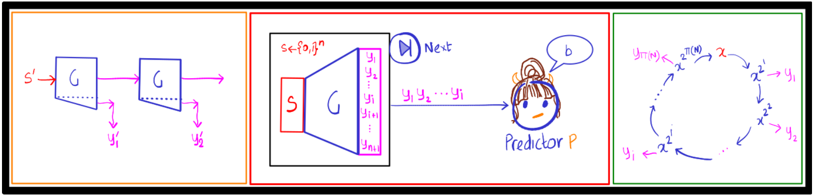
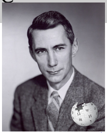


CS783: Theoretical Foundations of Cryptography

Lecture 4 (09/Aug/24)

Instructor: Chethan Kamath

- We started off with Shannon's impossibility *

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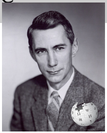

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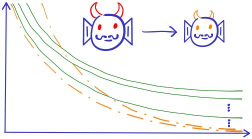
- One way around Shannon's impossibility is to settle for *computational* secrecy
  - Needed two new notions: PPT and negligible



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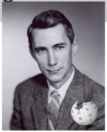
Recall from Last Lecture



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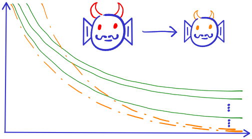

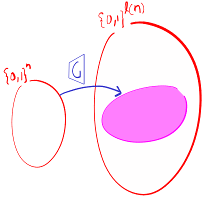

- Defined pseudo-random generators (PRGs)

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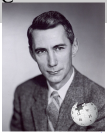

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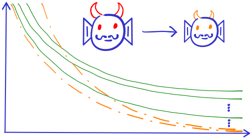
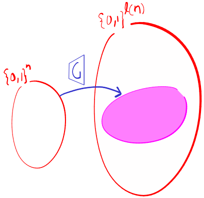
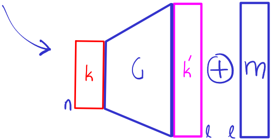




- Defined pseudo-random generators (PRGs)
- Saw construction of computational OTP from PRG
  - First security reduction!



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Applications of PRG

- Already saw application of PRG: constructing SKE that allows encrypting longer messages

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- Several other applications
  - Helps reduce the amount of uniform random bits required: crucial to cryptography since most algorithms are randomised
  - *Derandomisation*, i.e., convert a randomised algorithm into a deterministic algorithm
    - Yao: if "strong" PRGs exist, then $\mathsf{BPP} = \mathsf{P}$


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*

- Non-cryptographic PRGs (e.g., LFSR): physics simulation

  ⚠ But not pseudorandom in cryptographic sense

*Credit: Kyoto-prize.org

Do PRGs Exist?

- Recall from Lecture 3: they don't if $\mathsf{P} = \mathsf{NP}$.
  - Thus PRGs only exist *conditioned* on $\mathsf{P} \neq \mathsf{NP}$.

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$$\begin{array}{ccccccccc} x & \to & x^2 & \to & x^{2^2} & \to & x^{2^3} & \to & \cdots \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & b_1 & & b_2 & & b_3 & & \end{array}$$

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  - Theoretical e.g.: Based on hardness of *factoring* integers
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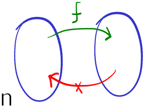
  - E.g.: one-way function (OWF) and one-way permutation
  - OWF is the *minimal* assumption required for cryptography



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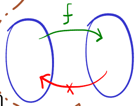
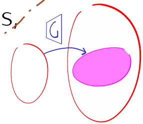
- Thus, seemingly different notions of pseudorandomness, unpredictability and hardness are the same!





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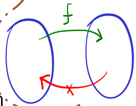

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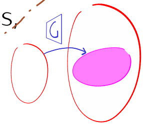


- Note. If PRGs against *fixed-poly.* distinguishers suffices, then: *complexity-theoretic* assumptions → PRG
  - Look up Nisan-Wigderson PRG!

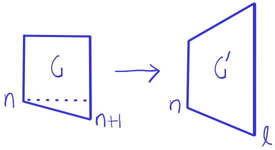
Hardness vs Randomness*

NOAM NISAN† AND AVI WIGDERSON‡

Institute of Computer Science, Hebrew University of Jerusalem, Israel

Plan for this Lecture

1. Length-Extension of PRG



Plan for this Lecture

a) Get a feel for pseudorandomness

b) We'll get to see another reduction

c) Introduces "hybrid argument"

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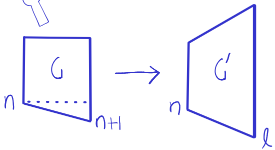

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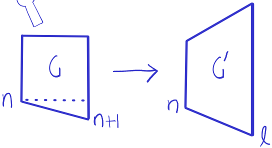

2 Unpredictability

- Unpredictability is Equivalent to Pseudorandomness
- Unpredictable Sequence from Integer Factoring

PRG

Unpredictable sequence

Factoring

Plan for this Lecture

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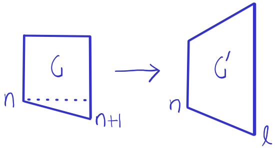

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- Recall: PRG is an expanding function whose output (on uniformly random input) "seems random" to PPT *distinguishers*.

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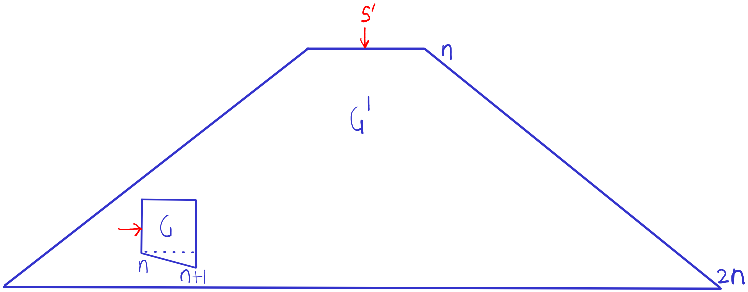

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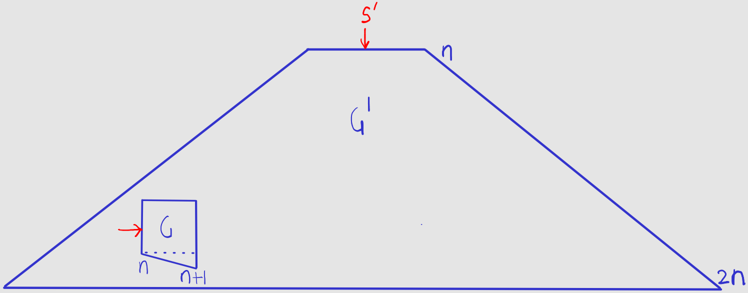

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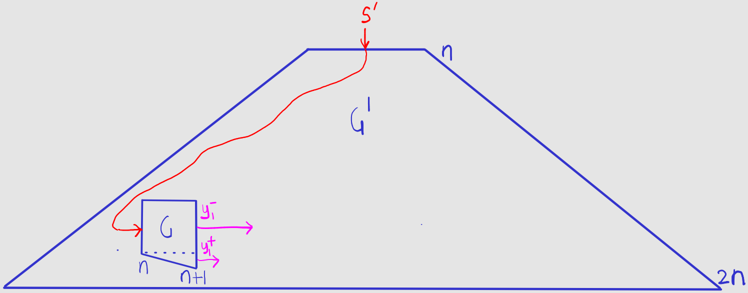

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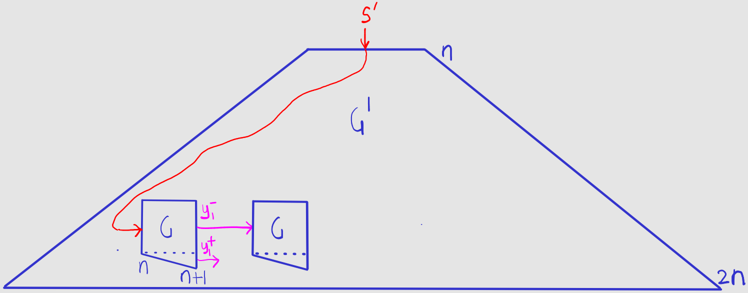

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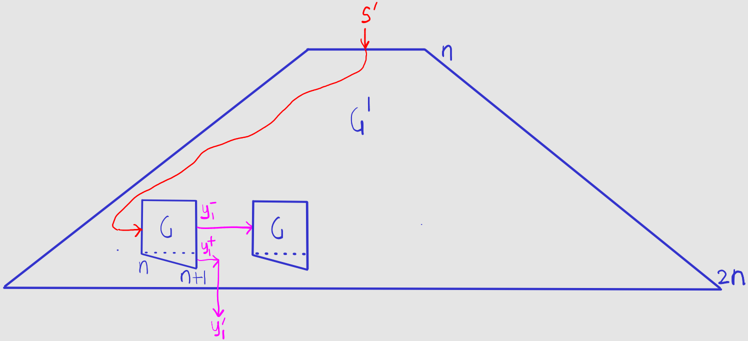

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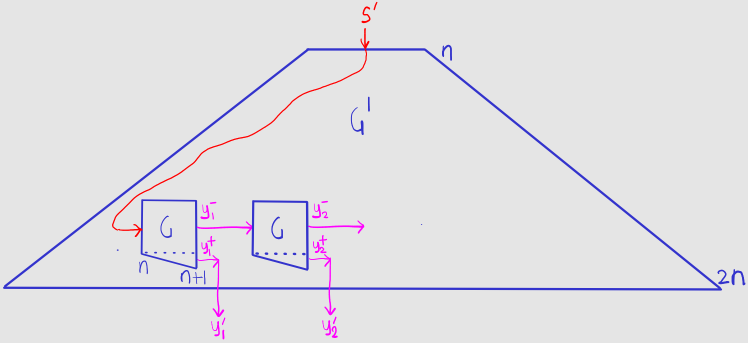

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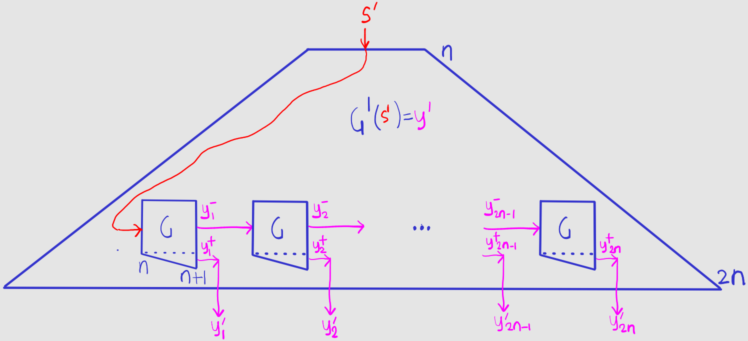

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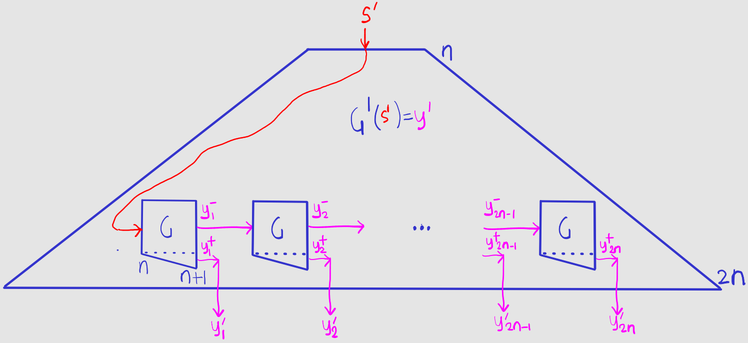

Exercise 1

Formally write down the construction of G′.

Before the Proof, Recall Definition of PRG

Defintion 1 (PRG, via Imitation Game)

Let G *be an efficient deterministic algorithm that for any* $n \in \mathbb{N}$ *and input* $s \in \{0,1\}^n$*, outputs a string of length* $\ell(n) > n$. ← "stretch/ expansion factor"
G *is PRG if for every PPT distinguisher* D

$$\delta(n) := \left| \Pr_{s \leftarrow \{0,1\}^n}[\mathsf{D}(G(s)) = 0] - \Pr_{r \leftarrow \{0,1\}^{\ell(n)}}[\mathsf{D}(r) = 0] \right|$$

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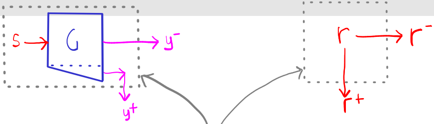


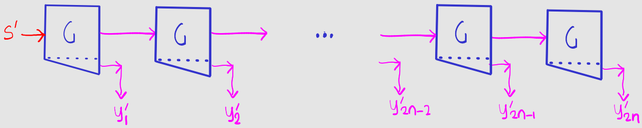
For G stretching by one bit

Proving Pseudorandomness: a Hybrid (Security) Argument

Theorem 1

If G *is a PRG, then so is* G′.

Proof. Intuition



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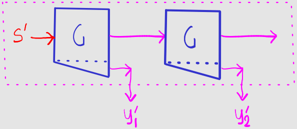
Let's focus on just two iterations



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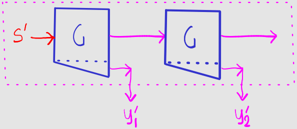
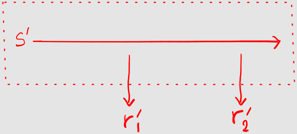


H0 pseudorandom world



H2 random world

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Proof. Intuition: consider hybrid worlds .

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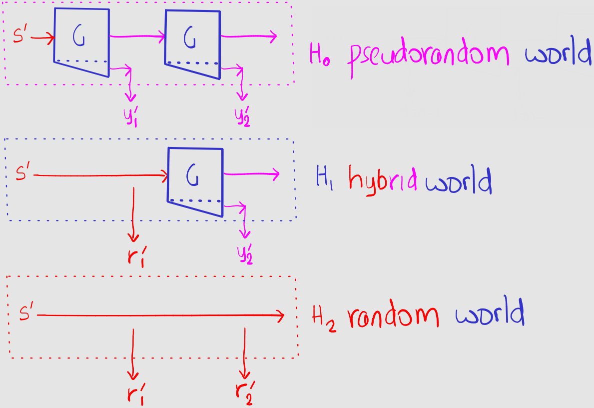

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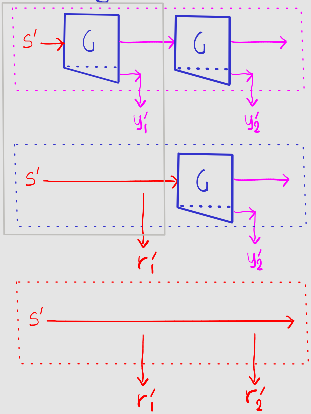

H_0 pseudorandom world

H_1 hybrid world

H_2 random world

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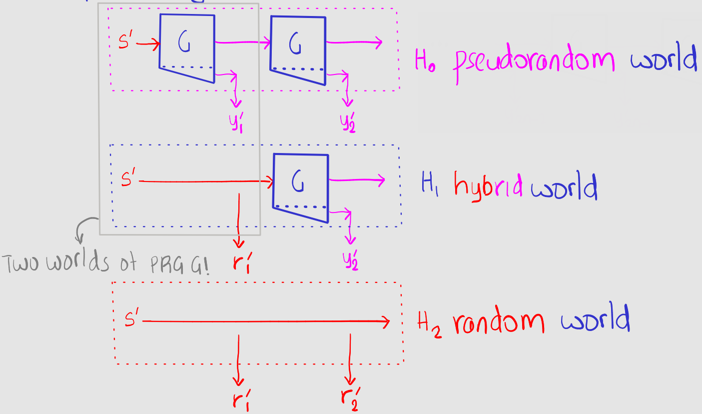

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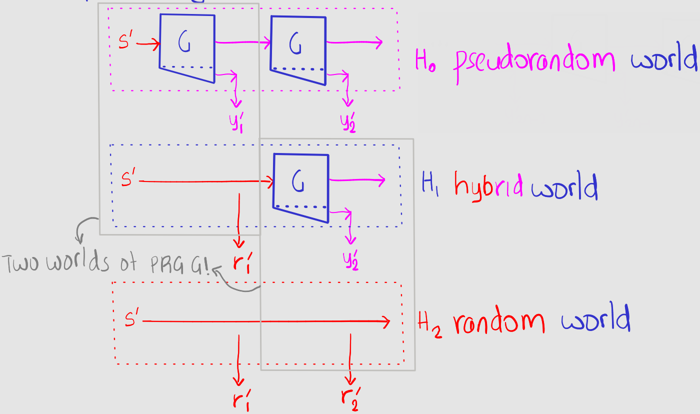

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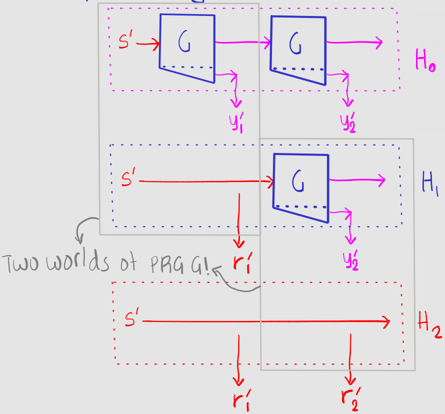

Claim 1

Distinguisher for G′

$$\Pr[D'(H_0)=0]-\Pr[D'(H_2)=0]\geq\delta$$

$$\Downarrow$$

$\exists i\in[0,1]$ such that

$$\Pr[D'(H_i)=0]-\Pr[D'(H_{i+1})=0]\geq\delta/2$$

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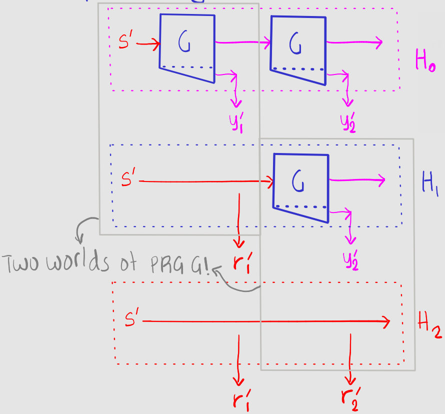

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$$\Downarrow$$

$\exists i \in [0,1]$ such that ?

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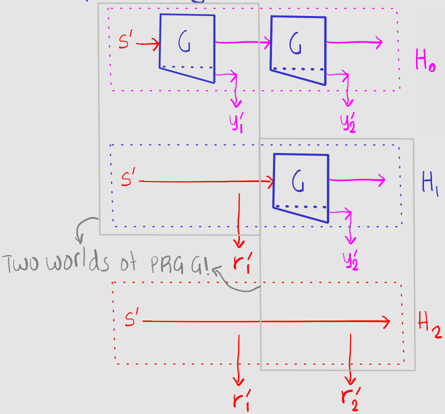

Claim 1 Distinguisher for G′

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$\Downarrow$

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$\Pr[D'(H_i)=0] - \Pr[D'(H_{i+1})=0] \geq \delta/2$ (∗)

Claim 2

(∗) $\Rightarrow$ Distinguisher D for G !

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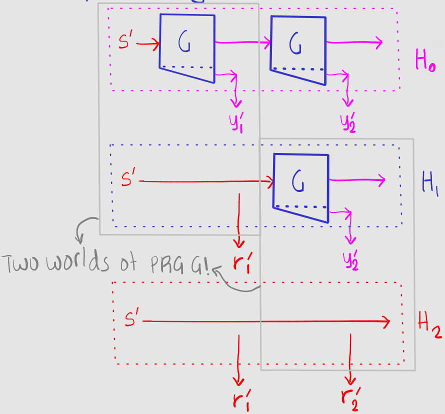

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Distinguisher for G′

$$\Pr[D'(H_0)=0] - \Pr[D'(H_2)=0] \geq \delta$$

$$\Downarrow$$

$\exists i \in [0,1]$ such that

$$\Pr[D'(H_i)=0] - \Pr[D'(H_{i+1})=0] \geq \delta/2 \quad (*)$$

Claim 2

$(*) \Rightarrow$ Distinguisher D for G !

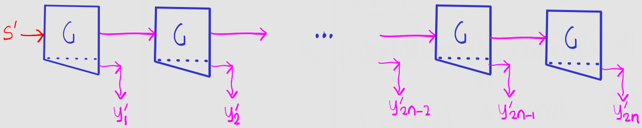
Claims 1 & 2 $\Rightarrow$ Theorem 1

Proving Pseudorandomness: a Hybrid (Security) Argument

Theorem 1

If G *is a PRG, then so is* G′.

Proof. Intuition: consider hybrid worlds .



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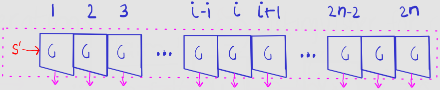

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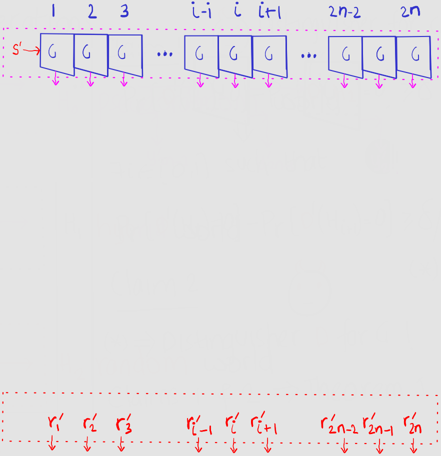

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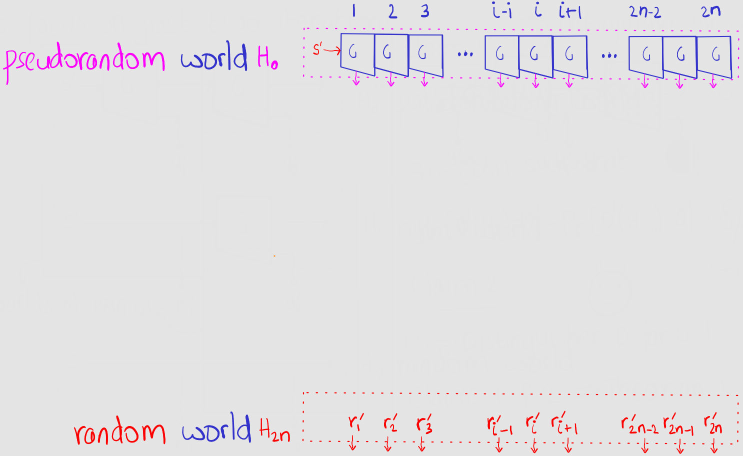

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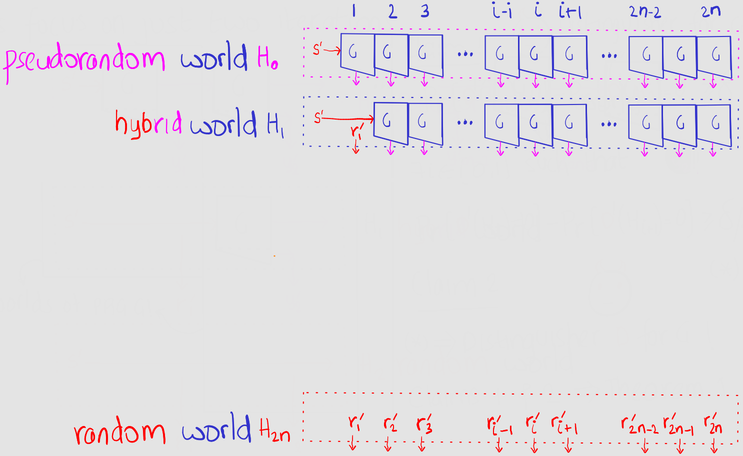

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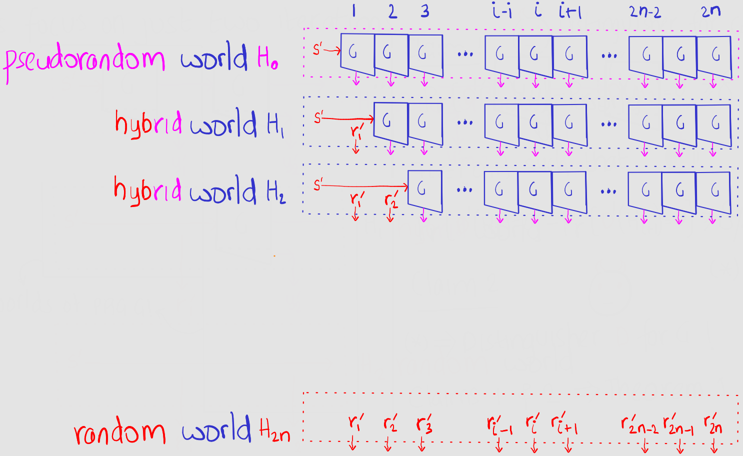

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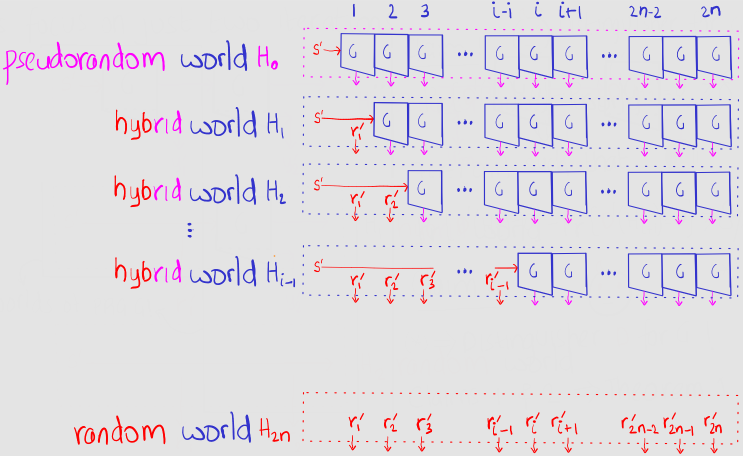

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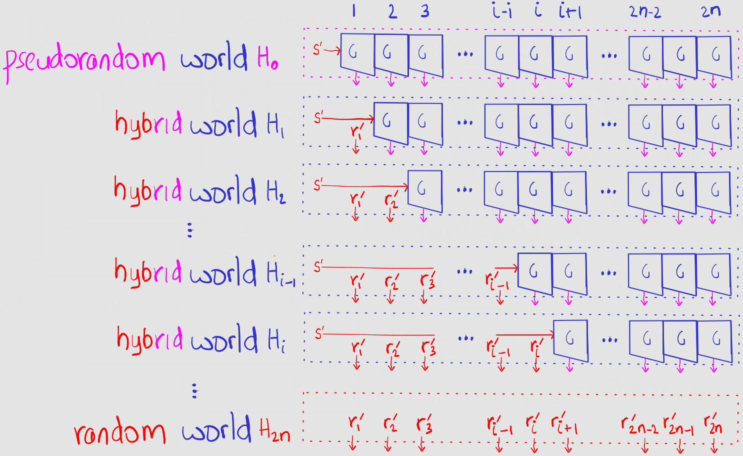

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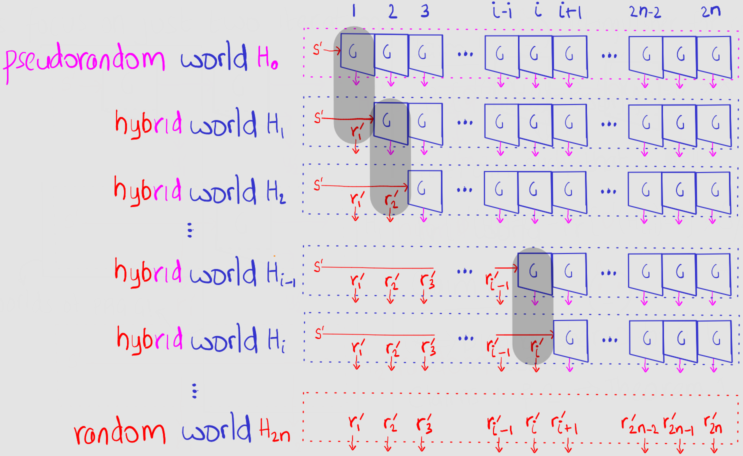

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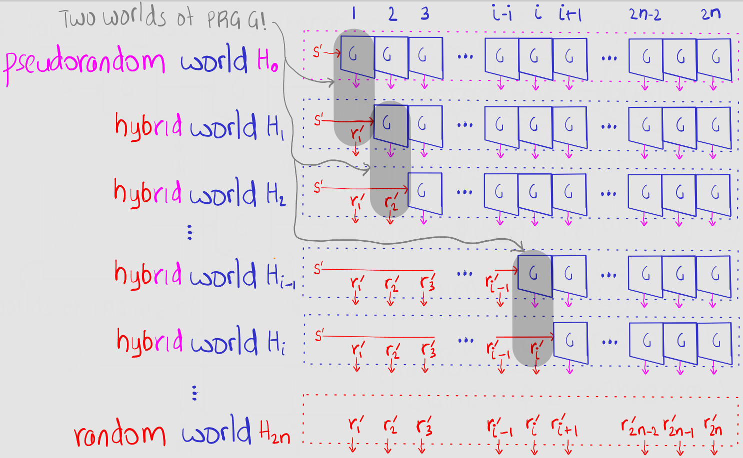

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"Reduction"

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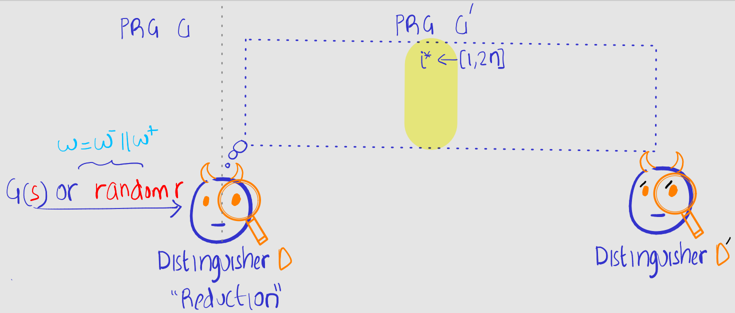

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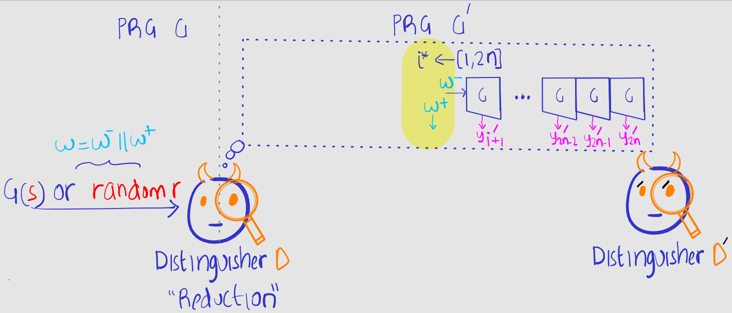

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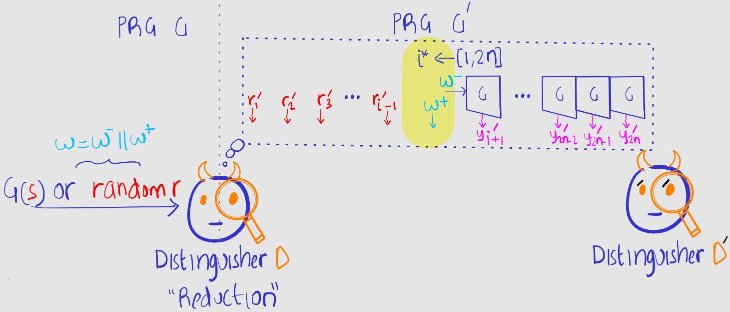

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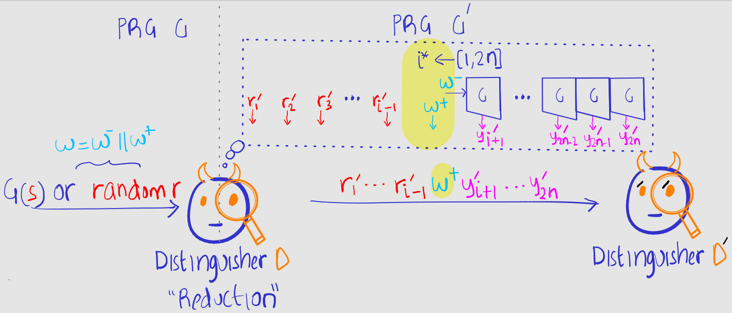

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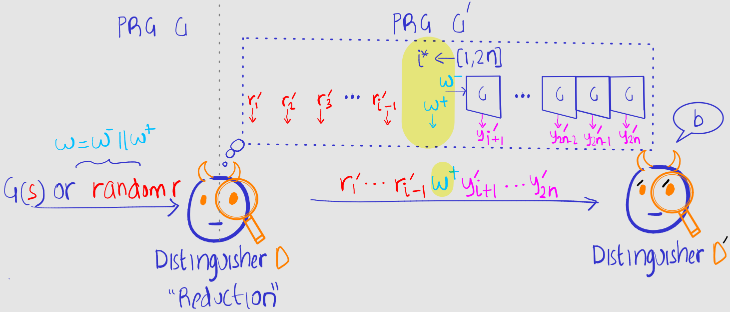

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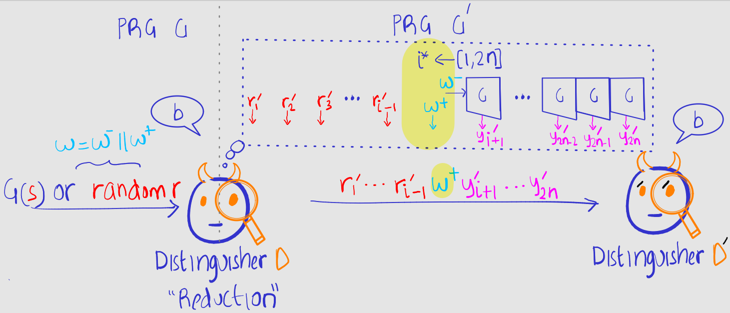

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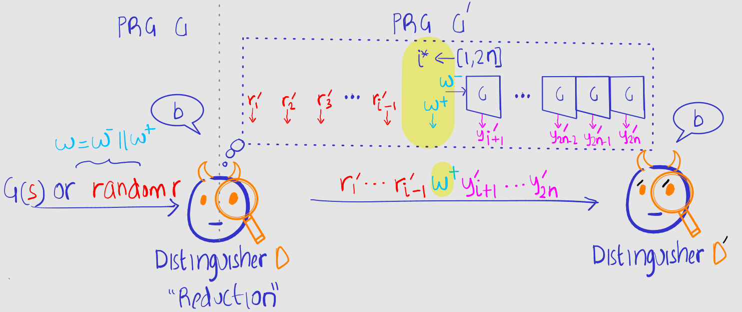

Analysis: similar to claims 1 and 2.

$\Pr[D'(H_0)=0] - \Pr[D'(H_{2n})=0] \geq \delta \Rightarrow$ $\qquad \Pr[i^*=i] = 1/2n$

$\exists i \in [0,2n-1]$ such that $\Pr[D'(H_i)=0] - \Pr[D'(H_{i+1})=0] \geq \delta/2n$ ∎

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- Construction 1 and Theorem 1 work for any polynomial stretch
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- *Maybe need a different construction?*

Plan for this Lecture

1 Length-Extension of PRG

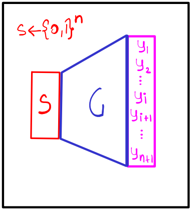
2 Unpredictability

- Unpredictability is Equivalent to Pseudorandomness
- Unpredictable Sequence from Integer Factoring

PRG
↑
Unpredictable sequence
↑
Factoring

Let's Define Unpredictability...

- Intuition: no PPT *predictor* can, given first i output bits, predict $i+1$-th output bit



Predictor P

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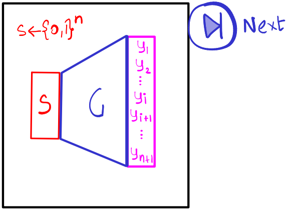

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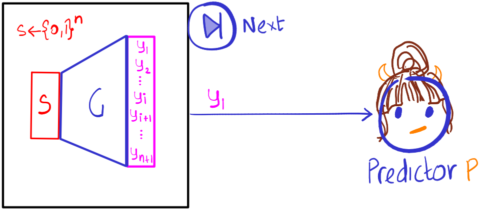

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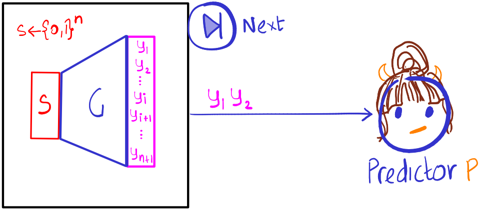

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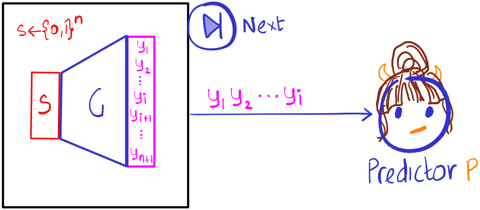

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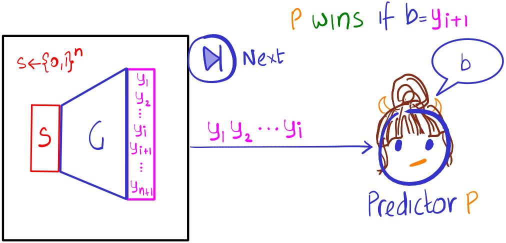

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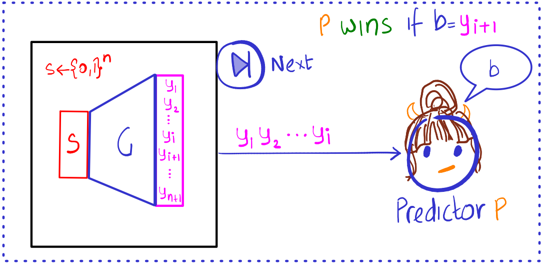

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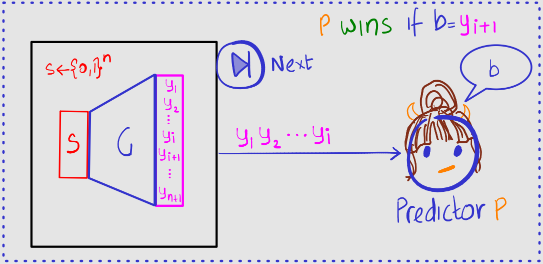
Defintion 2 (Tailored for expanding functions of stretch $n+1$)

Let G *be an efficient deterministic algorithm that for any $n \in \mathbb{N}$ and input $s \in \{0,1\}^n$, outputs a string of length $n+1$.*



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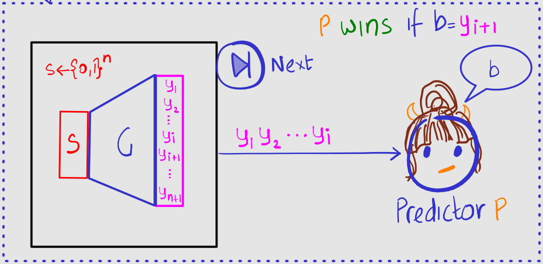
"on the right"

$$\Pr[\mathsf{P}\ \mathit{wins}] - \frac{1}{2}$$

is negligible.



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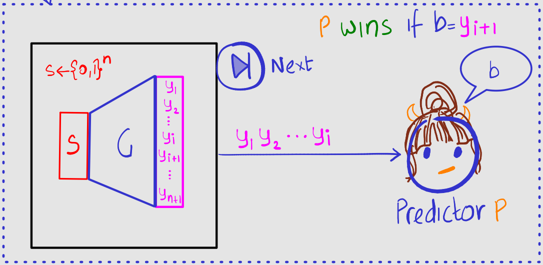

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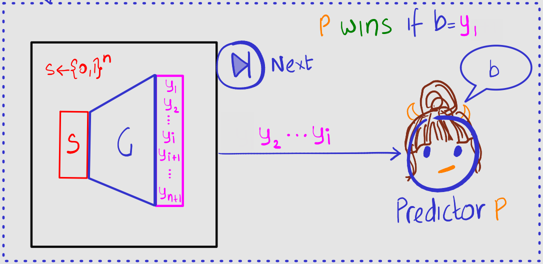

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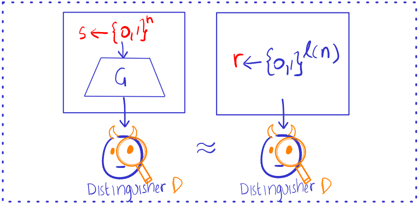
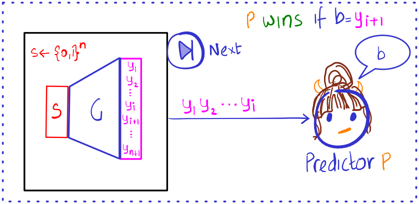
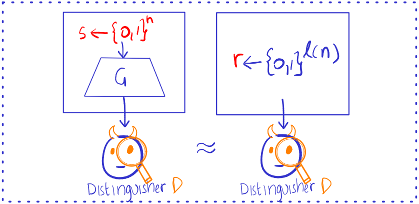
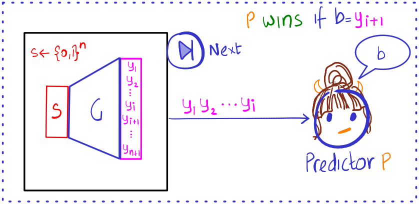
Unpredictability is Equivalent to Pseudorandomness





Which definition do you feel is easier to achieve?





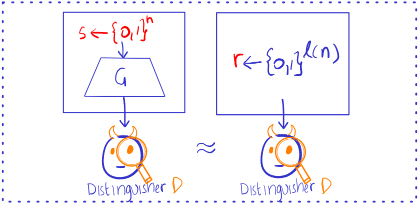
Which definition do you feel is easier to achieve?

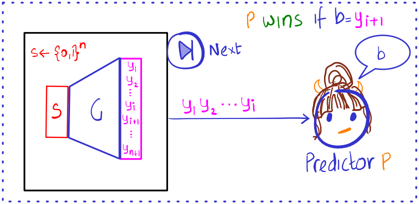
- Easier direction:

Exercise 3

Show that pseudorandomness (Defintion 1) implies next-bit unpredictability (Defintion 2).

Unpredictability is Equivalent to Pseudorandomness





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Show that pseudorandomness (Defintion 1) implies next-bit unpredictability (Defintion 2). Hint:

- *Goal:* $\exists$ *distinguisher* D *for* G $\Leftarrow$ $\exists$ *predictor* P *for* G
- *Feed* P *with prefix of challenge* w *(*r *or* $\mathsf{G}(s)$*) of random length.*
- *If* P *predicts the next bit of* w *correctly, then we're likely in the pseudorandom world*

Unpredictability Implies Pseudorandomness

Theorem 2

If G *is next-bit unpredictable, then it is a pseudorandom.*

Proof Sketch. Intuition: 1) hybrid argument .

$\square$

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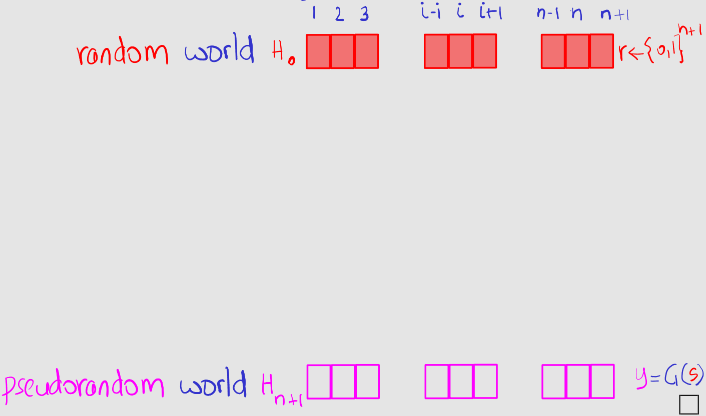

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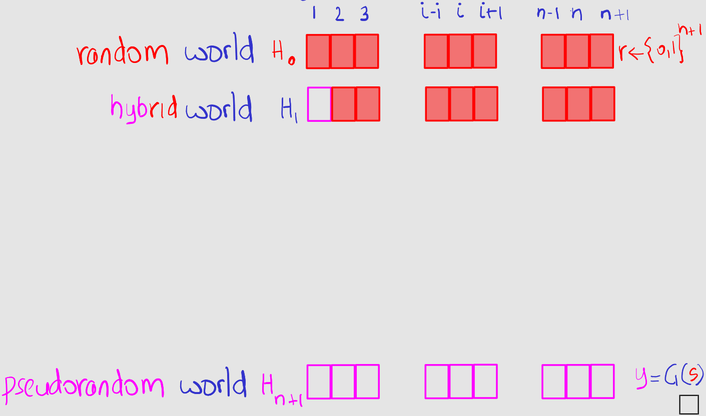

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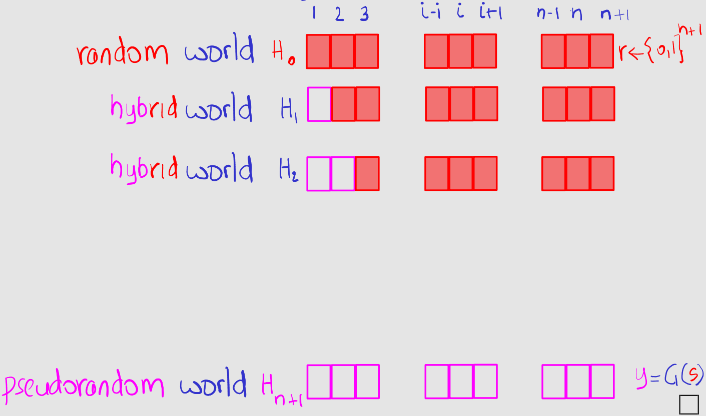

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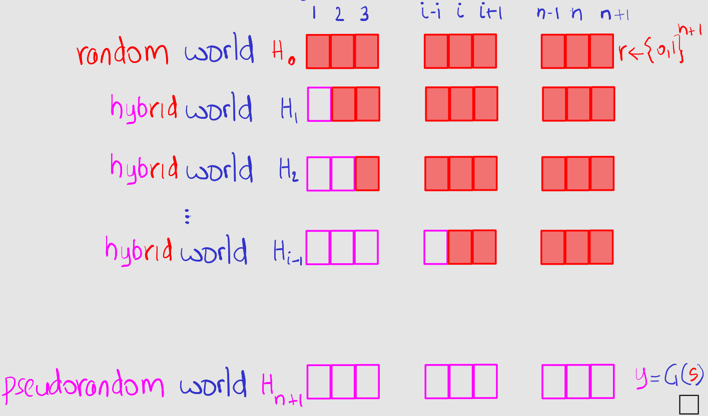

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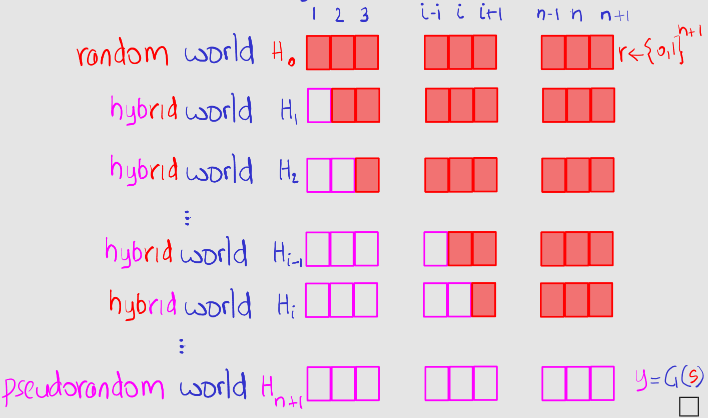

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Unpredictability Implies Pseudorandomness

Theorem 2

If G *is next-bit unpredictable, then it is a pseudorandom.*

Proof Sketch. Intuition: 1) hybrid argument

Claim 1 Distinguisher for G

$\Pr[D(H_{n+1})=0] - \Pr[D(H_0)=0] \geq \delta$

$\Downarrow$

$\exists i \in [1, n+1]$ such that ?

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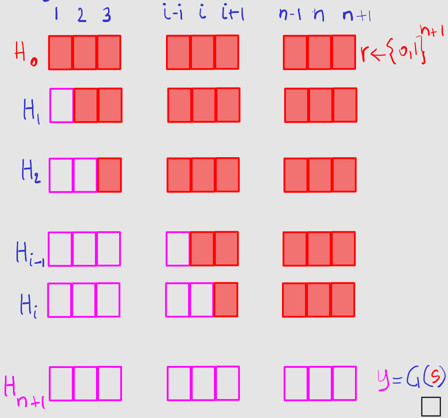

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Claim 1

Distinguisher for G

$\Pr[D(H_{n+1})=0] - \Pr[D(H_0)=0] \geq \delta$

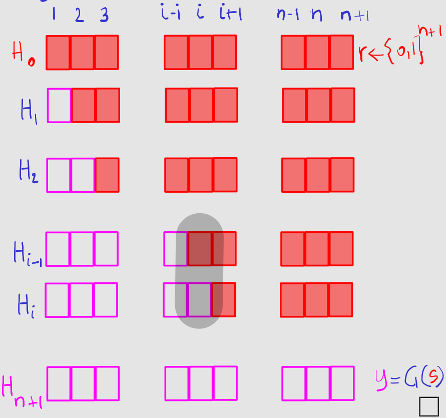
$\Downarrow$

$\exists i \in [1, n+1]$ such that ?

$\Pr[D(H_i)=0] - \Pr[D(H_{i-1})=0] \geq \delta/n$

Claim 2 :

D outputs 0 more often when given correct bit y_i than wrong bit $\bar{y}_i$!



□

Unpredictability Implies Pseudorandomness

Theorem 2

If G *is next-bit unpredictable, then it is a pseudorandom.*

Proof Sketch. $\exists$ predictor P for G $\Leftarrow$ $\exists$ distinguisher D for G.

Pseudorandomness

Predictor P

Distinguisher D

$\square$

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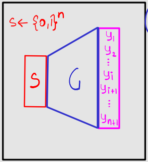

Next

Predictor P
"Reduction"

Distinguisher D

□

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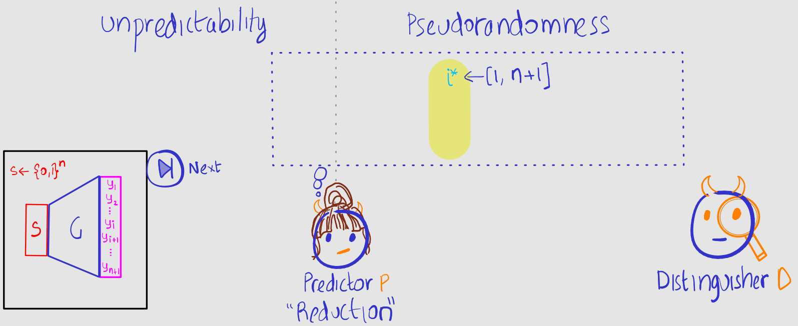

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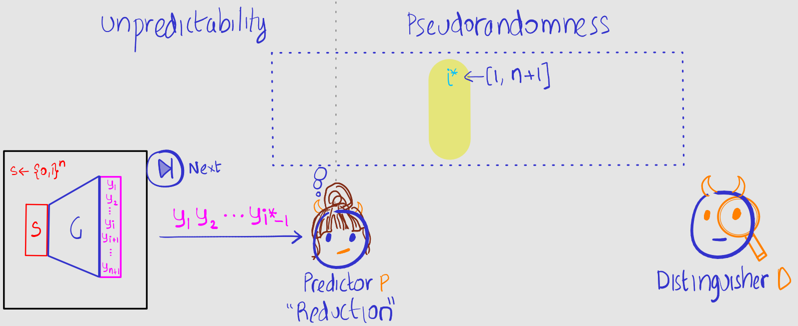

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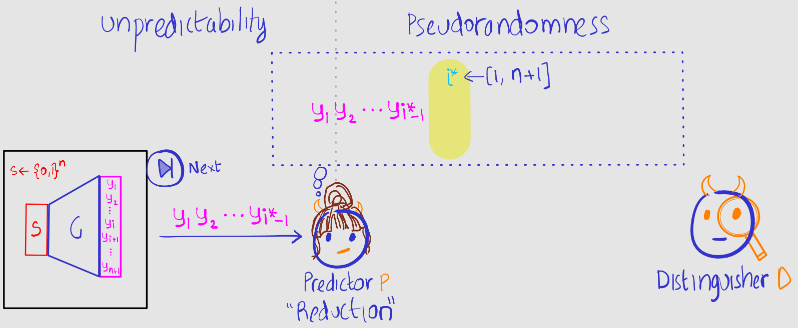

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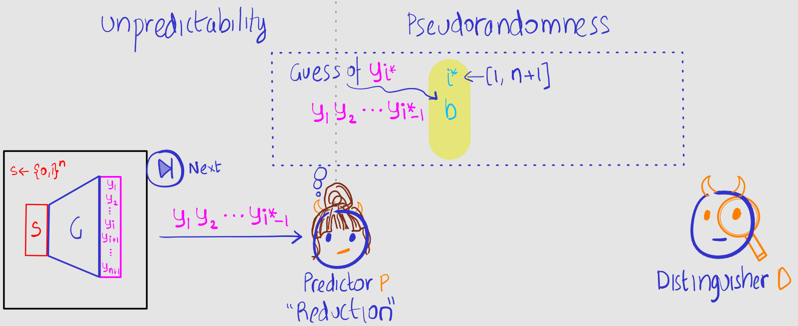

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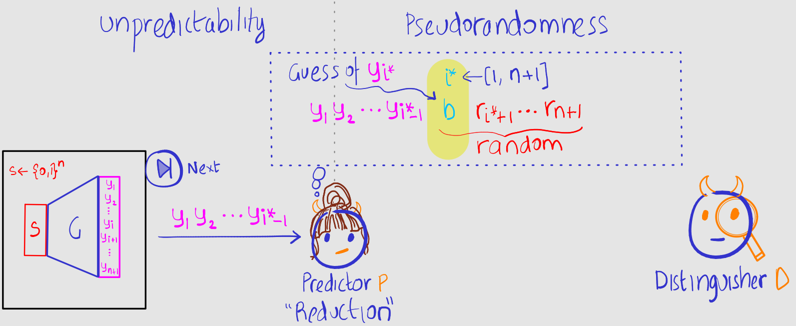

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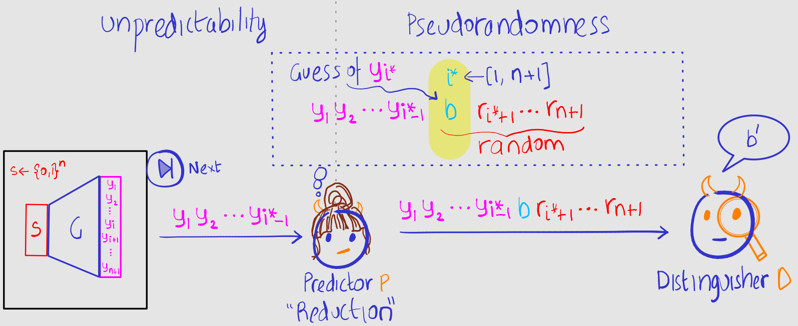

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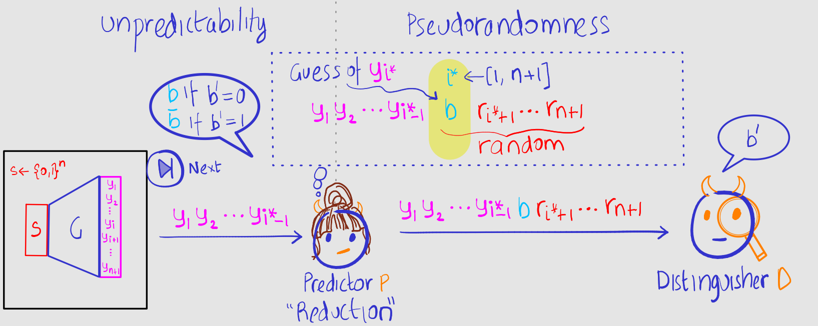

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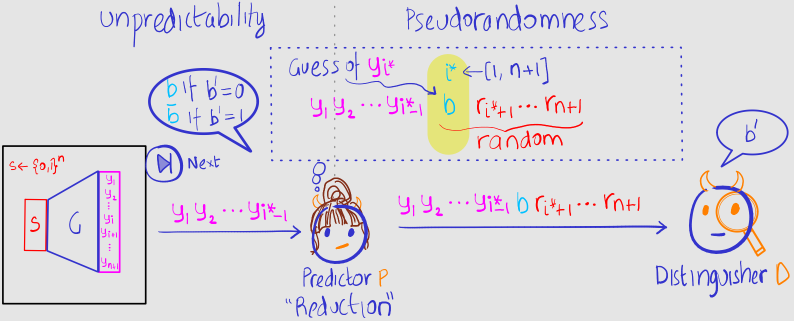

Analysis: If $b = y_{i^*}$ then D more likely to output 0 (claim 2).

$\square$

Plan for this Lecture

Integer Factoring...

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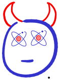

  - Assumption does not hold against quantum adversaries! Shor's algorithm computes factors in quantum polynomial time

Exercise 4

Show that taking square roots modulo N allows you to factor N

Integer Factoring...

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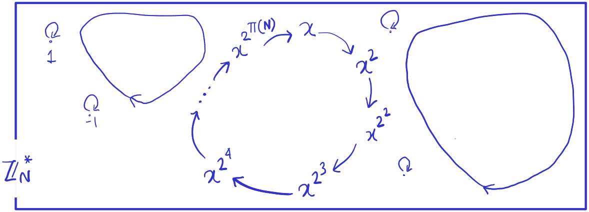

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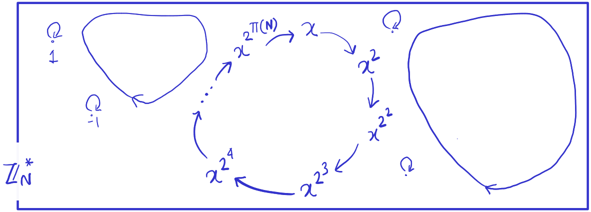
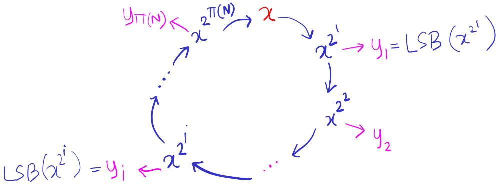


Exercise 5

Show that the squaring map cycles, and has super-polynomially-long period π (with overwhelming probability)

- Given a random square (quadratic residue) as seed, square in each step and output the LSB (or parity bit)



The Squaring (Blum-Blum-Shub) Generator

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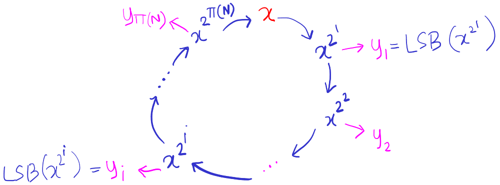

Theorem 3 (Blum, Blum and Shub'84, Vazirani-Vazirani'82)

Assuming factoring (Blum) integers is hard, the squaring generator is unpredictable (on the left).

- Intuition: Why is the sequence unpredictable?
  - Non-linear operation in each step (linearity can be exploited)
  - Taking square root is hard (Exercise 4)
  - Period of the cycle hidden (which can be exploited: e.g., LFSR)

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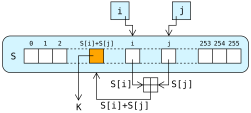
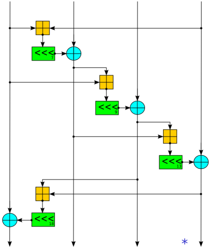
  - Drawback: sometimes broken (e.g., RC4)





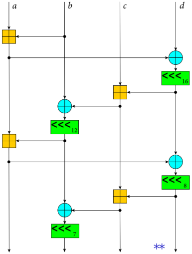


Image credit: Wikipedia (*Sissssou) (**Tony Arcieri)

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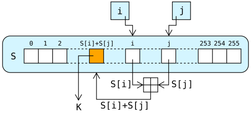

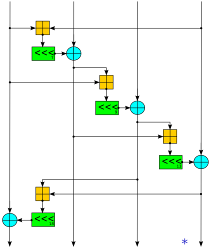

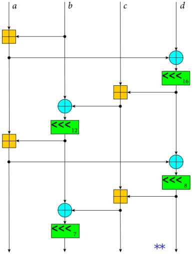

- Salsa20 implemented in eStream, NACL, OpenSSL etc

Image credit: Wikipedia (*Sissssou) (**Tony Arcieri)

To Recap

- We saw an equivalent formulation of pseudorandomness via unpredictability
- Described construction of an unpredictable sequence under factoring assumption
  - Actually yields PRG of arbitrary stretch
- Saw how length-extension for PRG works
  - Reduces task to constructing PRG that stretches by single bit
  - Modular design always useful: will re-use theorem in Lecture 6
  - Proof technique: hybrid argument!

References

1. [Gol01, §3.3] for a formal proof of Theorems 1 and 2
2. Next-bit unpredictability was introduced in [BM84]. Yao introduced pseudorandomness [Yao82], and then proved its equivalence to unpredictability
3. The squaring pseudo-random generator was studied in [BBS86, VV84]
4. You can read about how PRGs are used for derandomisation in [AB09, Chapter 20]. This is also a great source for reading about complexity-theoretic (i.e., Nisan-Wigderson) PRG. Yao's result on derandomisation of BPP is from [Yao82].

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