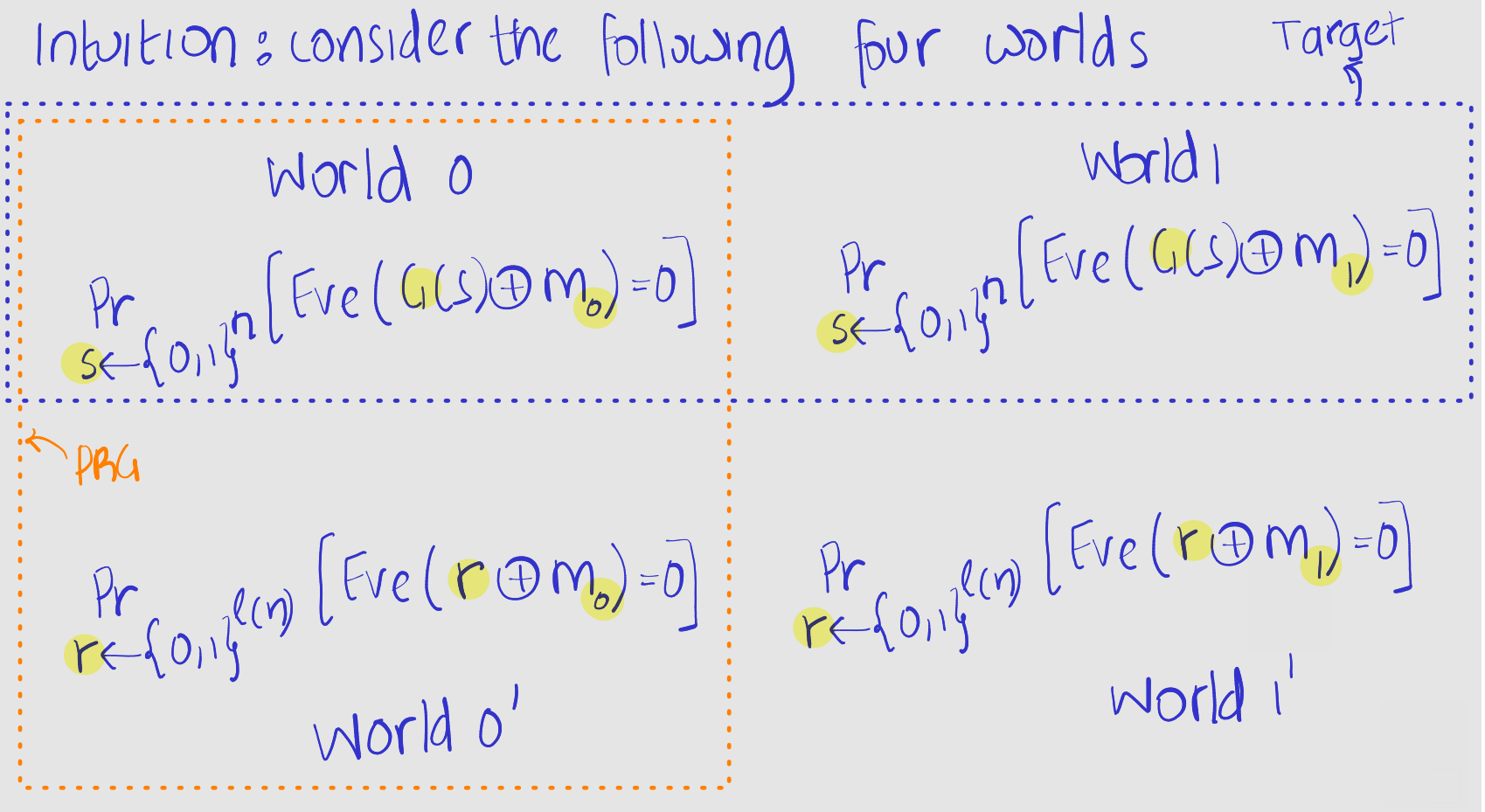

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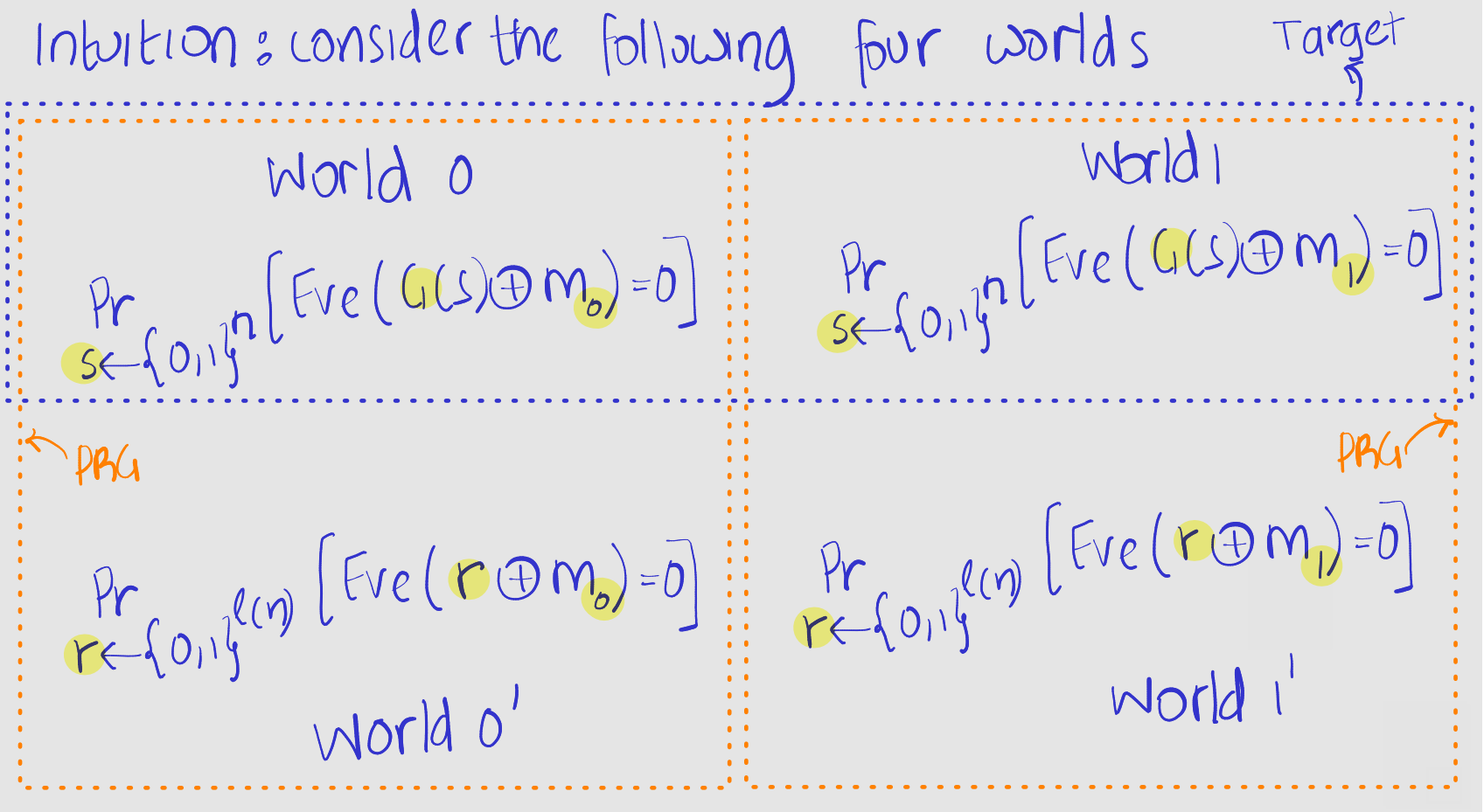

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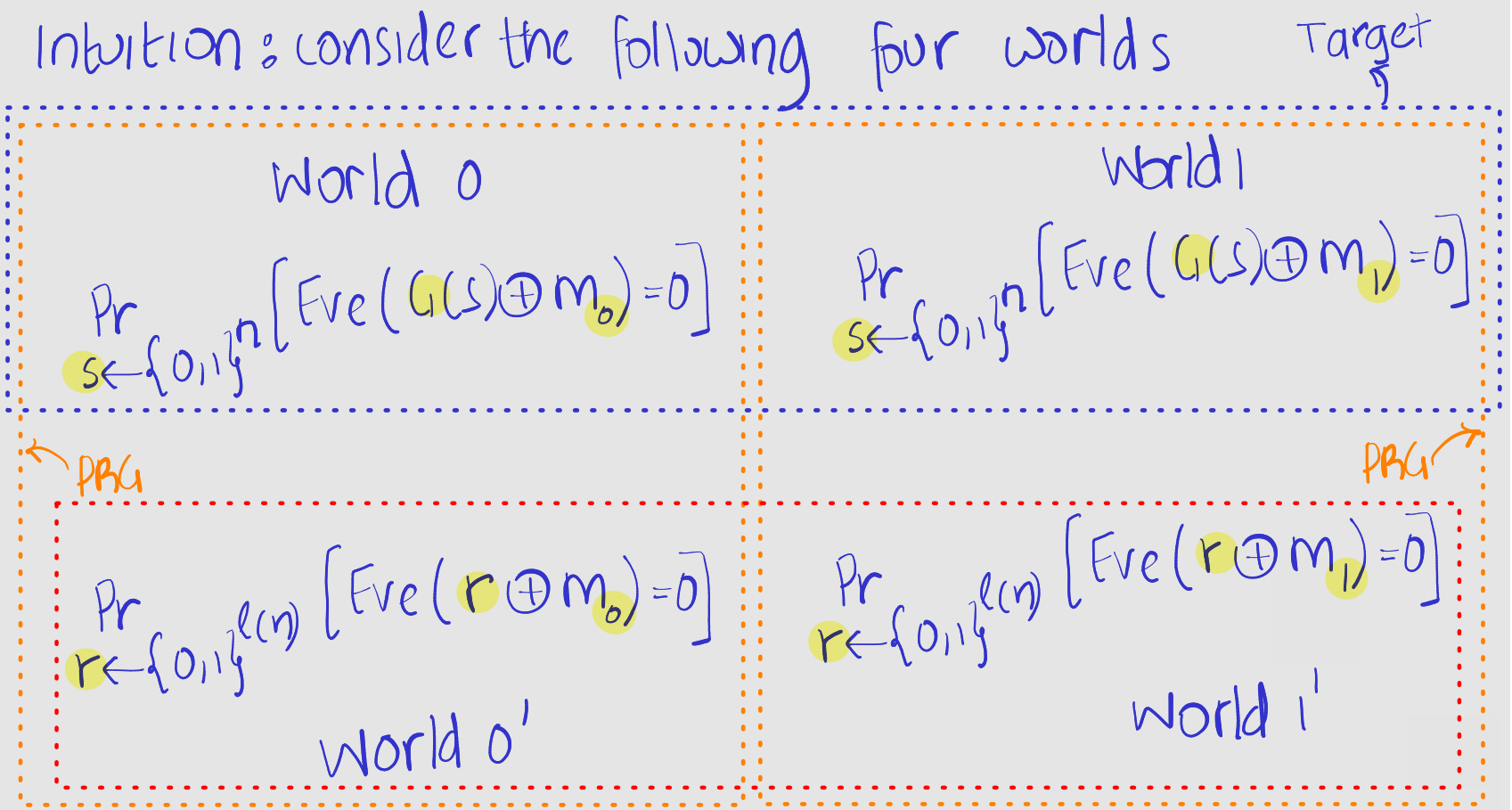

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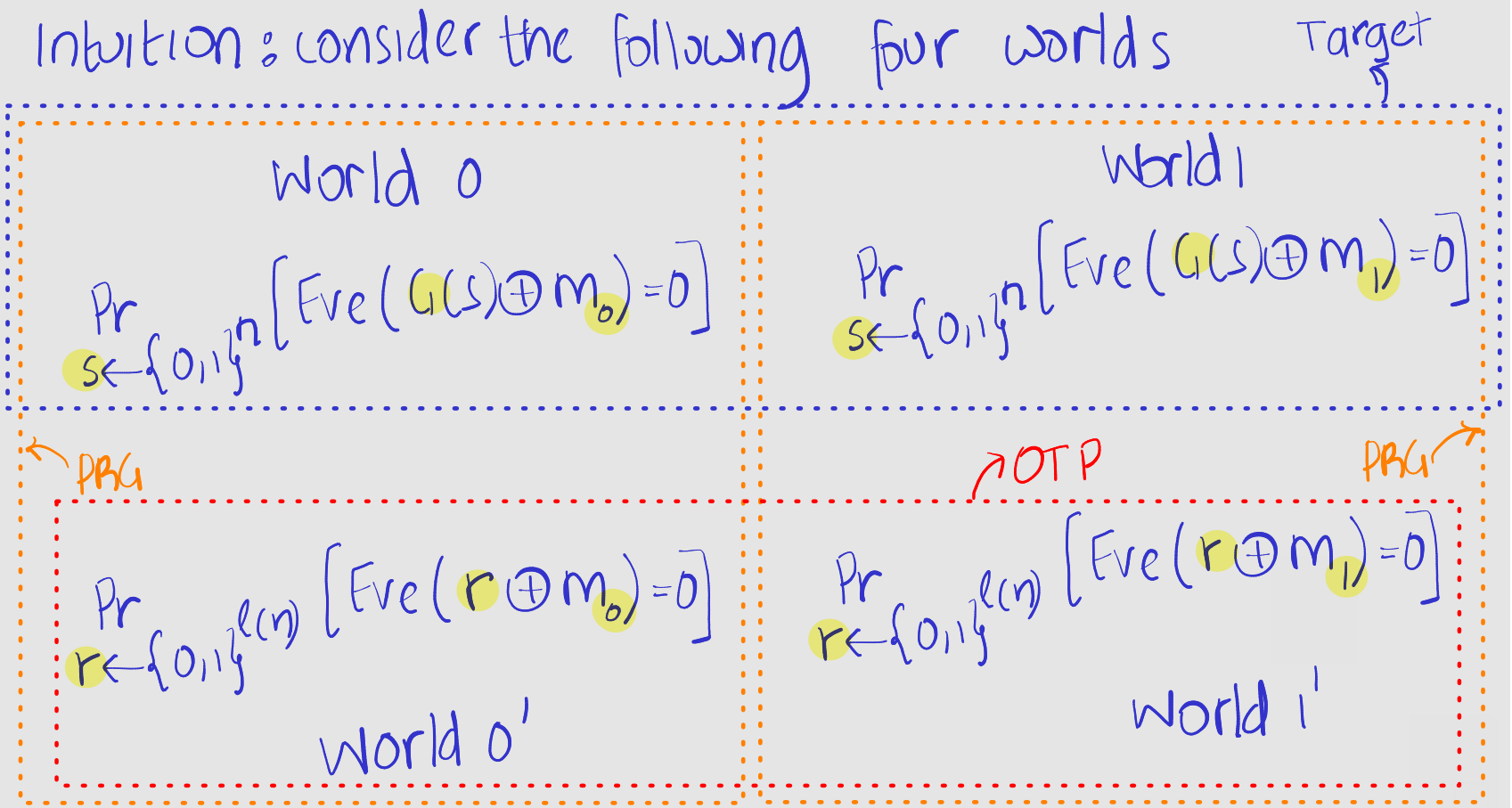

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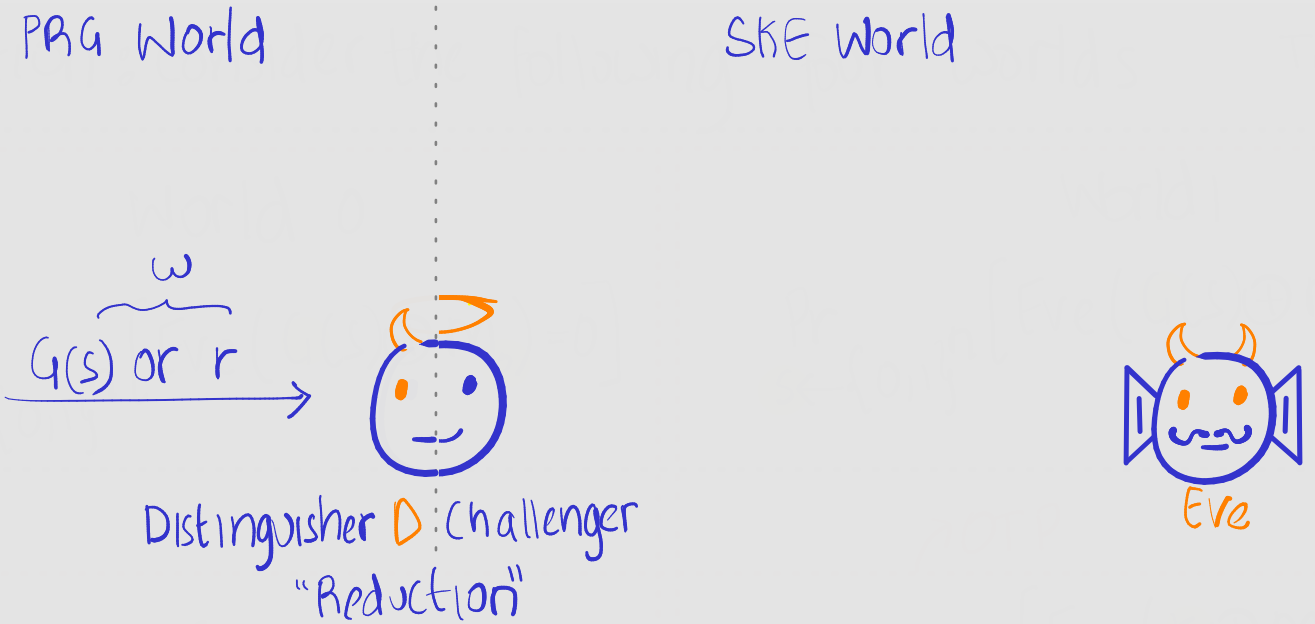

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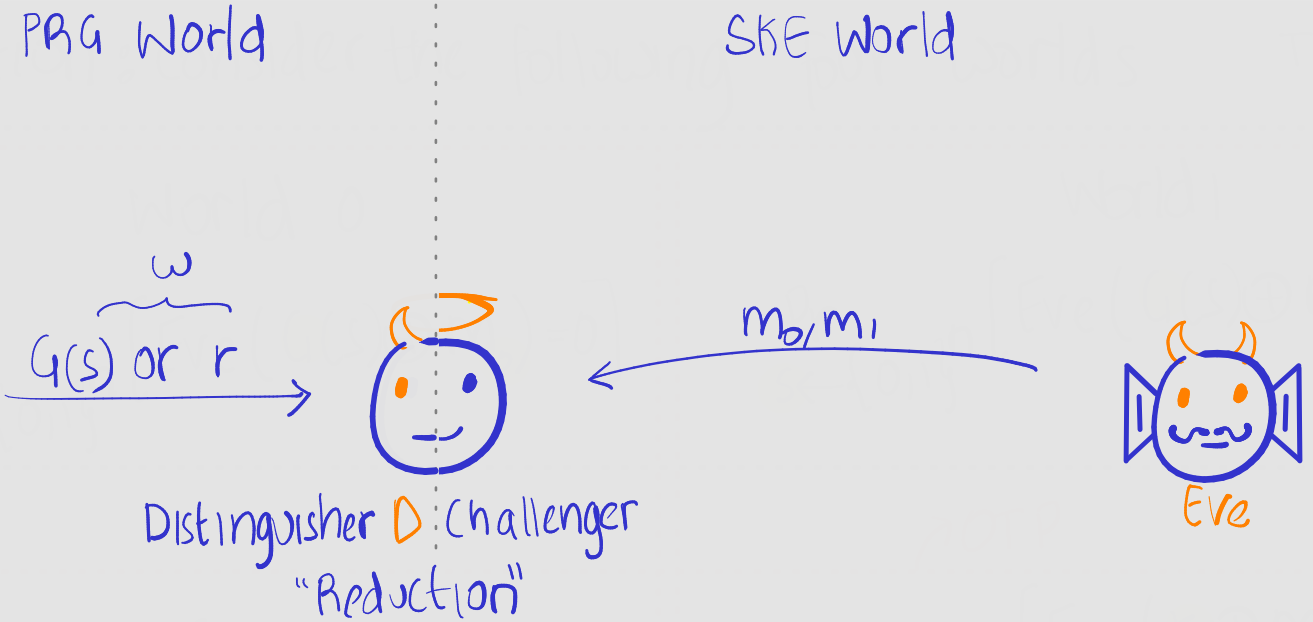

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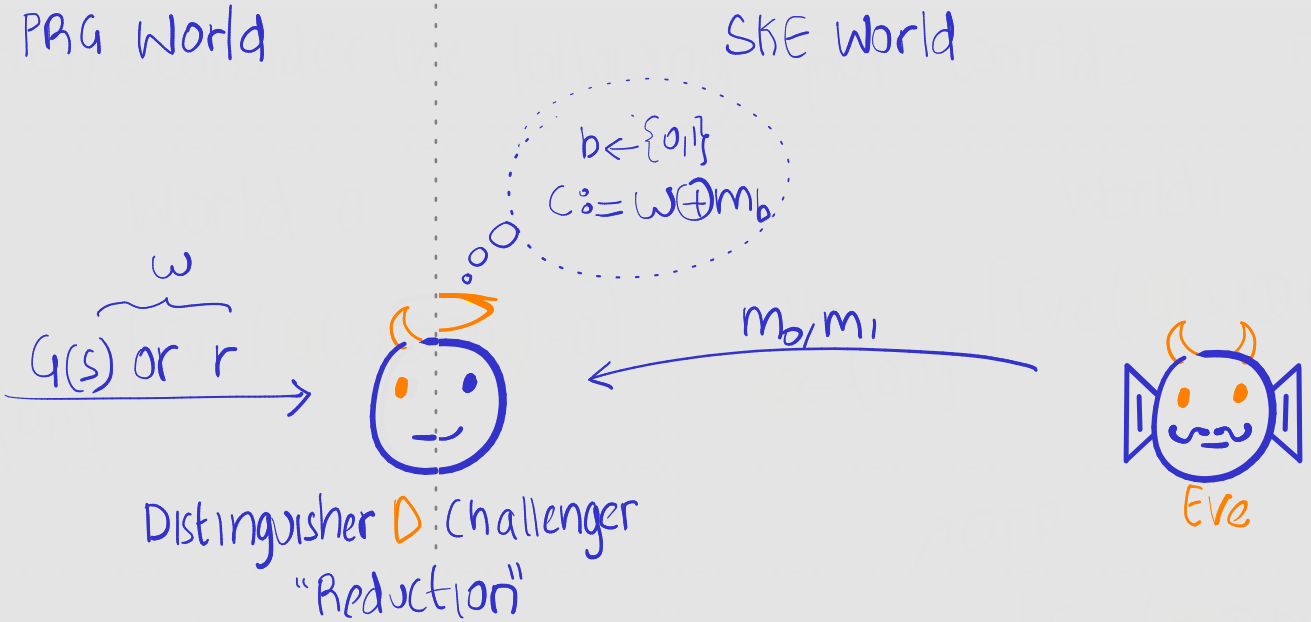

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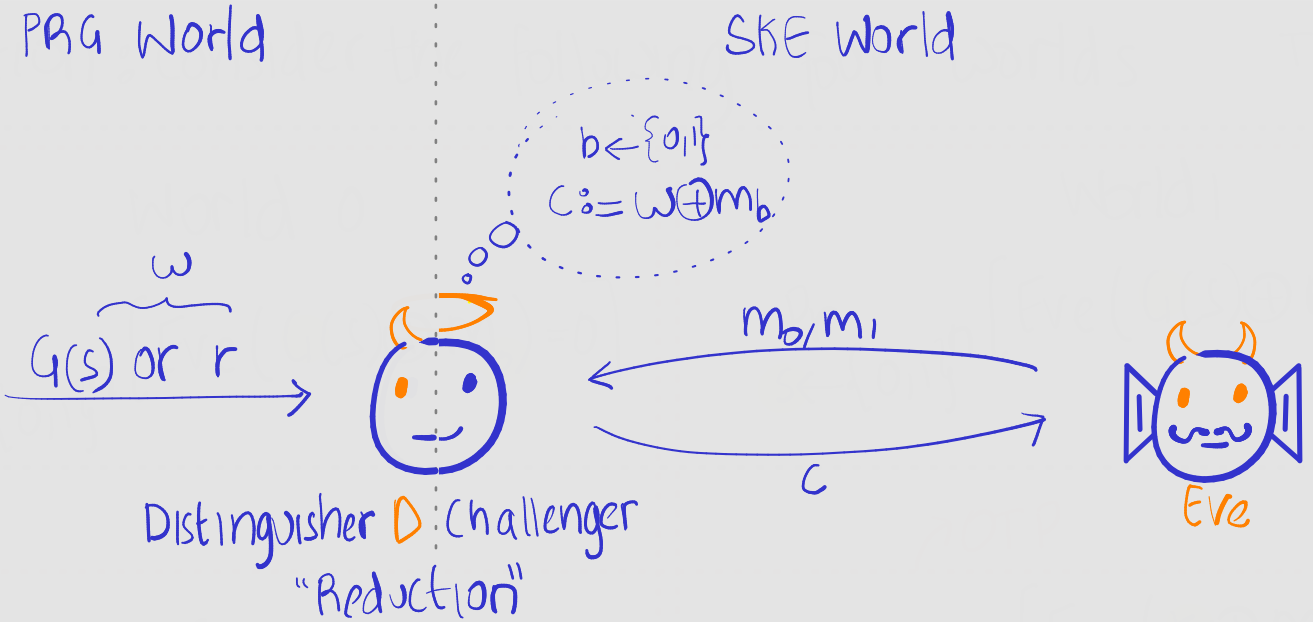

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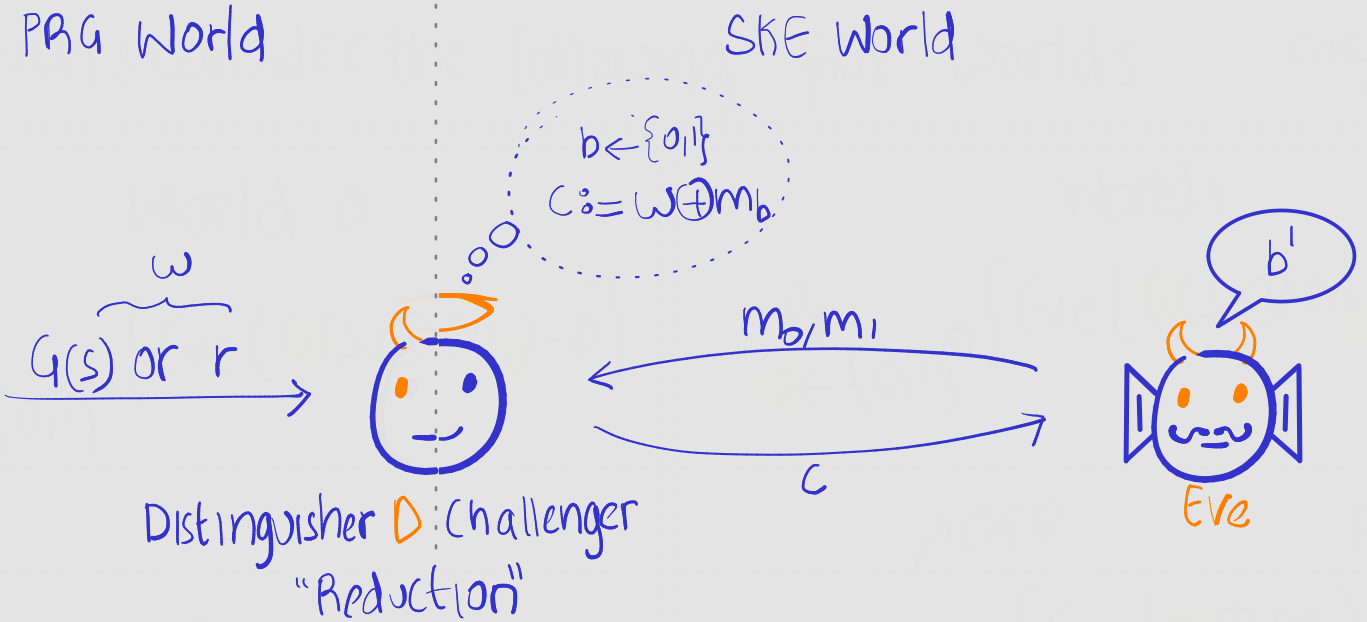

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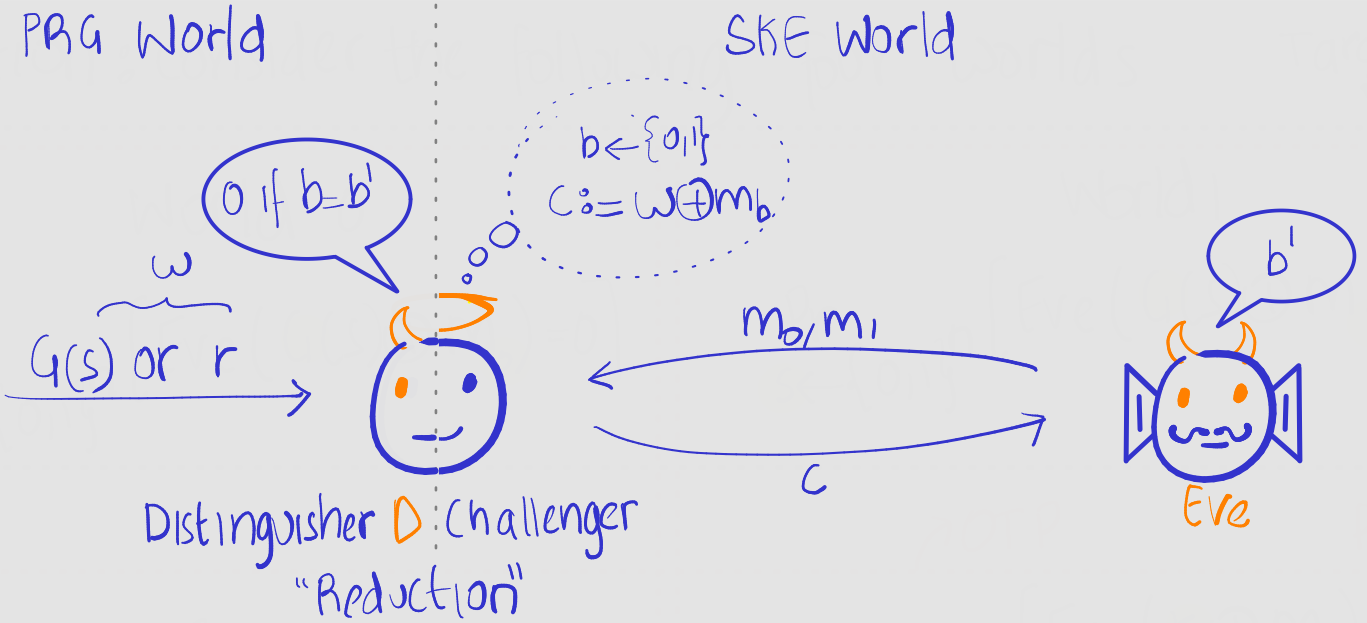

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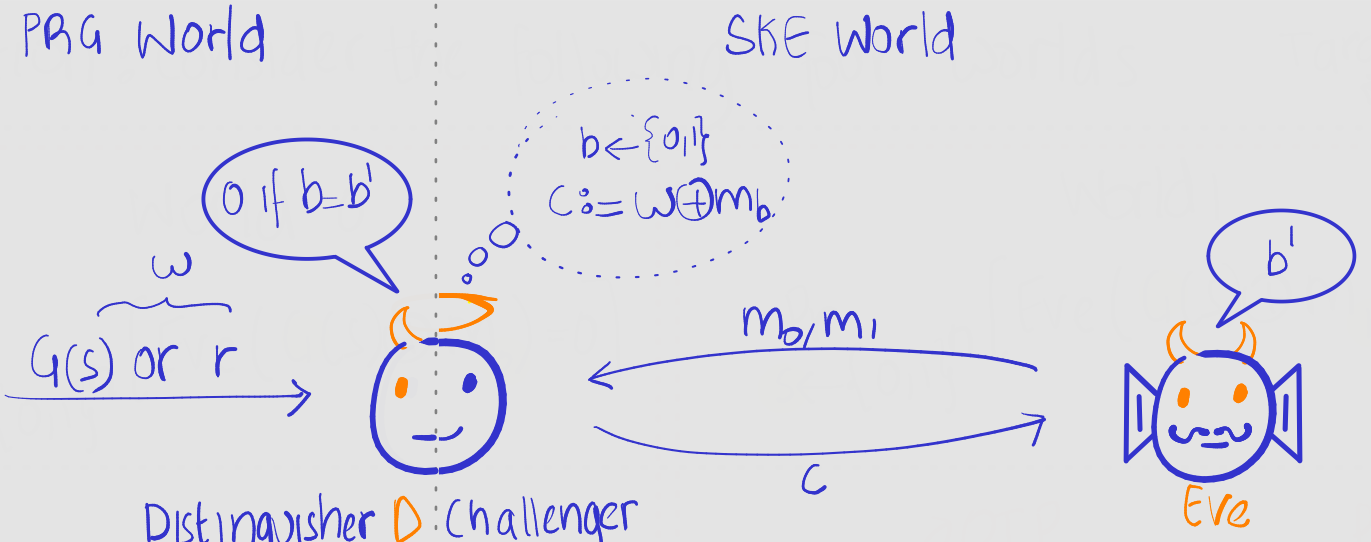

Analysis

We want to show: $\left| \Pr_{s \leftarrow \{0,1\}^n}[D(G(s))=0] - \Pr_{r \leftarrow \{0,1\}^{\ell(n)}}[D(r)=0] \right|$ not negl

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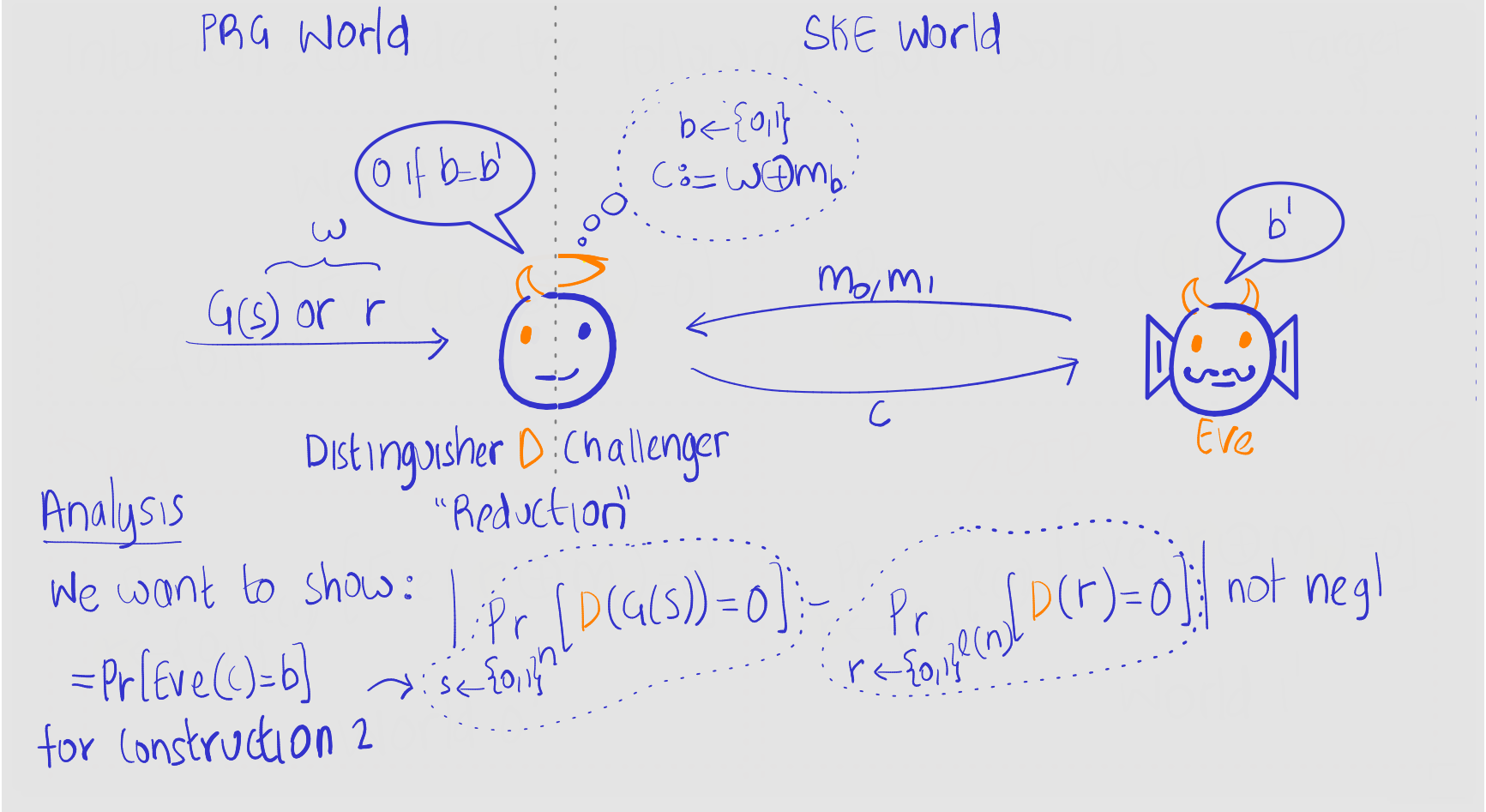

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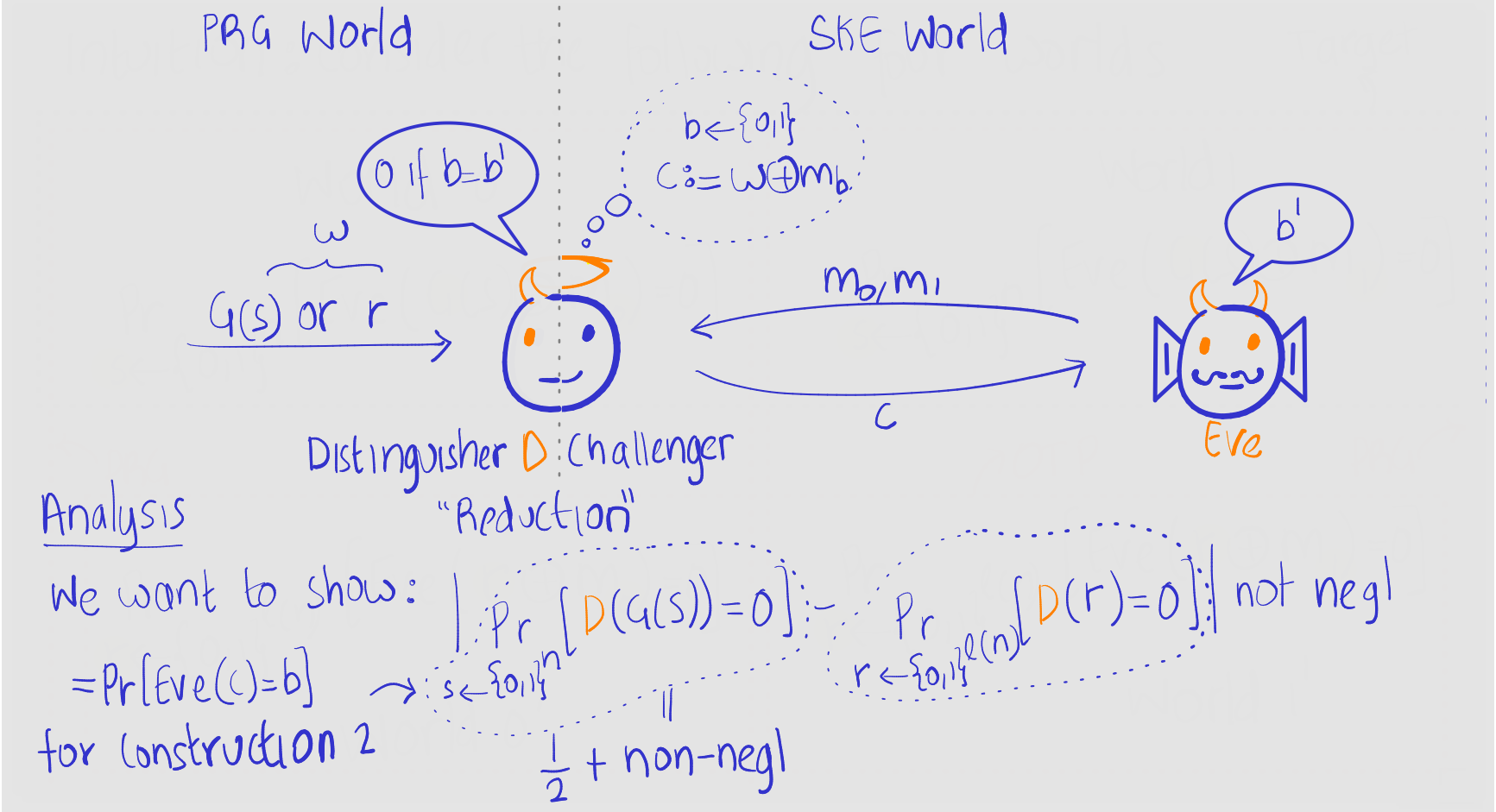

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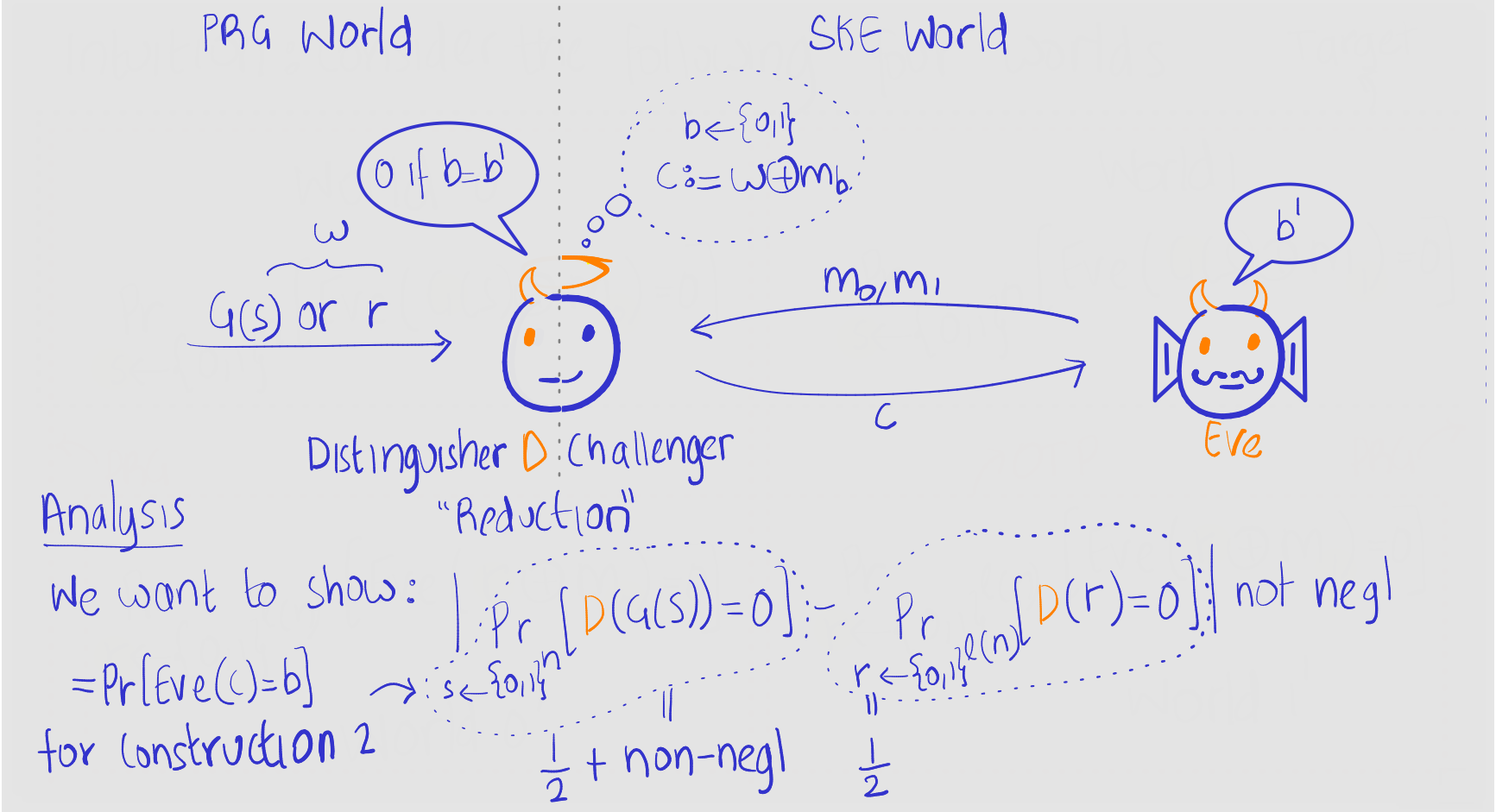

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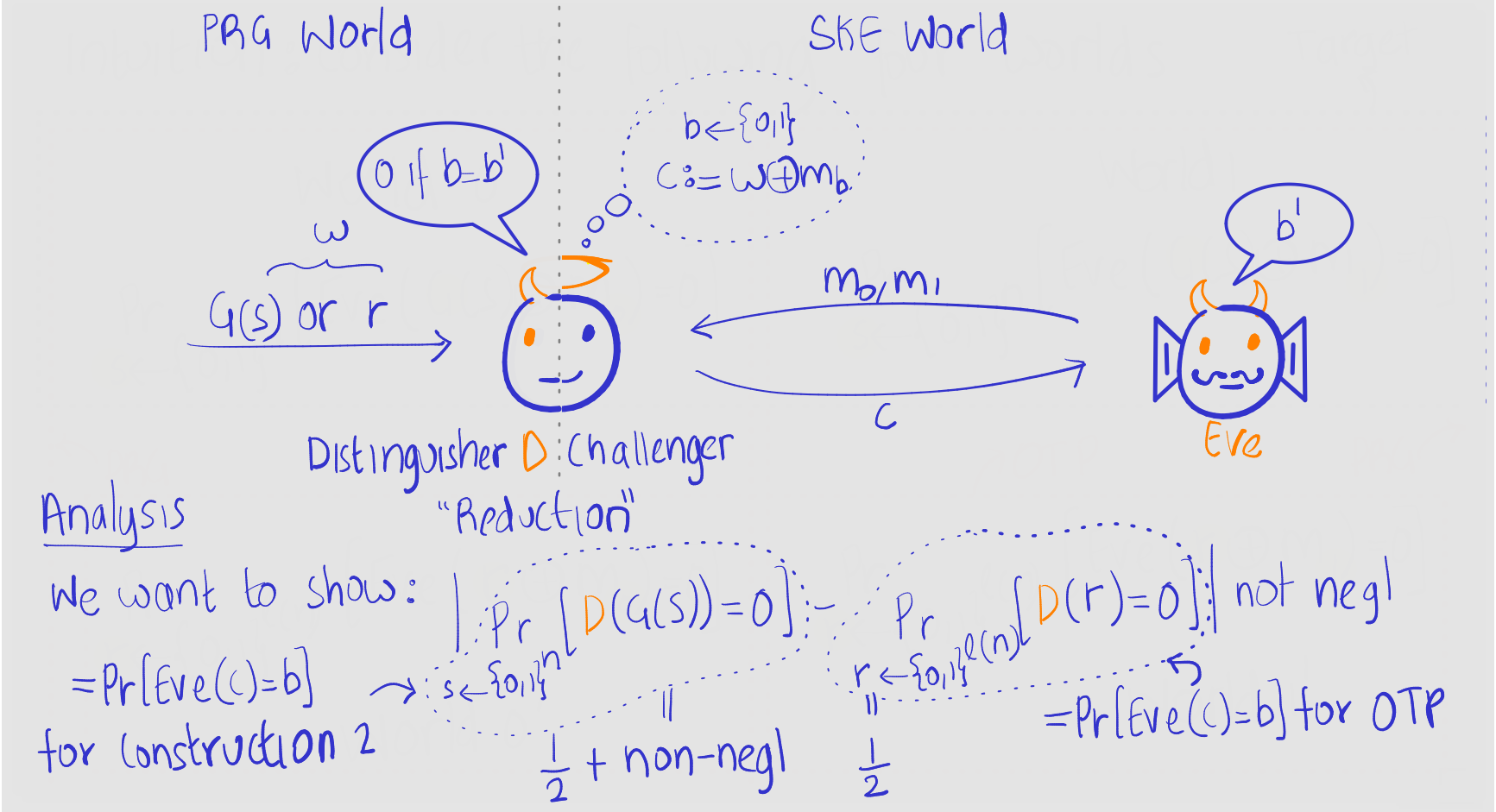

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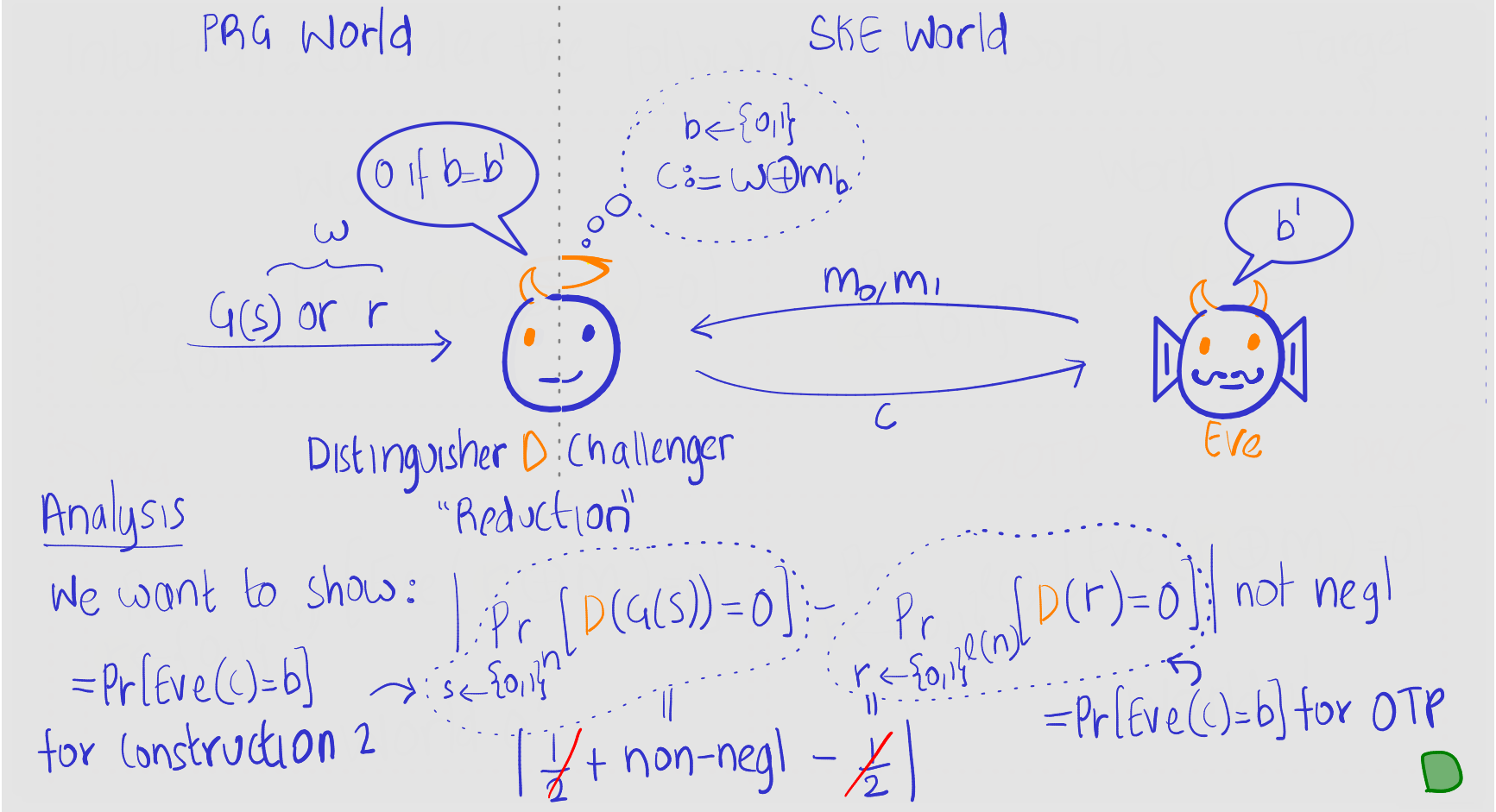

Proof of Computational Secrecy...

Exercise 3 (Formalise proof of Theorem 3)

Write down the proof formally:

1. *Why does the reduction work?*
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Exercise 4 (Strengthening Theorem 3)

- *Understand how to model non-uniform adversaries in the circuit model.*

- *Show that assuming PRG secure against non-uniform distinguishers, one can achieve computational OTP against non-uniform eavesdroppers.*
- *Does the reduction work for quantum adversaries?*

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- Recall from earlier that they don't if $\mathsf{P} = \mathsf{NP}$.
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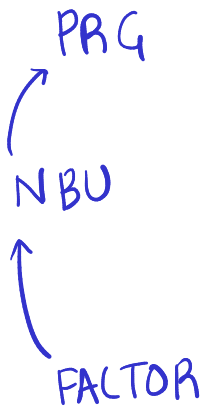

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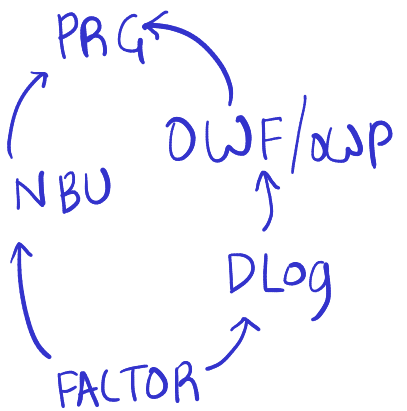

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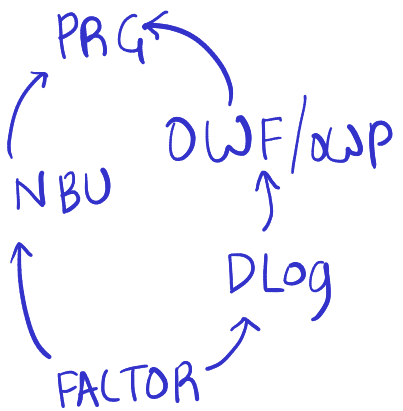

- Note. If pseudorandomness against *fixed-poly.* distinguishers suffices, then we can construct PRG under *complexity-theoretic* assumptions
  - Look up Nisan-Wigderson PRGs!

Applications of PRG

- Saw application of PRG to construct SKE

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- Other applications
  - Fundamental to cryptography since most algorithms are randomised: helps reduce the amount of "pure" randomness required
  - Derandomisation, i.e., turn a randomised algorithm into deterministic
  - Non-cryptographic PRGs (e.g., LFSR): simulation in physics
    - But broken in cryptographic sense

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  - First security reduction!
- The security definitions can be seen through the lens of:

Defintion 6 (computational indistinguishability)

Two distributions X_0 and X_1 are computationally indistinguishable if for every PPT distinguisher D,

$$\delta(n) := \left| \Pr_{x \leftarrow X_0}[D(x) = 0] - \Pr_{x \leftarrow X_1}[D(x) = 0] \right|$$

is negligible.

Next Lecture

- Yet another way to look at PRGs: next-bit unpredictability (stream ciphers)
  - Concrete construction of PRG
- Length-extension for PRG
- First hybrid (security) argument!
- Introduce pseudo-random functions (PRFs)

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More Questions?

References

1. [KL14, §3.1-3.3] for details about this lecture
2. [Gol01, §1.3] for a thorough treatment of the computational model used in cryptography (including a discussion of the non-uniform circuit model).
3. Foundational works on pseudorandomness were done by Blum and Micali [BM84] Yao [Yao82].
4. You can read about how PRGs are used for derandomisation in [AB09, Chapter 20]. This is also a great source for reading about complexity-theoretic (i.e., Nisan-Wigderson) PRG.
5. You can read about how Shannon's impossibility can be bypassed in the bounded storage model in [Mau92]


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