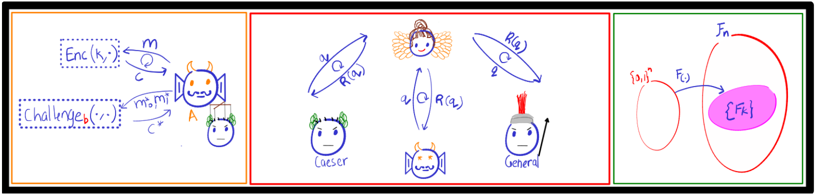
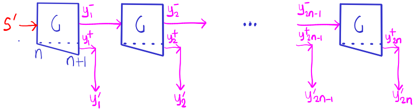


CS783: Theoretical Foundations of Cryptography

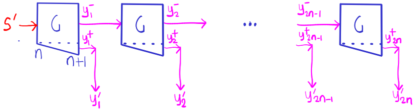
Lecture 5 (13/Aug/24)

Instructor: Chethan Kamath

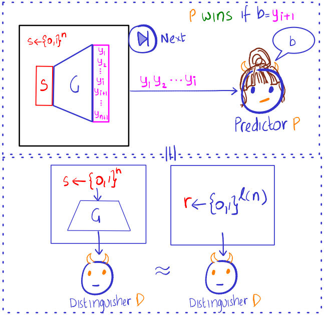
Recall from Last Lecture...

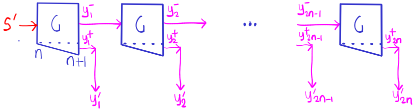


- Length extension of PRG and hybrid argument

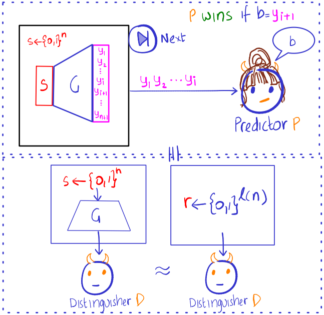


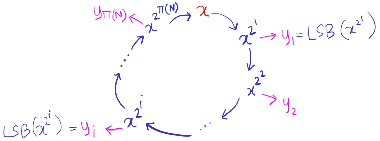
- Length extension of PRG and hybrid argument
- Pseudorandomness vs unpredictability
  - Equivalence between the two notions





- Length extension of PRG and hybrid argument
- Pseudorandomness vs unpredictability
  - Equivalence between the two notions
- Unpredictable sequences from integer factoring





Recall from Last Lecture...

secret communication of longer messages

General *template*:

1. Identify the task
2. Come up with precise threat model M (a.k.a security model)
   - Adversary/Attack: What are the adversary's capabilities? Eavesdropper
   - Security Goal: What does it mean to be secure? Computational secrecy
3. Construct a scheme Π Computational one-time pad from PRG
4. Formally prove that Π in secure in model M

Security reduction from PRG

Plan for This Lecture

multiple

secret communication of ~~longer~~ messages

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Plan for This Lecture

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Security reduction from PRF

Plan for This Lecture

secret communication of ~~longer~~ multiple messages

General *template*:

1. Identify the task
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Security reduction from ~~PRG~~ PRF

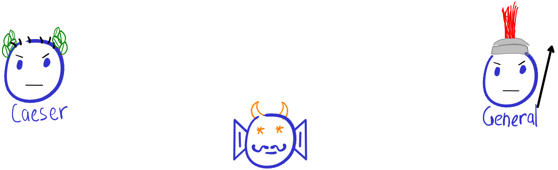
Plan for This Lecture

1. Pseudo-Random Function (PRF)

2. Goldreich-Goldwasser-Micali (GGM) Construction

Let's Encrypt Many Messages Using PRG G...

- Setting: Caeser and his general share a key $k \in \{0,1\}^n$ and want to secretly communicate n messages from $\{0,1\}^n$ in presence of eavesdropper Eve*



Let's Encrypt Many Messages Using PRG G

$= n$

$n \to n^2$

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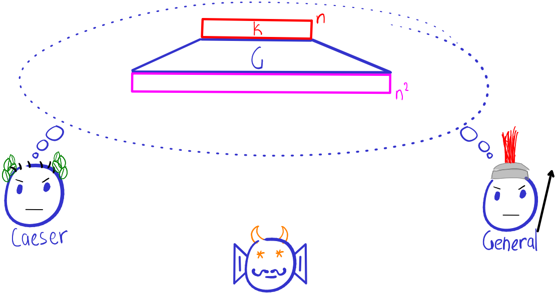

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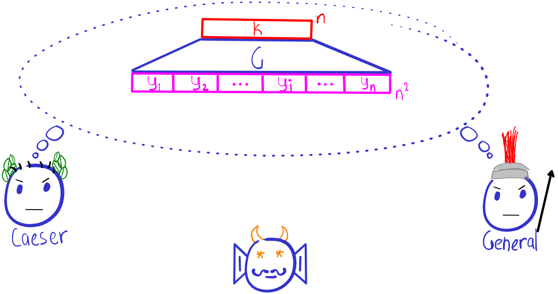

- SKE construction: use output of G as n pseudorandom OTPs

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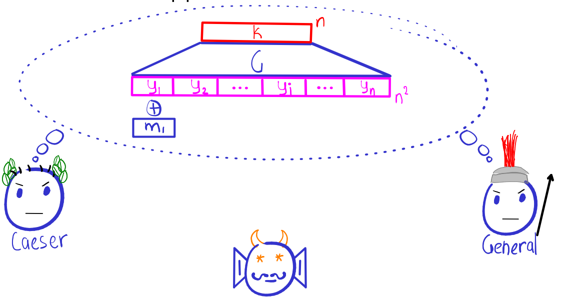

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Let's Encrypt Many Messages Using PRG G

$\overset{\shortparallel}{n}$

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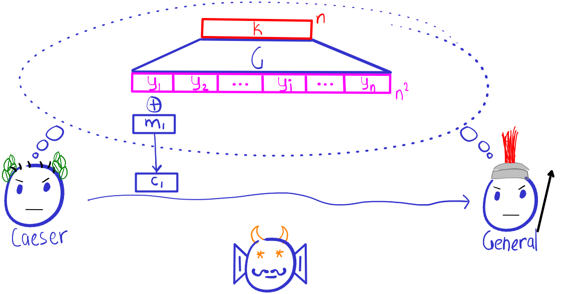

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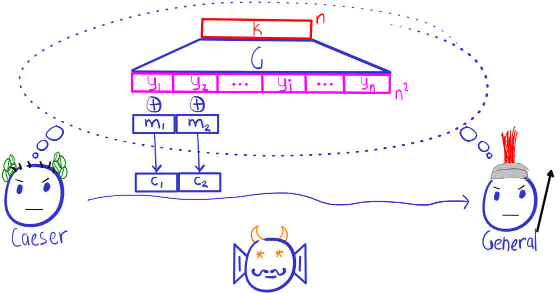

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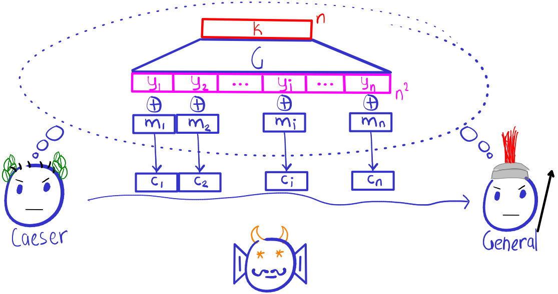

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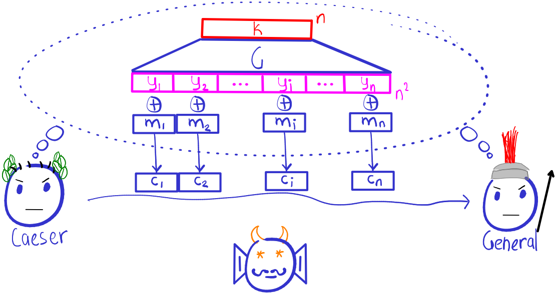

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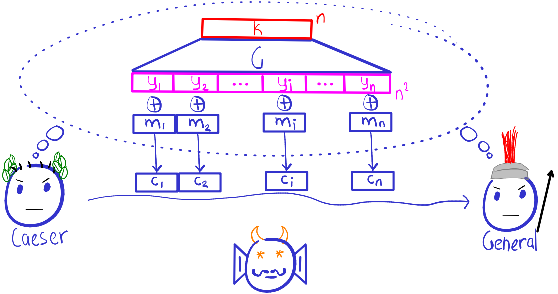

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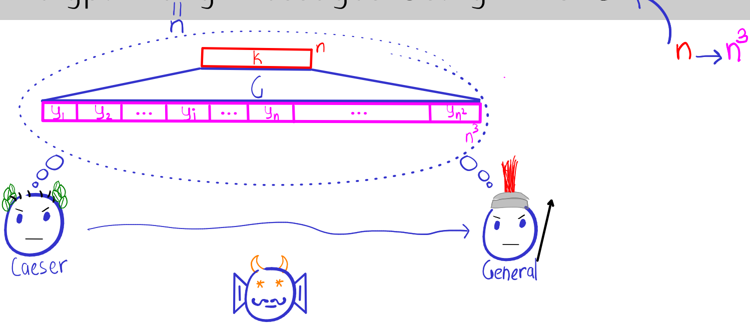
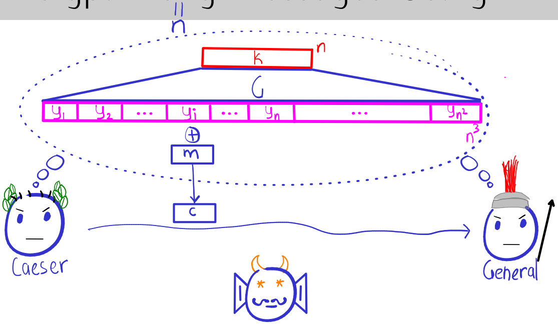
- Problem: construction *stateful*; synchrony must be maintained
  - We lose correctness if (e.g.) ciphertexts delivered out of order
  - ? Come up with a scenario that leads to loss of secrecy



What if the stretch is n^3?

Let's Encrypt Many Messages Using PRG G...

$n \to n^3$



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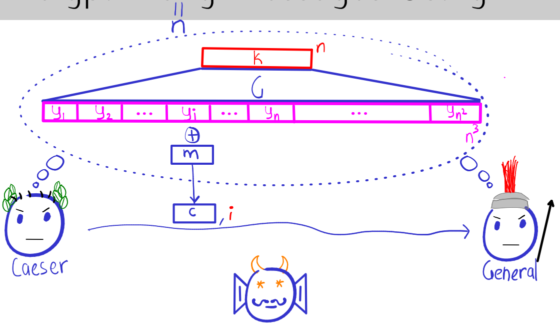

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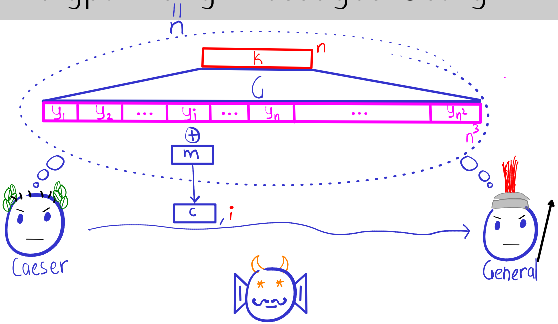

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- Underlying problem: only poly. pseudorandom OTPs available

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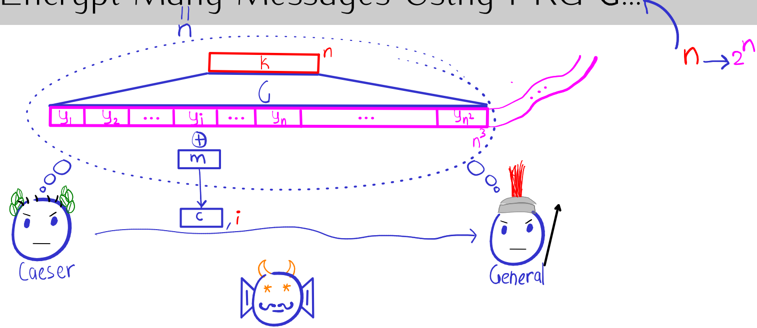

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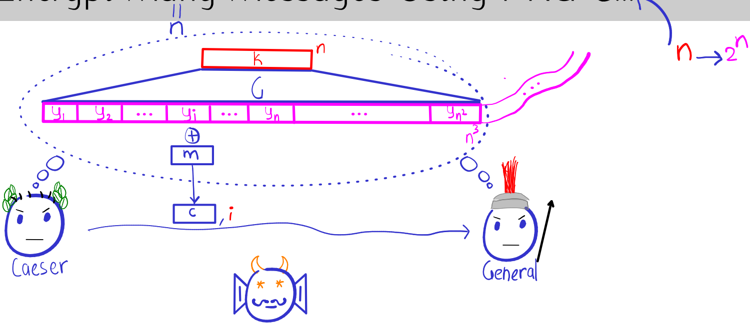

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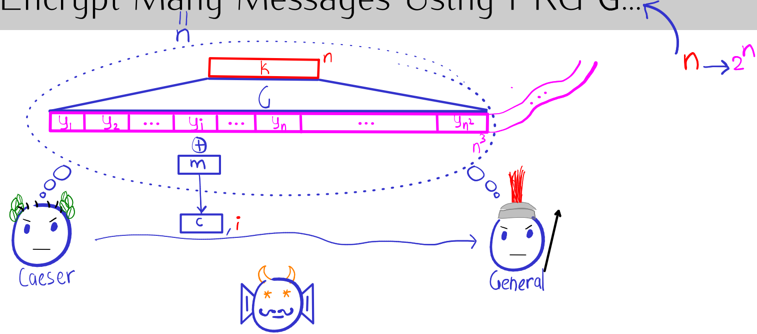

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- Not all pseudorandom OTPs are efficiently "accessible"

- Need "PRG" with
  1. Exponential stretch
  2. Output bits "efficiently" accessible (also called locality)

Let's Encrypt Many Messages Using an Oracle in the Sky

- Setting:
  - Caeser and his general have shared a key $k \in \{0,1\}^n$
  - Everyone (including Eve*) has access to a *random function oracle* $R : \{0,1\}^{2n} \rightarrow \{0,1\}^n$

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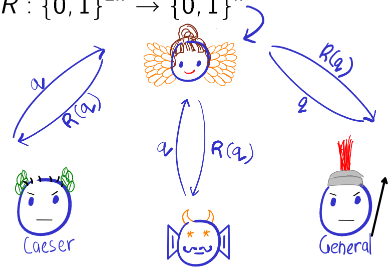

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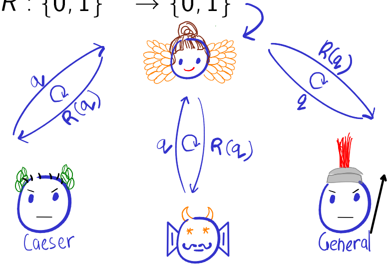

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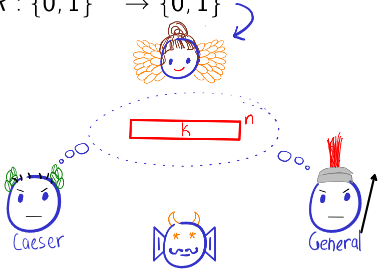

How will you construct a stateless encryption scheme given R?

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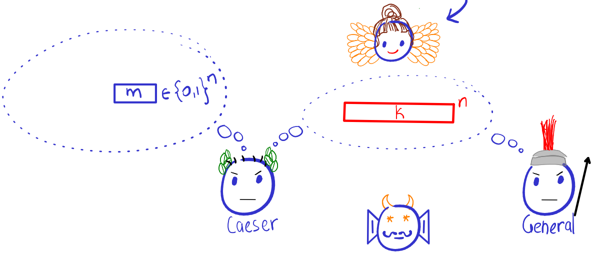

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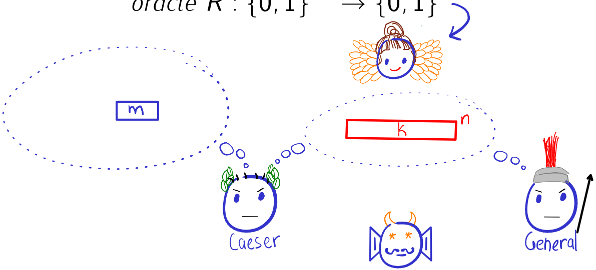

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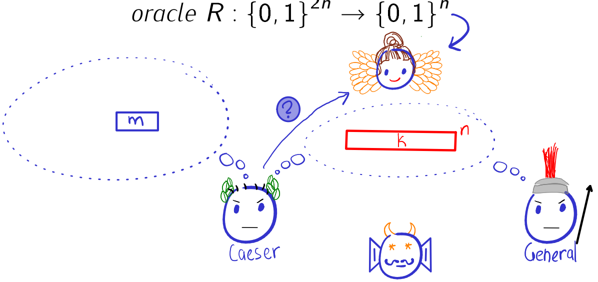

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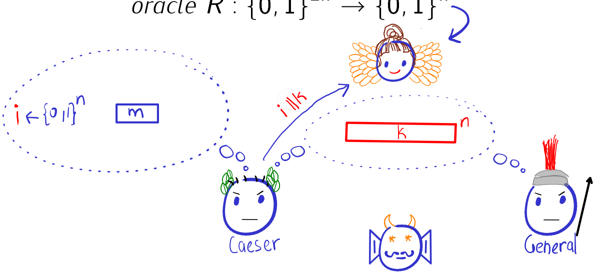

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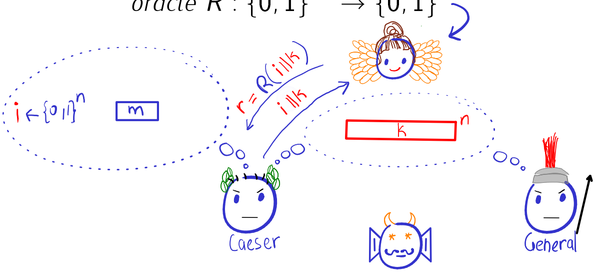

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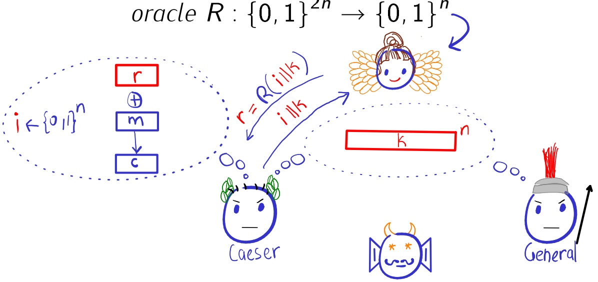

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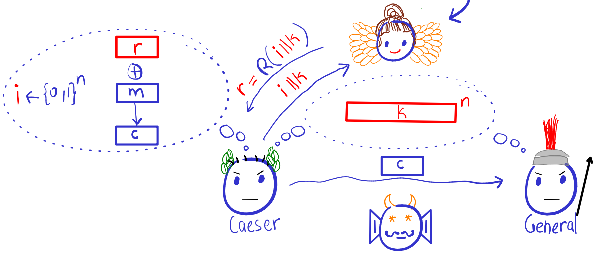

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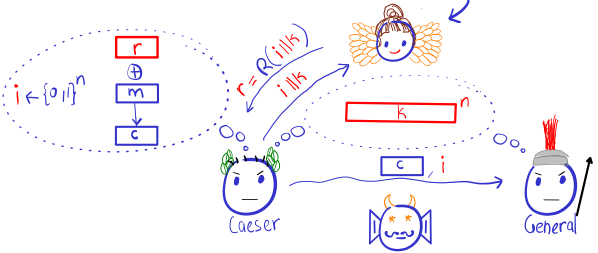

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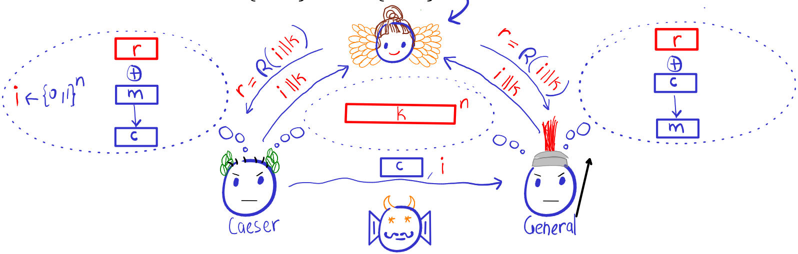

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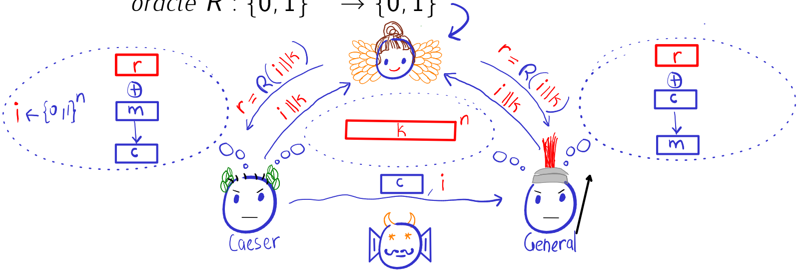

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Exercise 1

What if Caeser and his general did not have the shared key k? Can they still do something given the oracle in the sky?

Plan for This Lecture

Computational analogue of random oracle

1 Pseudo-Random Function (PRF)

2 Goldreich-Goldwasser-Micali (GGM) Construction

PRF: Computational Analogue of Oracle in the Sky

- A function F that "seems like" a random function oracle to PPT distinguishers

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- More formally:
  - F_k sampled at random from a (smallish) family of functions $\{F_k : \{0,1\}^n \to \{0,1\}^n\}_{k \in \{0,1\}^n}$
  - $\approx$
  - A random function, sampled from the set of *all* functions $\mathcal{F}_n$

$\{f : \{0,1\}^n \to \{0,1\}^n\}$

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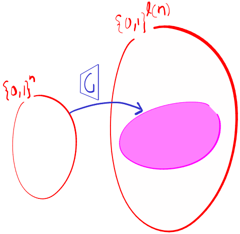

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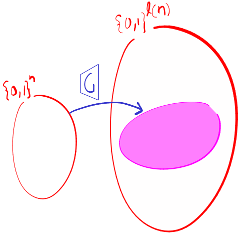

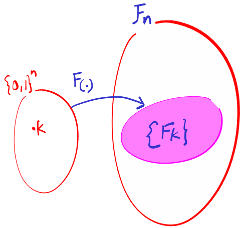

- A function F that "seems like" a random function oracle to PPT distinguishers
- More formally:
  - F_k sampled at random from a (smallish) family of functions $\{F_k : \{0,1\}^n \to \{0,1\}^n\}_{k \in \{0,1\}^n}$
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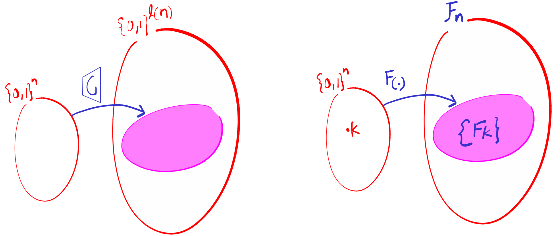

? Number of functions in $\{F_k\}$ vs. number of functions $\mathcal{F}_n$?

- Why is it still useful?
  - Helps generate exponentially-many *pseudorandom* OTPs

How Exactly to Define Pseudorandomess for Functions?...

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- Way around:
  - Distinguisher given *oracle* access to the functions
  - One query=one unit of running time → efficient PPT distinguisher can only make polynomially-many queries

How Exactly to Define Pseudorandomess for Functions?...

Defintion 1 (PRF, via Imitation Game)

A family of functions $\{F_k : \{0,1\}^n \to \{0,1\}^n\}_{k \in \{0,1\}^n}$ is a PRF if for every PPT oracle distinguisher D

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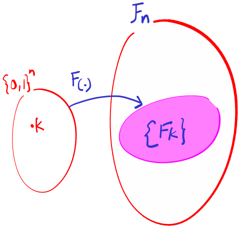

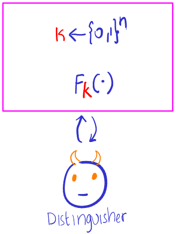

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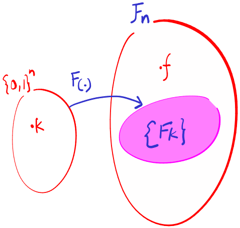

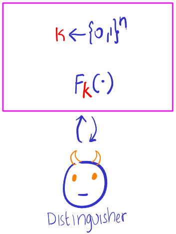

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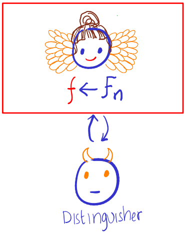

Let's Check if You Understood Defintion 1

PRF or not? Below $F^{(1)}$ and $F^{(2)}$ are PRFs

1. $F_k(x) := k \oplus x$
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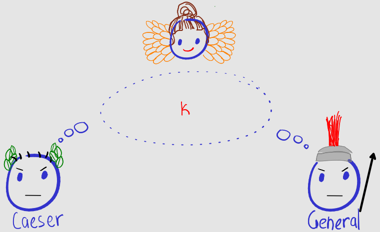
Exercise 2

In all the "yes" cases above, formally prove; in all the "no" cases, describe a counter-example.

Construction 1 (Replace random oracle with PRF)



Stateless Symmetric-Key Encryption from PRF

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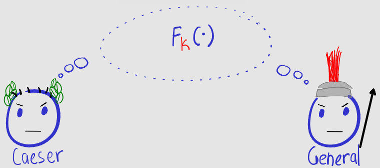

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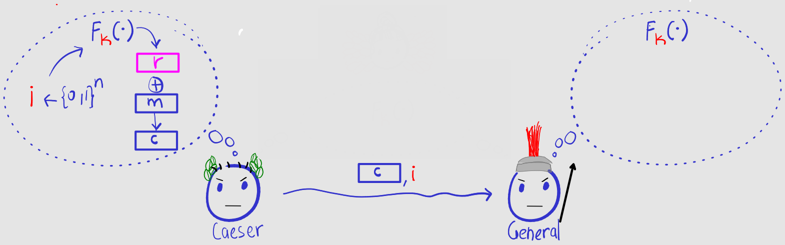

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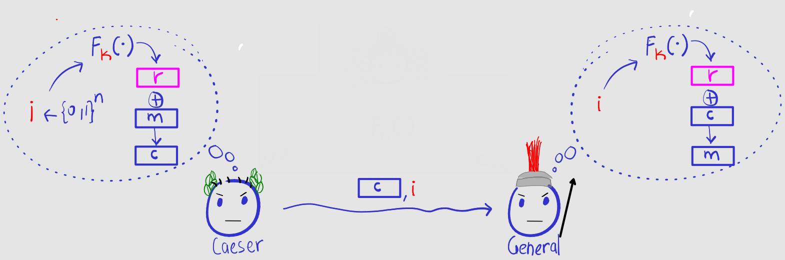

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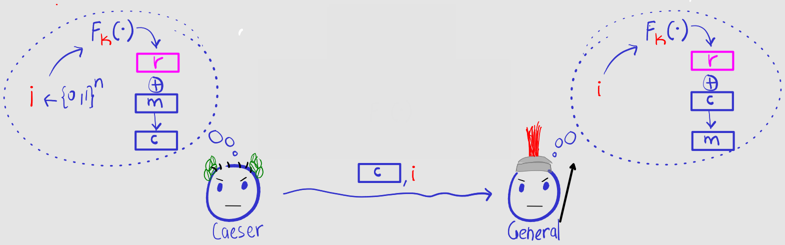

- Note: encryption is randomised and thus length of ciphertext is longer than plaintext (first such scheme in this course)

Exercise 3 (Hint: reduction similar to pseudorandom OTP)

Prove that Construction 1 is secure against eavesdroppers.

In Fact you Get More: CPA-Secret SKE!

- Stronger adversaries who can influence Caeser's messages

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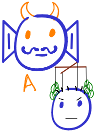
$k \leftarrow Gen(1^n)$



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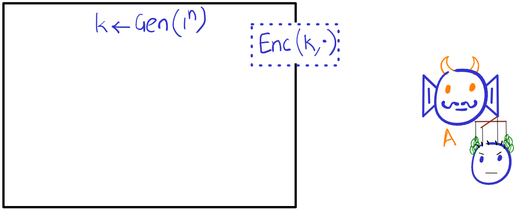

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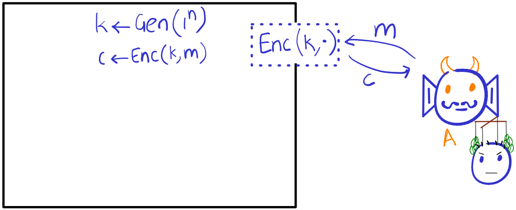

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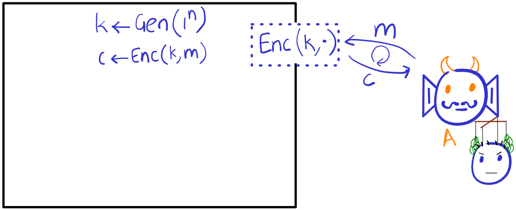

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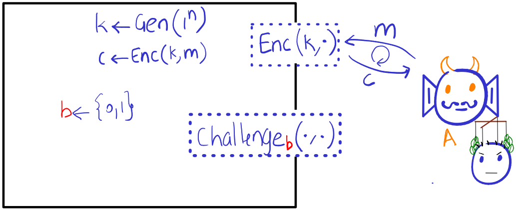

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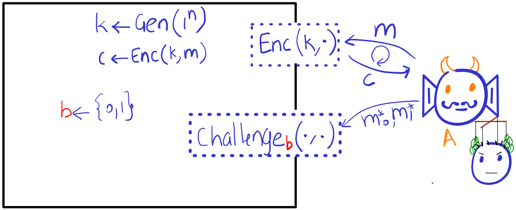

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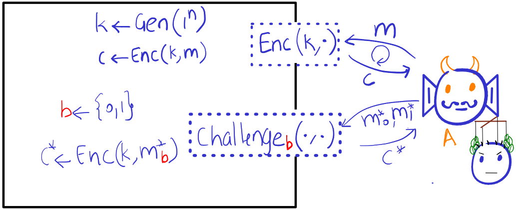

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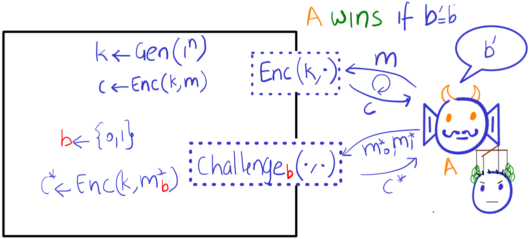

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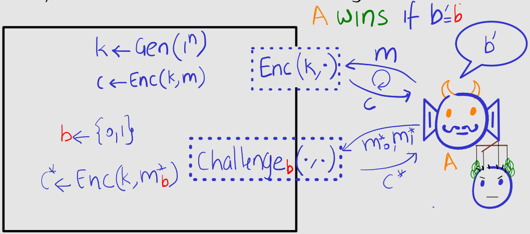
Defintion 2 (Secrecy against chosen-plaintext attack (CPA))

An SKE $\Pi = (\mathsf{Gen}, \mathsf{Enc}, \mathsf{Dec})$ is CPA-secret if for every PPT CPA adversary A

$$\Pr[A \text{ wins}] - 1/2$$

is negligible.



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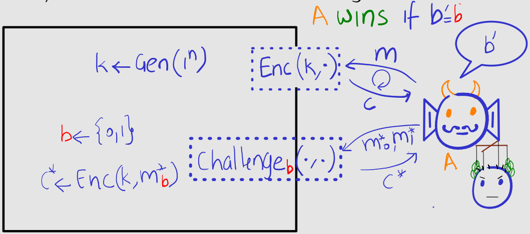

Exercise 4 (CPA model)

1. *Show that computational OTP (Lecture 3) is not CPA-secret*
2. *Prove that Construction 1 is CPA-secret*

PRFs IRL

- Coming up: theoretical construction, but inefficient for practice
- Practical PRFs: block ciphers like AES, which however only support certain key-sizes (128, 192, 256)
  - Supported by most libraries (e.g., OpenSSL, NaCl) and even implemented on modern processors (AES-NI)

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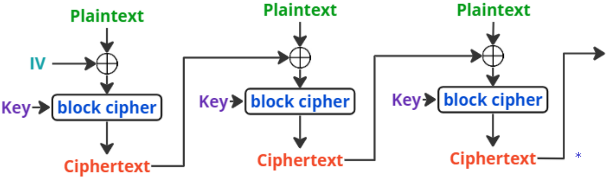

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*Credit: Wikipedia/Epachamo

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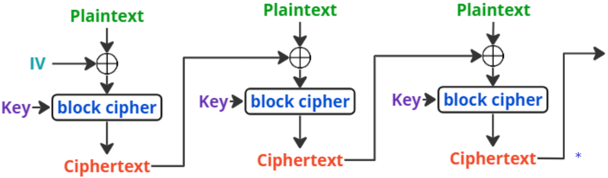
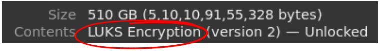


*

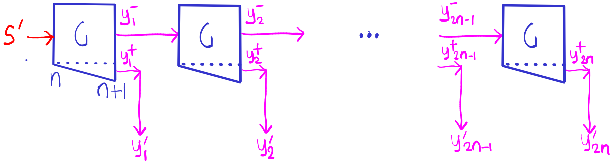
  - My laptop uses LUKS for disk encryption, which uses AES-XTS



Plan for this Lecture

1. Pseudo-Random Function (PRF)
2. Goldreich-Goldwasser-Micali (GGM) Construction

Let's Try to Construct a PRF



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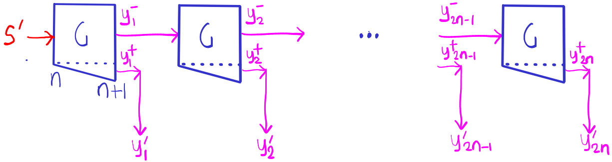

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  - Hint: Use length-doubling PRG

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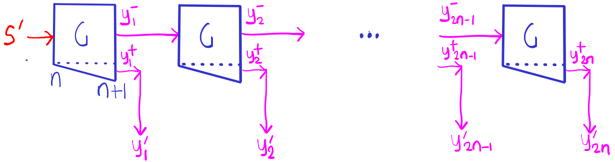

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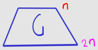
- How to reconcile the two requirements?
  - Hint: Use length-doubling PRG
  - Use binary tree instead of chain!

Tree-Based Construction from Length-Doubling PRG G

$n \to n^2$

Construction 2 (GGM PRF $\{F_k : \{0,1\}^n \to \{0,1\}^n\}_{k \in \{0,1\}^n}$)



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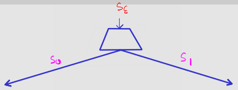

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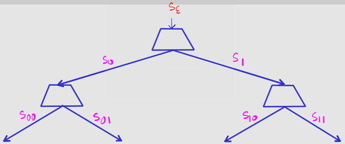

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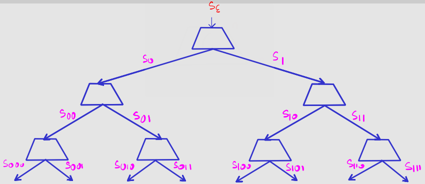

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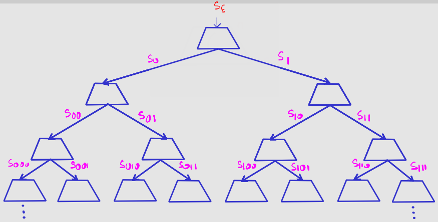

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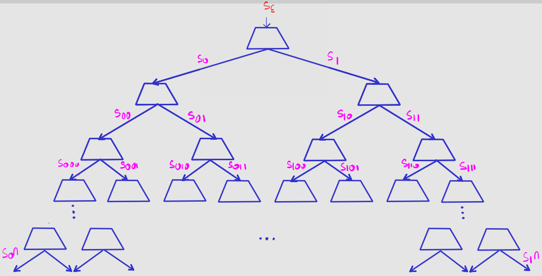

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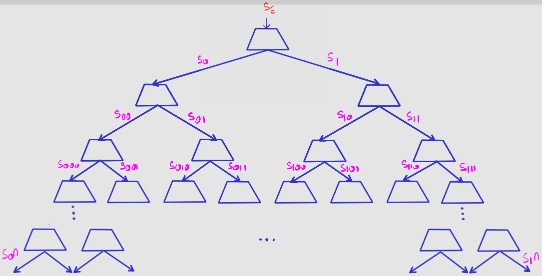

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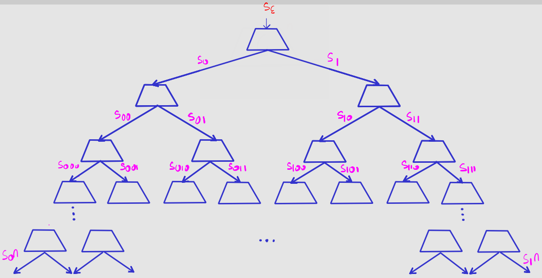

- *Define* $F_k(x) = s_x$ *with* $s_\varepsilon := k$

Exercise 5

1. *Write down the construction formally.*
2. *What if we use* d*-ary tree instead of binary tree?*

How do We Prove that Construction 2 is a PRF?...

Theorem 1

If G *is a length-doubling PRG, then Construction 2 is a PRF.*

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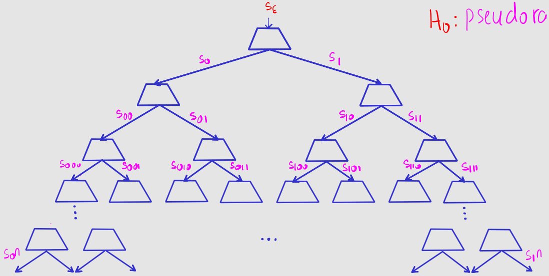

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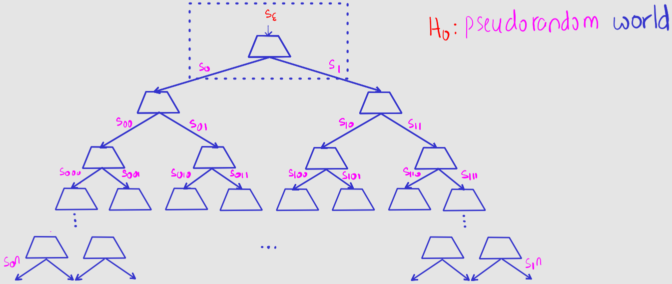

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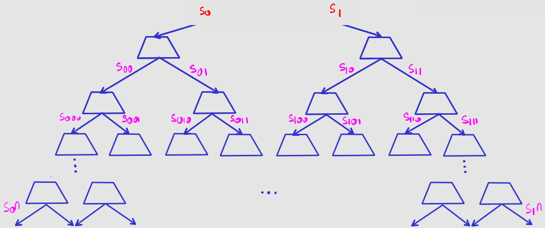

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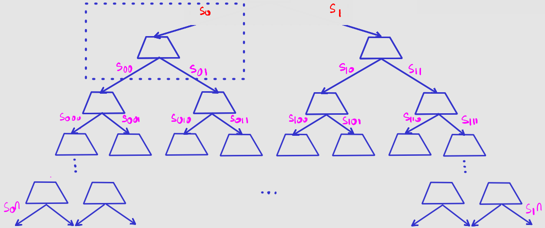

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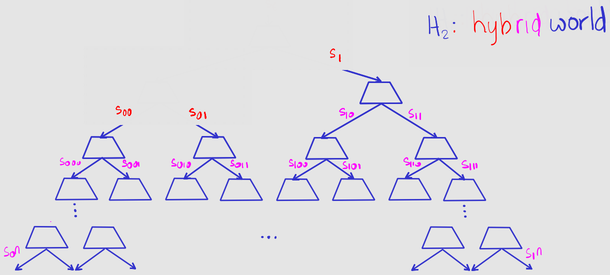

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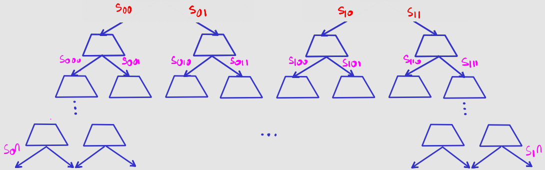

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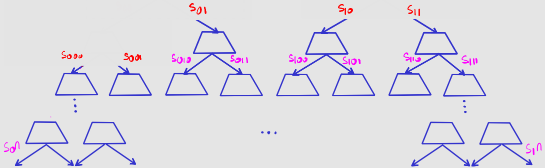

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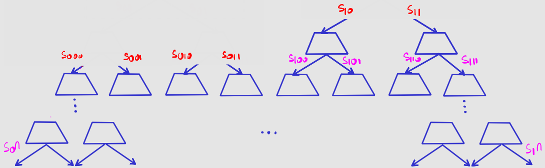

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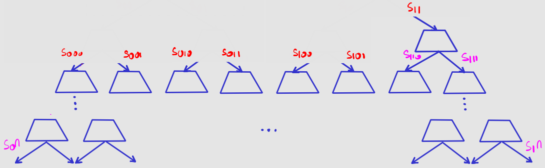

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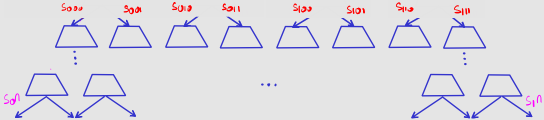

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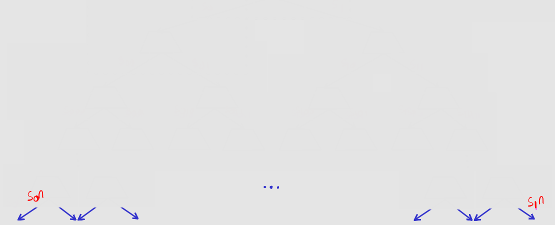

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Problem: exponential number of hybrids ×

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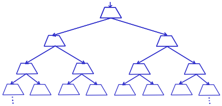
- Only switch values required to answer distinguisher's queries
  - Distinguisher makes at most Q queries $\Rightarrow$ number of switches per level of the tree is at most Q
- The hybrid worlds:
  - Each level $i \in [1, n]$ has *at most* Q hybrid worlds
  - Hybrid worlds at level $i \in [1, n]$ (think of $2^i \gg Q$):
    - $H_{i,0}, \cdots, H_{i,Q}$, where in $H_{i,q}$ the values used to answer first q queries are switched from pseudorandom to random

□

To Recap



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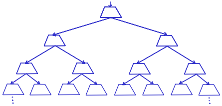
To Recap



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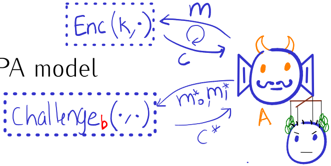

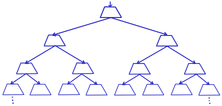

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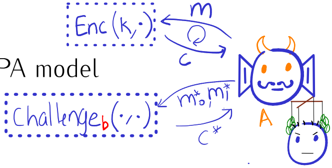
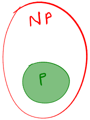
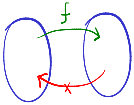


- Constructed a stateless SKE from PRF
    - It is actually secret in the stronger CPA model
- Other applications of PRFs
    - Authentication (coming up: Lecture 7)
    - Natural proofs: barrier to resolving the $\mathsf{P} \overset{?}{=} \mathsf{NP}$ question



Next Lecture

- Hardness vs. pseudorandomness
- One-way function and one-way permutation
- Hardcore predicates



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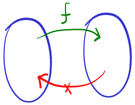

More Questions?

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