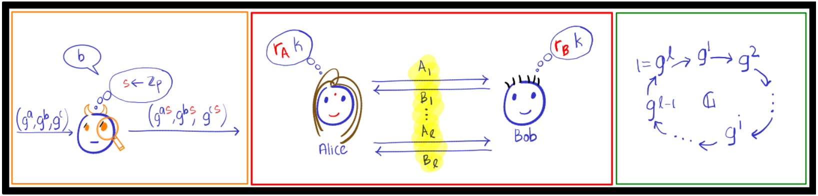

CS783: Theoretical Foundations of Cryptography

Lecture 8 (23/Aug/24)

Instructor: Chethan Kamath

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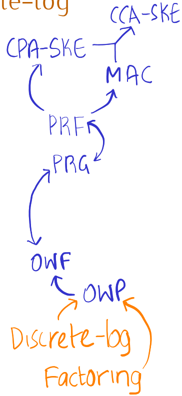
- Primitives encountered: PRG, PRF, OWF, OWP, PI hash, MAC
- Computational hardness assumptions: factoring, discrete-log



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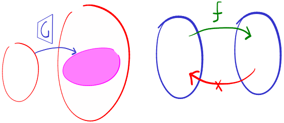
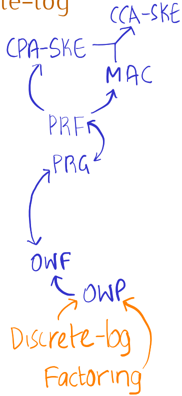


$$x \to x^2 \to x^{2^2} \to x^{2^3} \to \cdots$$
$$b_1 \quad b_2 \quad b_3$$



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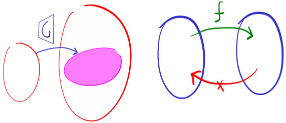

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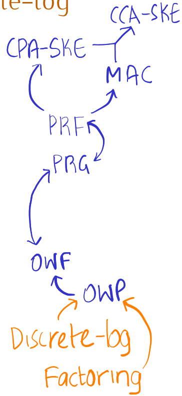

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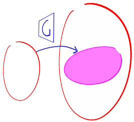

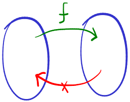

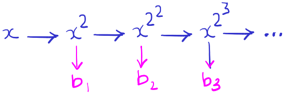

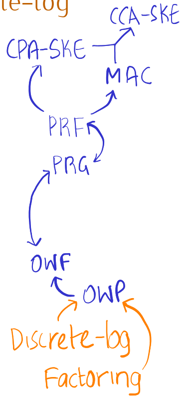

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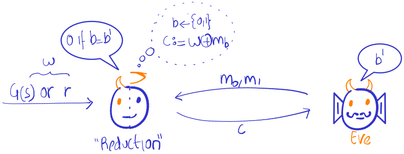
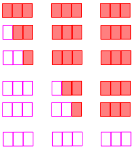
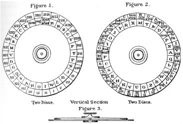


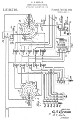


MODULE I

(Shared keys)

For a large part of history



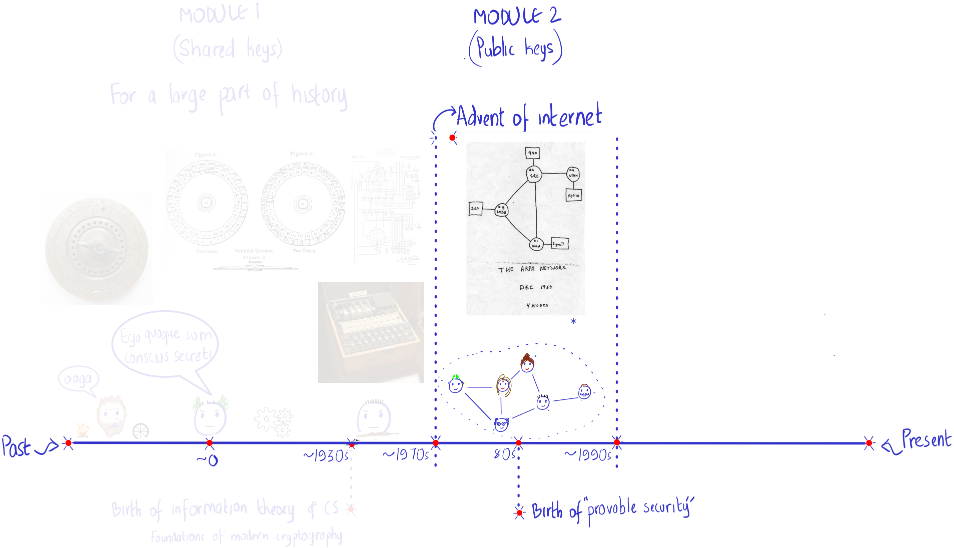


Birth of information theory & CS

Foundations of modern cryptography

Credit for images: Wikipedia (*User:DARPA)

This Module...



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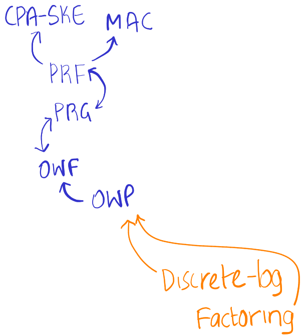

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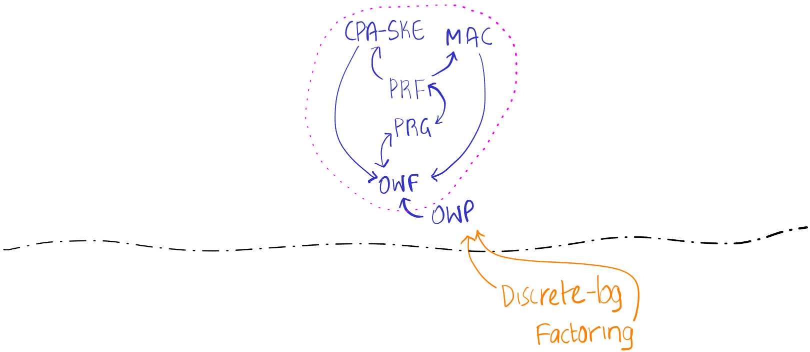

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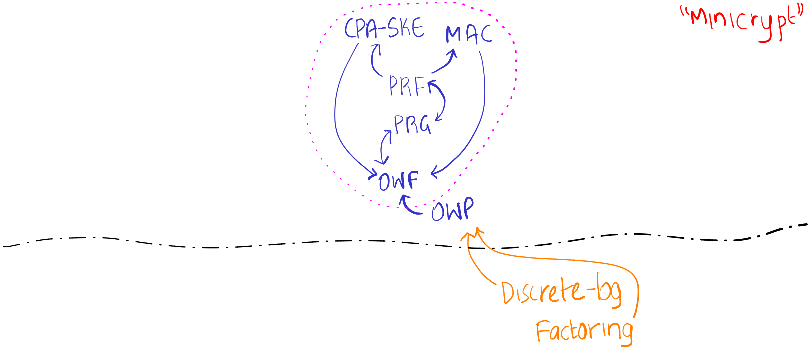

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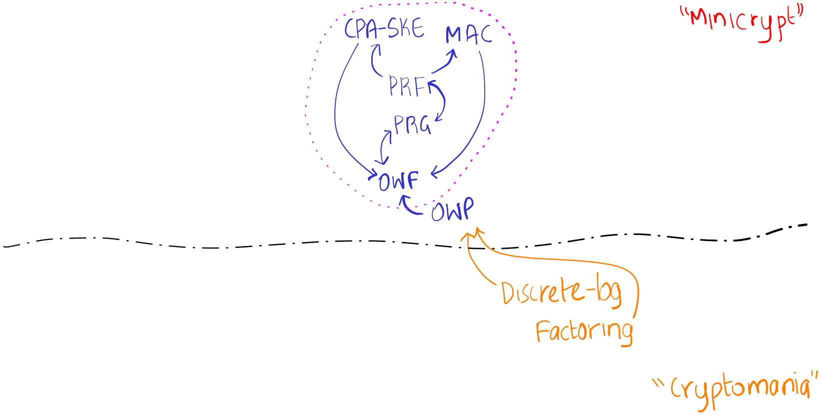

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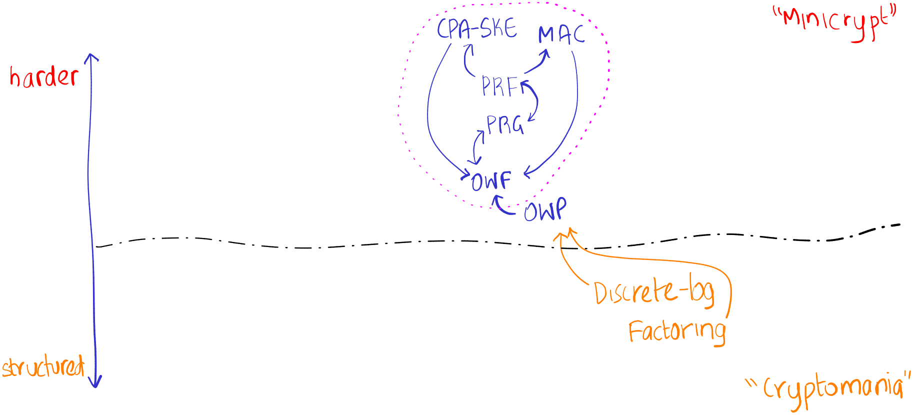

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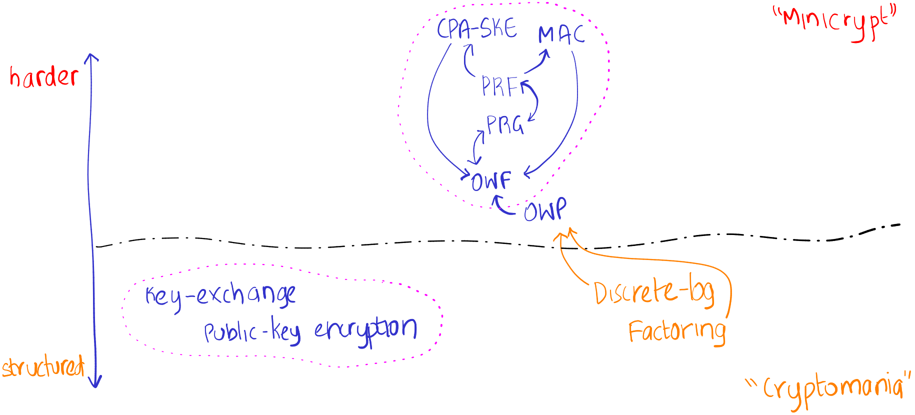

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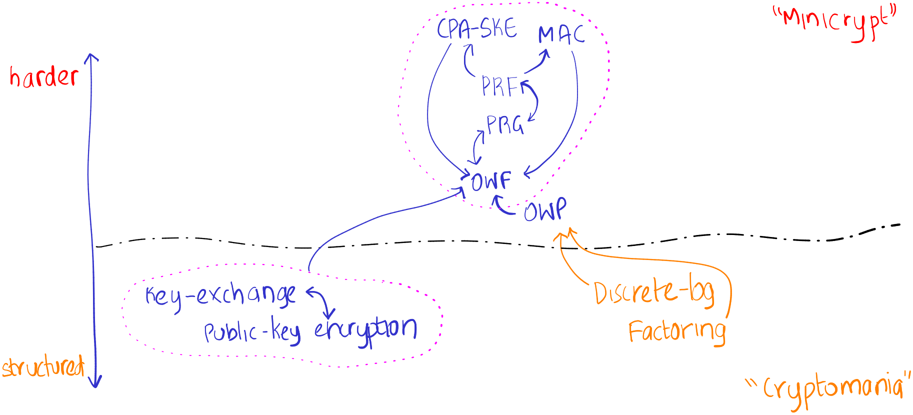

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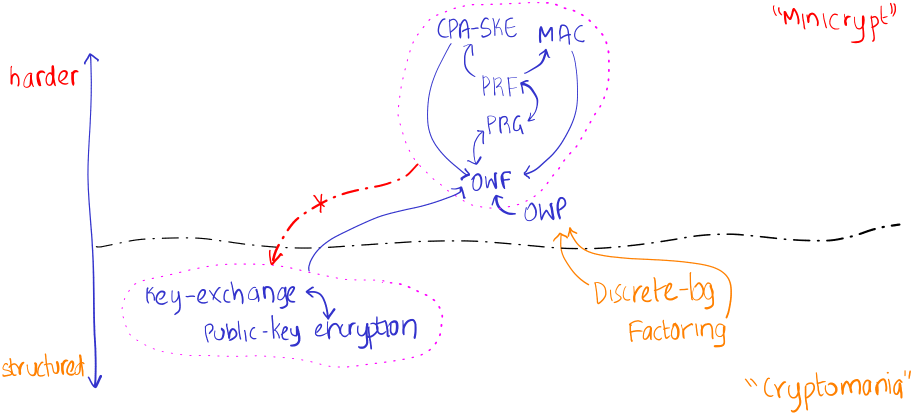

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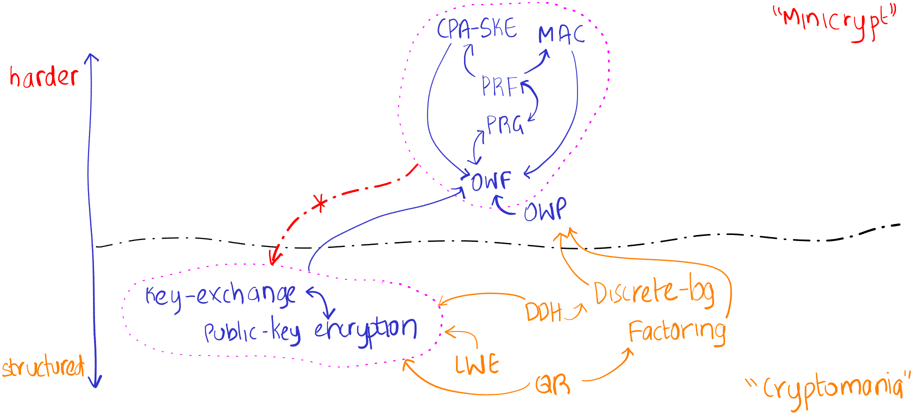

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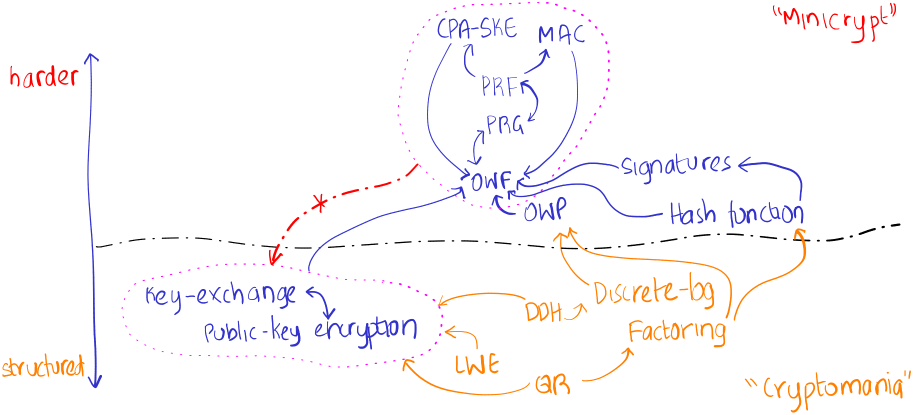

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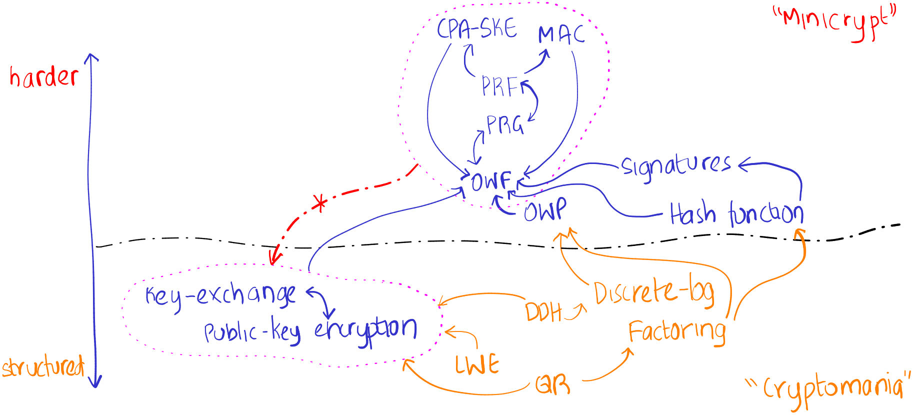

- Today we focus on Task 3: how does one establish a shared key in the first place?

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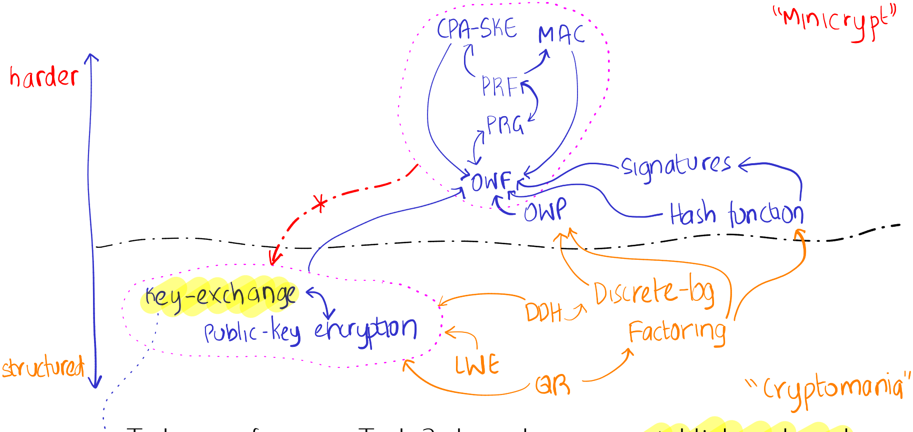

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Plan for This Lecture...

General *template*:

1. Identify the task

Key exchange

2. Come up with precise threat model M (a.k.a security model)
   - Adversary/Attack: What are the adversary's capabilities?
   - Security Goal: What does it mean to be secure?
3. Construct a scheme Π
4. Formally prove that Π in secure in model M

computational secrecy

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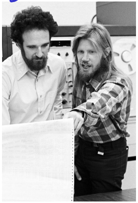

Credit for image: icour.fr

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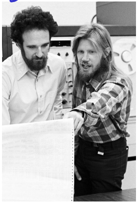
↳ Assumption = proof!



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Plan for This Lecture...

1. Key Exchange Protocol
2. Diffie-Hellman Key-Exchange Protocol
3. Exchanging Multiple Keys

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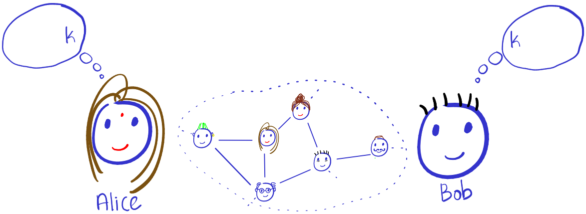
How To Establish a Shared Key in the First Place?



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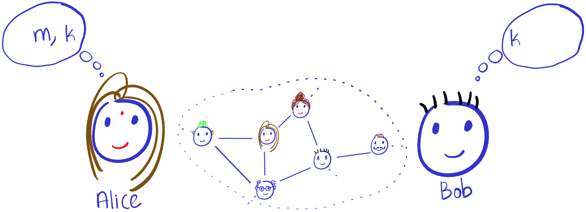

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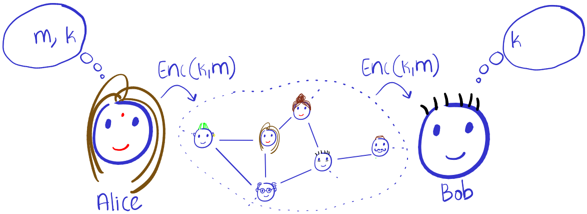

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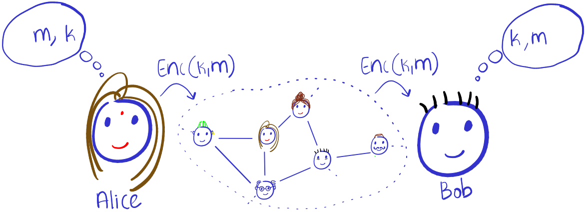

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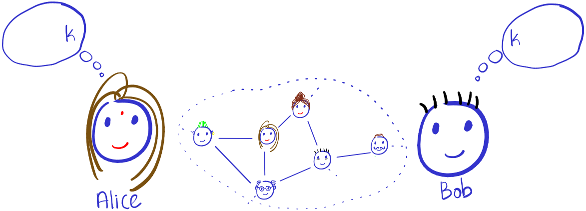

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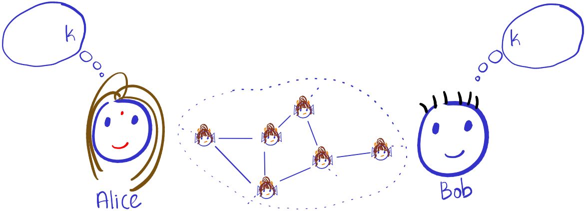

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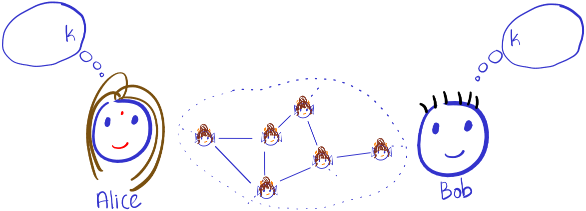

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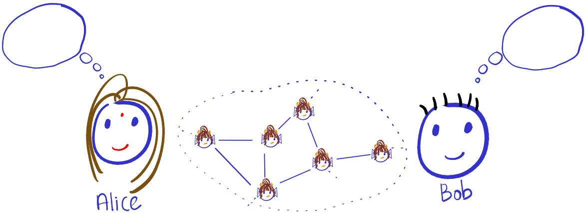

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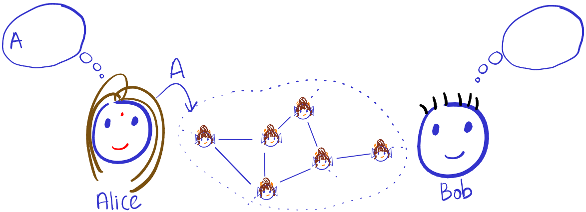

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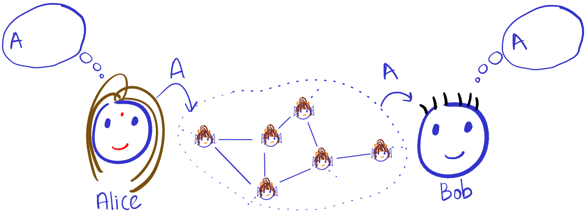

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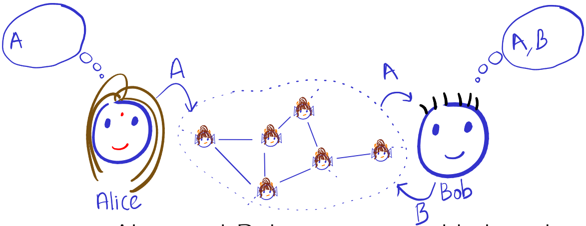

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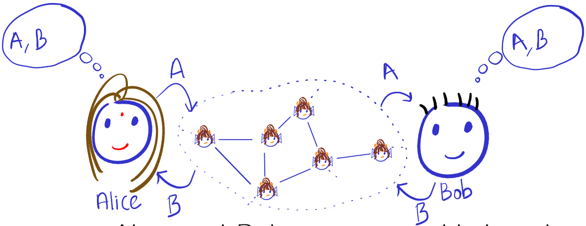

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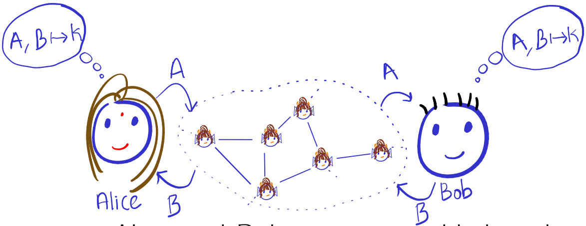

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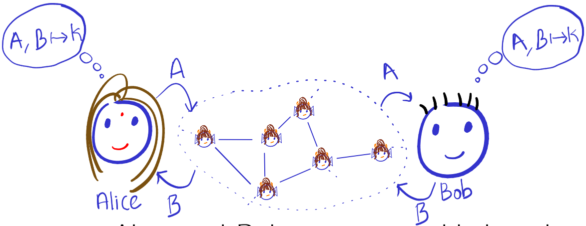

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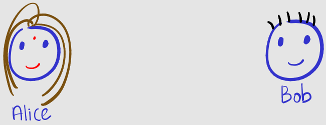
- Key Exchange IRL: HTTPs, TLS, SSH

Syntax of Key Exchange Protocol

Defintion 1 (Key Exchange Protocol)

A (two-party) key-exchange protocol Π *is a probabilistic protocol between two parties* A *and* B *at the end of which party* A *locally outputs* $k_A \in \{0,1\}^n$ *and party* B *locally outputs* $k_B \in \{0,1\}^n$.



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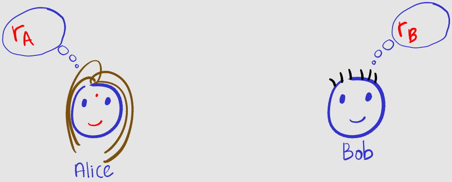

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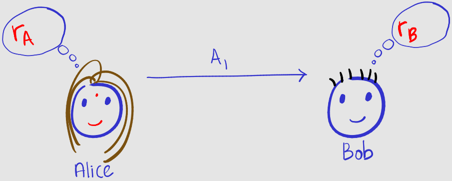

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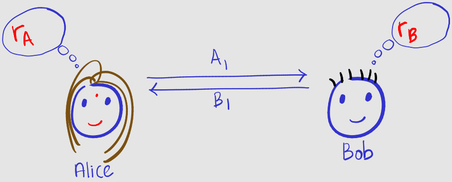

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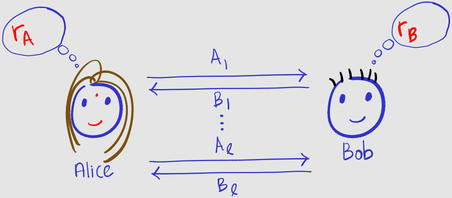

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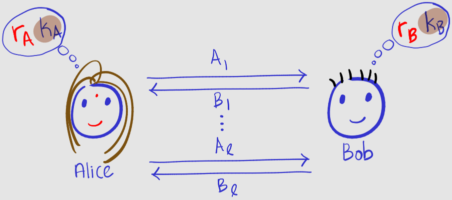

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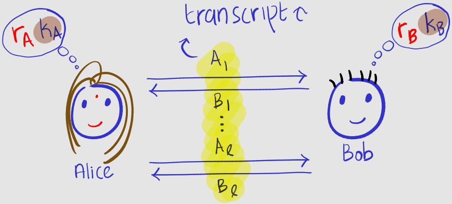

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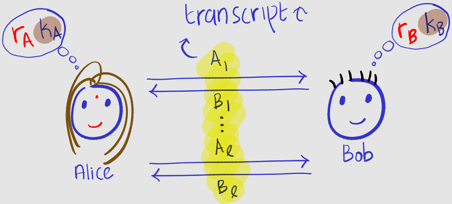

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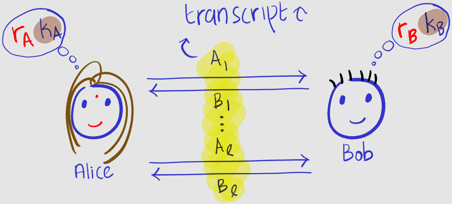

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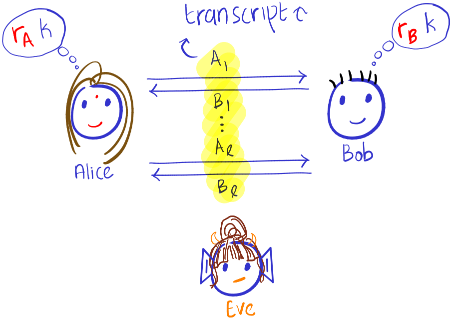
- Intuitively, what is the security requirement?



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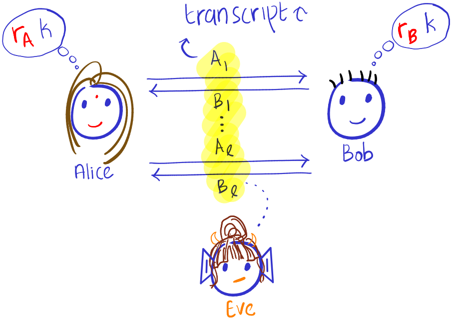

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Defintion 2 (Secrecy Against Eavesdroppers)

A key-exchange protocol Π is secret against eavesdroppers if for every PPT eavesdropper Eve the following is negligible.

$$\delta(n) := \left| \Pr_{(k,\tau)\leftarrow\Pi(1^n)}[\mathsf{Eve}(\tau, k) = 0] - \Pr_{\substack{(k,\tau)\leftarrow\Pi(1^n)\\ r\leftarrow\{0,1\}^n}}[\mathsf{Eve}(\tau, r) = 0] \right|$$

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Exercise 1

How can an unbounded eavesdropper Eve break secrecy?

Plan for this Lecture

But First Some Group Theory...

What are some properties of $(\{0,1\}^n, \oplus)$ we have exploited?

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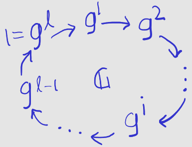

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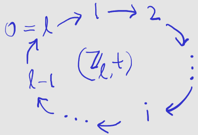

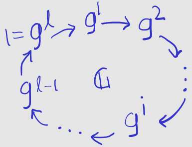

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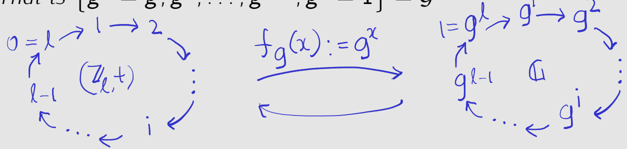

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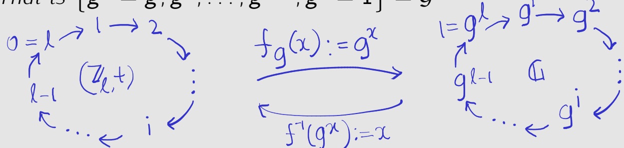

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But First Some Group Theory...

Exercise 2 (Lagrange's theorem)

Prove that the order of an element divides order of the (finite) group.

Exercise 3

For a group $\mathbb{G}$ *of order* ℓ *with generator* g*, show using group axioms that for all* $a, b \in \mathbb{Z}_\ell$*,* $(g^a)^b = g^{ab} = (g^b)^a$

Exercise 4

Prove that a prime-order group is cyclic. Are all cyclic groups of prime order?

Addition modulo prime p

$(\mathbb{Z}_p, +)$

$\{0, \cdots p-1\}$ $\quad$ $g_1 + g_2 := g_1 + g_2 \pmod{p}$

Some Examples of Groups

Addition modulo prime p

$(\mathbb{Z}_p, +)$

$\{0, \dots p-1\}$ $g_1 + g_2 := g_1 + g_2 \pmod{p}$

- order p
- cyclic

Addition modulo prime p

$0 = p \to 1 \to 2 \to \cdots \to i \to \cdots \to p-1 \to 0 = p$

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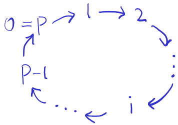

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Multiplication modulo $N = pq$

primes

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"RSA group"

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Finding order as hard as factoring N

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Multiplication modulo $N = pq$

primes

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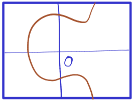

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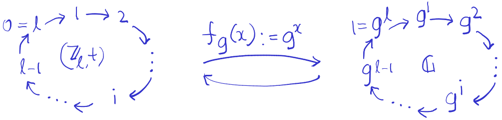
A Hard(?) Computational Problem Over Cyclic Groups...

- Recall our exp. map



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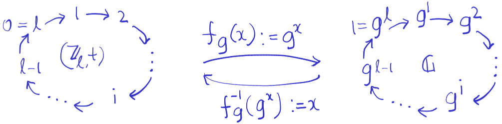

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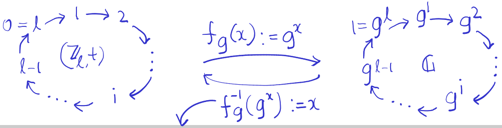

Defintion 5 (Discrete logarithm (DLog) problem in $\mathbb{G}$ w.r.to S)

- *Input:*
  1. $(\mathbb{G}, \ell, g)$ *sampled by a group sampler* $\mathsf{S}(1^n)$
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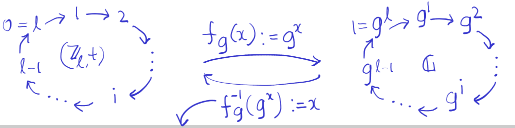

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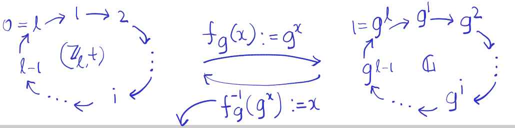

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The DLog assumption in $\mathbb{G}$ w.r.to S holds if solving the DLog problem in $\mathbb{G}$ w.r.to S is hard for all PPT inverters Inv. That is, for all Inv, the following is negligible:

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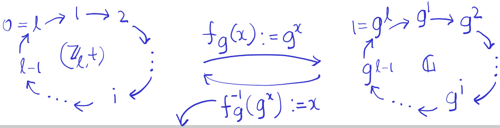

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A Hard(?) Computational Problem Over Cyclic Groups...

Addition modulo prime p

$0 = p \to 1 \to 2 \to \cdots \to i \to \cdots \to p-1 \to 0$

$(\mathbb{Z}_p, +)$

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- cyclic

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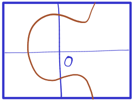

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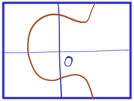

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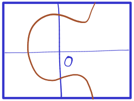

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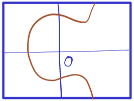

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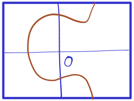

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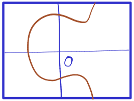

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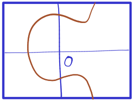

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Believed hard; no known sub-exponential algos!

Diffie-Hellman Key-Exchange Protocol

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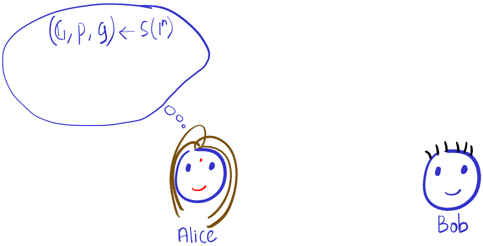

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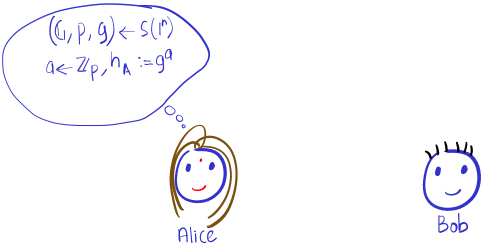

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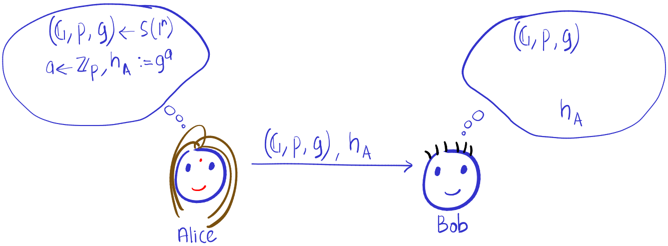

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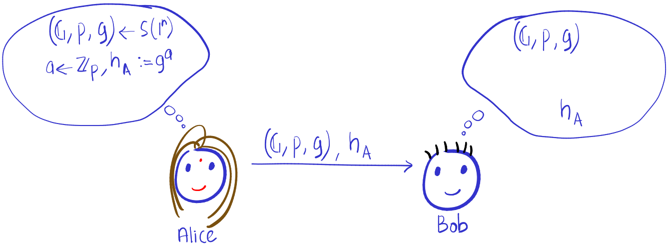

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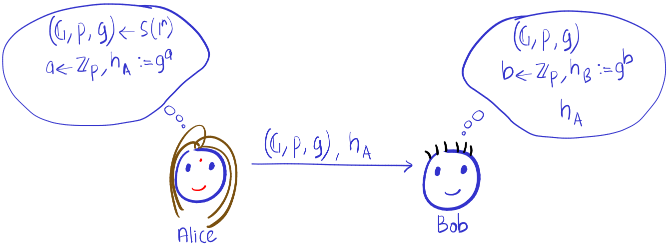

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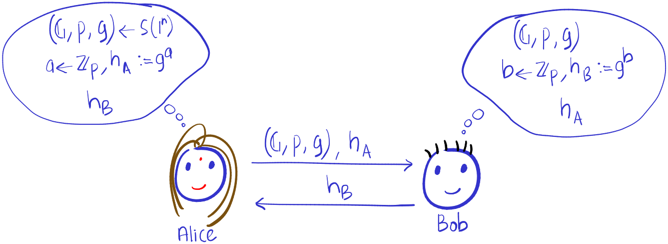

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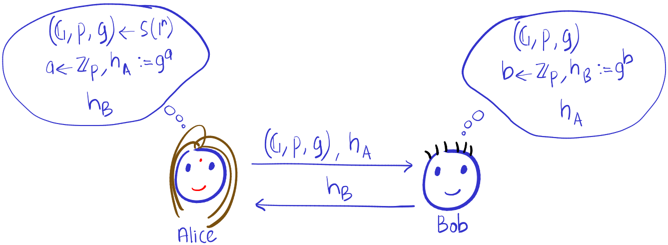

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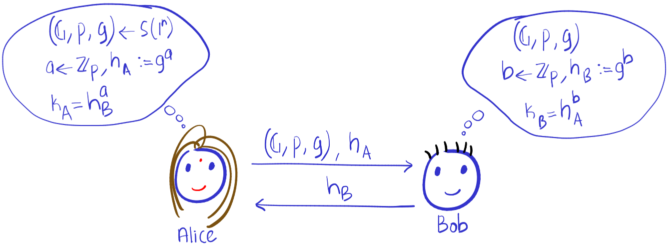

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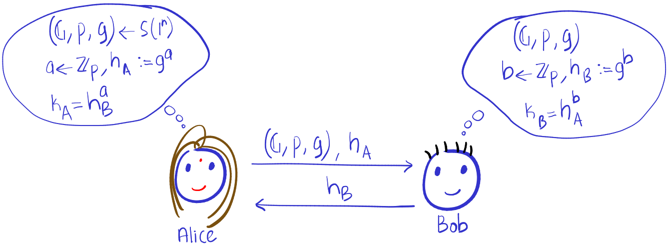

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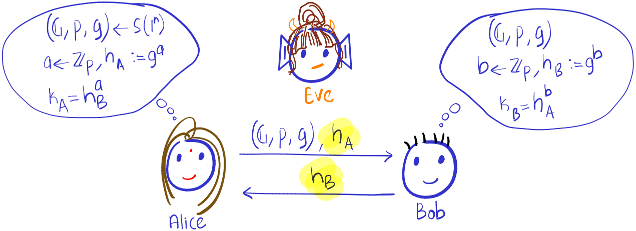
  3. Alice outputs $k_A := (h_B)^a$; Bob outputs $k_B := (h_A)^b$
- Correctness of key generation:

$$k_A = h_B^a = (g^b)^a = g^{ab} = (g^a)^b = h_A^b = k_B$$

When is it Secret Against Eavesdroppers?...



- What does Eve see? The transcript is $(h_A := g^a, h_B := g^b)$

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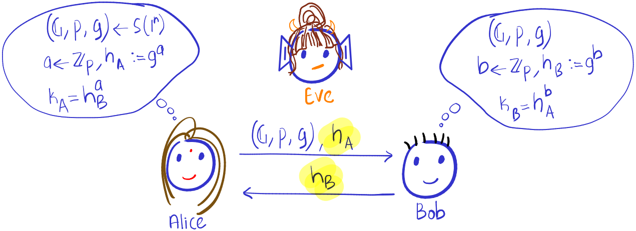

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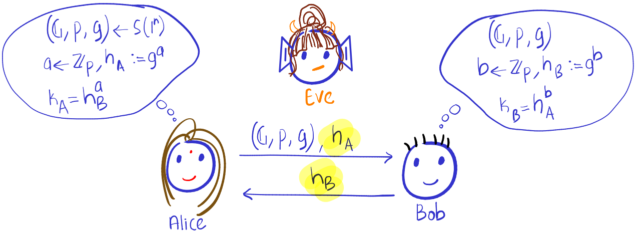

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Then Eve can invert h_A to get a and compute $k = h_B^a$

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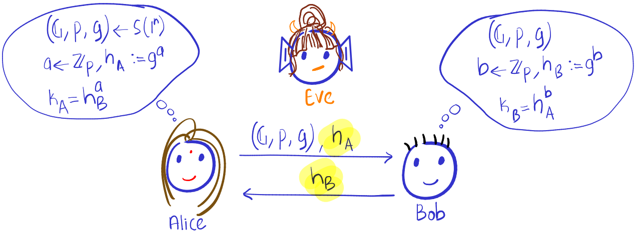

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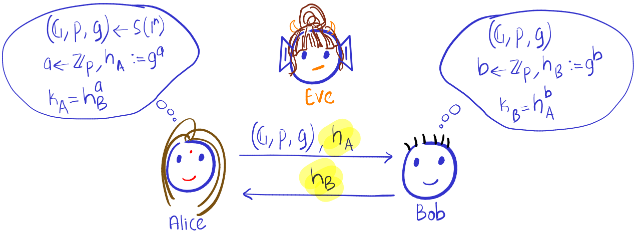

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What if DLog problem is easy over $\mathbb{G}$?

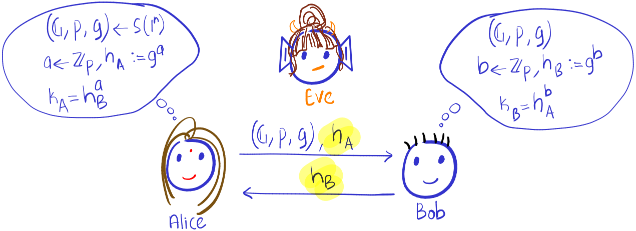
Then Eve can invert h_A to get a and compute $k = h_B^a$

Is DLog problem being hard sufficient?

No, what if Eve can compute g^{ab} given g^a and g^b?

- This is the "computational Diffie-Hellman" (CDH) problem

When is it Secret Against Eavesdroppers?...



- What does Eve see? The transcript is $(h_A := g^a, h_B := g^b)$
- What if DLog problem is easy over $\mathbb{G}$?
  - Then Eve can invert h_A to get a and compute $k = h_B^a$
- Is DLog problem being hard sufficient?
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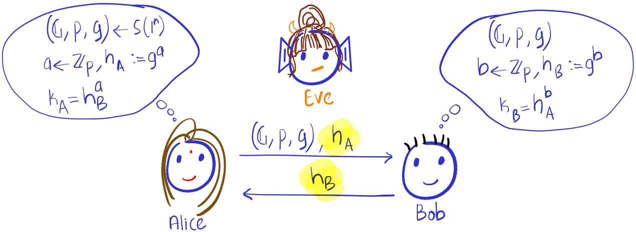

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Is CDH problem being hard sufficient?

What if Eve can distinguish g^{ab} from random group elements?

- There exist such groups!

When is it Secret Against Eavesdroppers?...

Assumption 2 (Decisional DH (DDH) assumption in in $\mathbb{G}$ w.r.to S)

The DDH assumption holds in $\mathbb{G}$ w.r.to S if for all PPT distinguishers D, the following is negligible:

$$\left| \Pr_{\substack{(\mathbb{G},\ell,g)\leftarrow \mathsf{S}(1^n)\\ a,b\leftarrow \mathbb{Z}_\ell}}[\mathsf{D}(g^a, g^b, g^{ab}) = 0] - \Pr_{\substack{(\mathbb{G},\ell,g)\leftarrow \mathsf{S}(1^n)\\ a,b,r\leftarrow \mathbb{Z}_\ell}}[\mathsf{D}(g^a, g^b, g^{r}) = 0] \right|$$

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Theorem 1

Diffie-Hellman key-exchange is secret against eavesdroppers under the DDH assumption in $\mathbb{G}$ w.r.to S.

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Secrecy requirement is same as the assumption! □

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Exercise 5

But I did slightly cheat! Figure out where.

Where is DDH Assumption Known to Hold?

Addition modulo prime p

$0 = p \to 1 \to 2 \to \cdots \to i \to \cdots \to p-1 \to 0$

$(\mathbb{Z}_p, +)$

$\{0, \ldots, p-1\}$ $\quad g_1 + g_2 := g_1 + g_2 \pmod p$

- order p
- cyclic

Multiplication modulo prime p

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Multiplication modulo $N = pq$

primes

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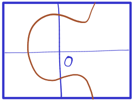
$\{1, \ldots, p-1, p+1, \ldots, 2p-1, \ldots, pq-1\}$ $\quad g_1 \cdot g_2 := g_1 \cdot g_2 \pmod N$

- order $(p-1)(q-1)$
- not cyclic

Elliptic curves modulo prime p

$(E, +)$ "curve points addition"

solutions to $y^2 = x^3 + Ax + B \pmod p$



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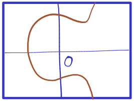

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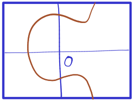

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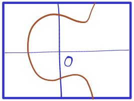

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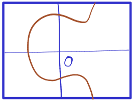

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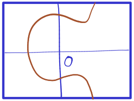

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Believed hard

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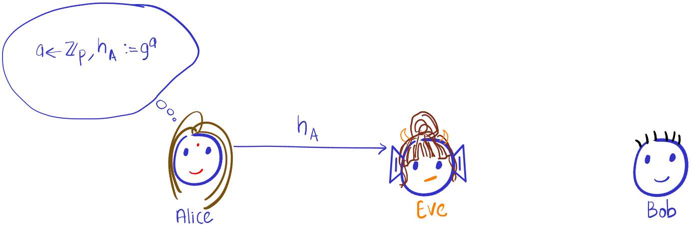
  - Recall that active Eve can intercept/tamper messages

What About Secrecy Against Active Eve?



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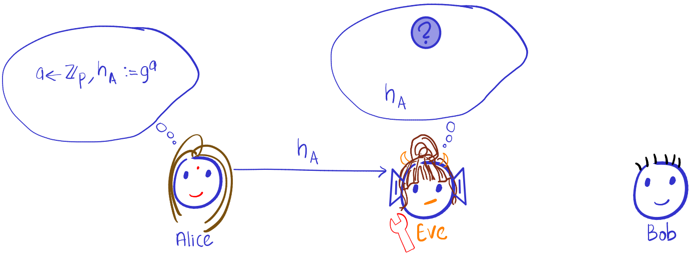

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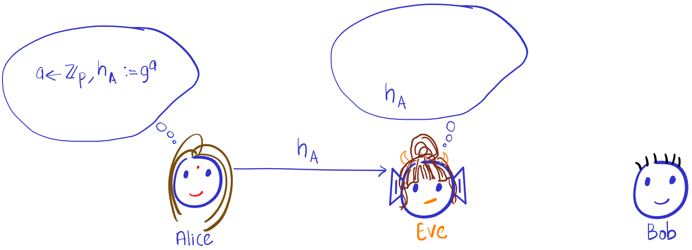

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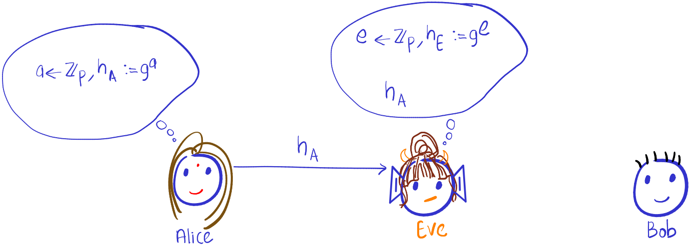

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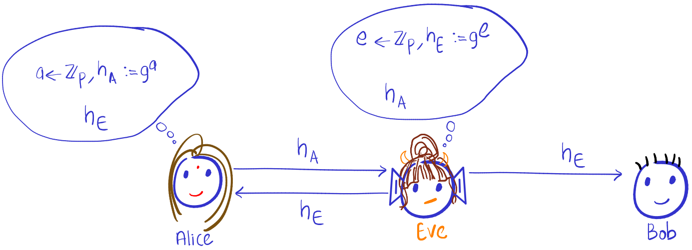

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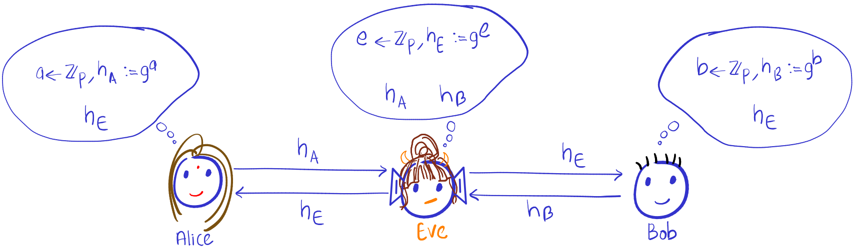

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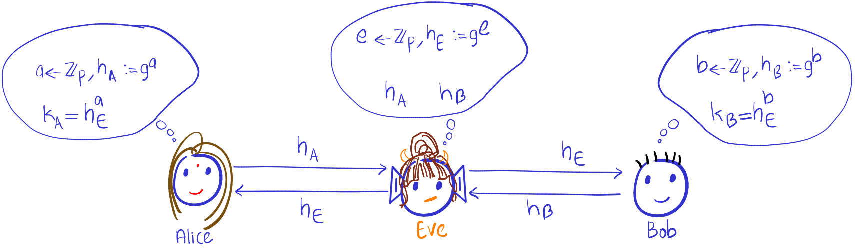

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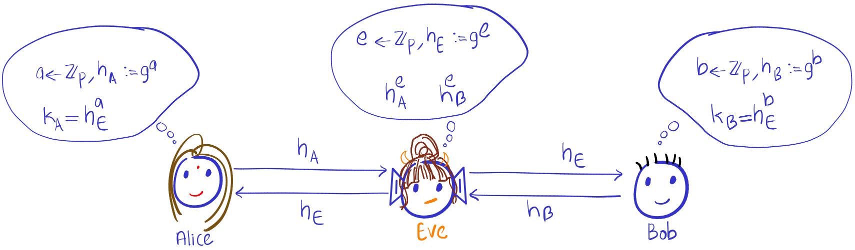

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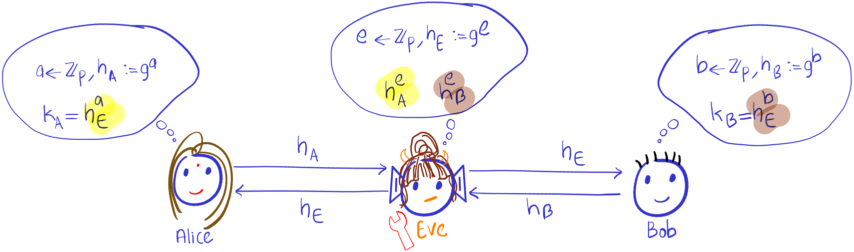

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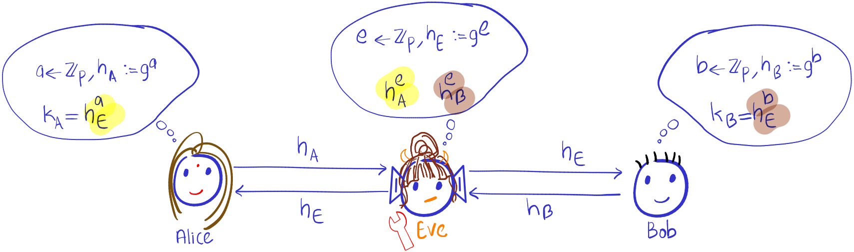

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⚠ Insecure against active adversary

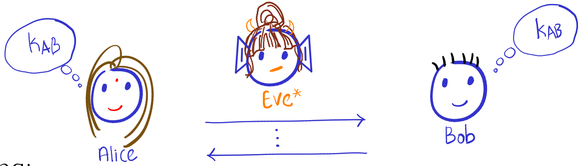
Plan for this Lecture

1 Key Exchange Protocol

2 Diffie-Hellman Key-Exchange Protocol

3 Exchanging Multiple Keys

Let's Try Exchanging Multiple Keys



- The setting:
  - Alice and Bob want to establish a shared key $k_{AB} \in \{0,1\}^n$ in presence of an *eavesdropper* Eve*

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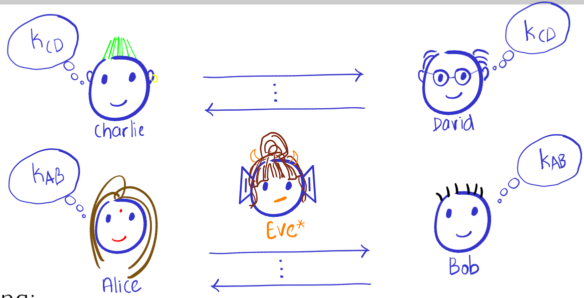

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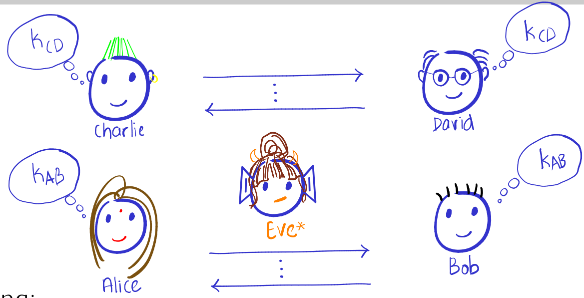

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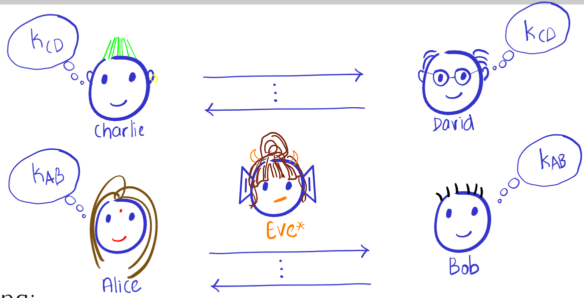

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- In general: t pairs of parties, t pairwise shared keys
- Secrecy: the t shared keys should be indistinguishable to Eve* from t random keys (given all transcripts)
- Solution: run t instances of DH key-exchange protocol
  - Can use same $(\mathbb{G}, p, g)$ across instances

Why is it Secret?

Theorem 2

Multiple instances of Diffie-Hellman key-exchange is secret against eavesdroppers under the DDH assumption in $\mathbb{G}$ w.r.to S.

Proof sketch.

Real world $(\mathbb{G}, p, g)$ (g^a, g^b, g^{ab})

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Why is it Secret? HYBRID ARGUMENT, OF COURSE!

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Random world H_t $(\mathbb{G}, p, g)$ $(g^{a_1}, g^{b_1}, g^{r_1})$ $(g^{a_2}, g^{b_2}, g^{r_2})$ $\cdots$ $(g^{a_i}, g^{b_i}, g^{r_i})$ $\cdots$ $(g^{a_t}, g^{b_t}, g^{r_t})$

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Why is it Secret? HYBRID ARGUMENT, OF COURSE!

Theorem 2

Multiple instances of Diffie-Hellman key-exchange is secret against eavesdroppers under the DDH assumption in $\mathbb{G}$ w.r.to S.

Proof sketch.

DDH real vs random

Real world H_0 $(\mathbb{G}, p, g)$ $(g^{a_1}, g^{b_1}, g^{a_1 b_1})$ $(g^{a_2}, g^{b_2}, g^{a_2 b_2})$ $\cdots$ $(g^{a_i}, g^{b_i}, g^{a_i b_i})$ $\cdots$ $(g^{a_t}, g^{b_t}, g^{a_t b_t})$

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$\vdots$

Hybrid H_i $(\mathbb{G}, p, g)$ $(g^{a}, g^{b}, g^{r_1})(g^{a_2}, g^{b_2}, g^{r_2}) \cdots (g^{a_i}, g^{b_i}, g^{r_i}) \cdots (g^{a_t}, g^{b_t}, g^{a_t b_t})$

$\vdots$

Random world H_t $(\mathbb{G}, p, g)$ $(g^{a_1}, g^{b_1}, g^{r_1})(g^{a_2}, g^{b_2}, g^{r_2}) \cdots (g^{a_i}, g^{b_i}, g^{r_i}) \cdots (g^{a_t}, g^{b_t}, g^{r_t})$

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- Hybrid argument with $t+1$ hybrids $H_0, \ldots, H_t$:
  - All keys real in H_0; all keys random in H_t
  - In hybrid world H_i, the first i keys are random and the rest real
  - Hybrids H_i and H_{i+1} indistinguishable by DDH assumption $\square$

Theorem 2 ← Loss in distinguishing advantage: $1/t$!

Multiple instances of Diffie–Hellman key-exchange is secret against eavesdroppers under the DDH assumption in $\mathbb{G}$ w.r.to S.

Proof sketch.

DDH real vs random

Real world H_0	$(\mathbb{G}, p, g)$	$(g^{a_1}, g^{b_1}, g^{a_1 b_1})$	$(g^{a_2}, g^{b_2}, g^{a_2 b_2})$	$\cdots$	$(g^{a_i}, g^{b_i}, g^{a_i b_i})$	$\cdots$	$(g^{a_t}, g^{b_t}, g^{a_t b_t})$
Hybrid H_1	$(\mathbb{G}, p, g)$	(g^{a}, g^{b}, g^{r_1})	$(g^{a_2}, g^{b_2}, g^{a_2 b_2})$	$\cdots$	$(g^{a_i}, g^{b_i}, g^{a_i b_i})$	$\cdots$	$(g^{a_t}, g^{b_t}, g^{a_t b_t})$
Hybrid H_2	$(\mathbb{G}, p, g)$	(g^{a}, g^{b}, g^{r_1})	$(g^{a_2}, g^{b_2}, g^{r_2})$	$\cdots$	$(g^{a_i}, g^{b_i}, g^{a_i b_i})$	$\cdots$	$(g^{a_t}, g^{b_t}, g^{a_t b_t})$
$\vdots$					$\vdots$		
Hybrid H_i	$(\mathbb{G}, p, g)$	(g^{a}, g^{b}, g^{r_1})	$(g^{a_2}, g^{b_2}, g^{r_2})$	$\cdots$	$(g^{a_i}, g^{b_i}, g^{r_i})$	$\cdots$	$(g^{a_t}, g^{b_t}, g^{a_t b_t})$
$\vdots$					$\vdots$		
Random world H_t	$(\mathbb{G}, p, g)$	$(g^{a_1}, g^{b_1}, g^{r_1})$	$(g^{a_2}, g^{b_2}, g^{r_2})$	$\cdots$	$(g^{a_i}, g^{b_i}, g^{r_i})$	$\cdots$	$(g^{a_t}, g^{b_t}, g^{r_t})$

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But We Can Do Better! Random Self-Reducibility...

- Random self-reducibility for DDH over $(\mathbb{G}, p, g)$:
  1. Given instance of DDH $\mapsto$ random instance of DDH
  2. Solve given instance of DDH $\Leftarrow$ solve random instance of DDH

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Claim 1

The DDH problem over $\mathbb{G}$ is random self-reducible

Proof. $\exists D'$ against given instance $\Leftarrow$ $\exists D$ against random instance.

$\square$

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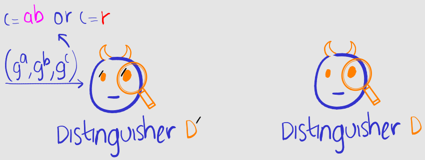

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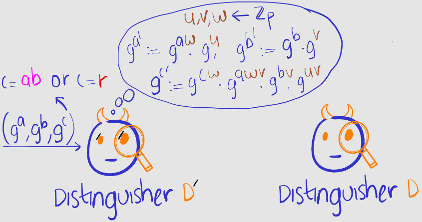

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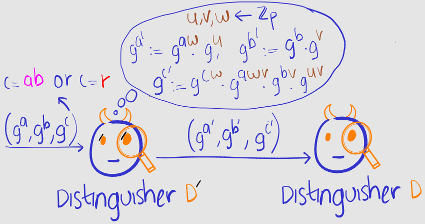

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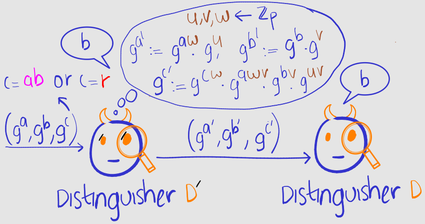

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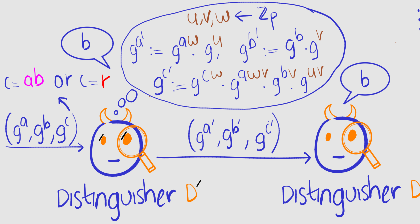
Proof. $\exists D'$ against given instance $\Leftarrow$ $\exists D$ against random instance.



Analysis

- case 1) $c = ab$ (real DDH) $\Rightarrow$
  $c' = abw + awv + bv + uv = (aw+u)(b+v) = a'b'$

$\square$

But We Can Do Better! Random Self-Reducibility...

- Random self-reducibility for DDH over $(\mathbb{G}, p, g)$:
  1. Given instance of DDH $\mapsto$ random instance of DDH
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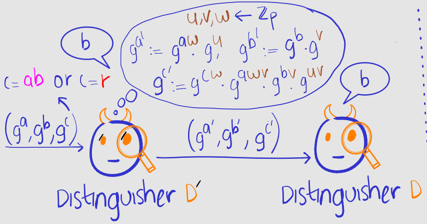
The DDH problem over $\mathbb{G}$ is random self-reducible

Proof. $\exists$D$'$ against given instance $\Leftarrow$ $\exists$D against random instance.



Analysis

- case 1) $c = ab$ (real DDH) $\Rightarrow$ real DDH $\Uparrow$

  $c' = abw + awv + bv + uv = (aw+u)(b+v) = a'b'$

$\square$

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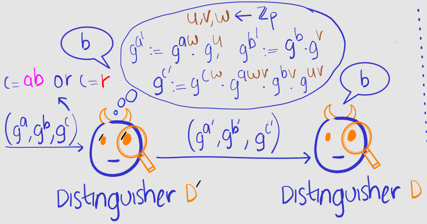

Analysis

real DDH $\Uparrow$

- case I) $c = ab$ (real DDH) $\Rightarrow$

  $c' = abw + awv + bv + uv = (aw+u)(b+v) = a'b'$

- case II) $c = r$ (random DDH) $\Rightarrow$

  $c' = rw + awv + bv + uv$

$\square$

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- Random self-reducibility for DDH over $(\mathbb{G}, p, g)$:
  1. Given instance of DDH $\mapsto$ random instance of DDH
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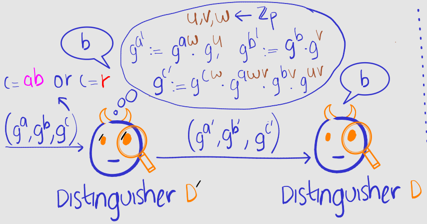

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  $c' = rw + awv + bv + uv$ $\Rightarrow$ random DDH

$\square$

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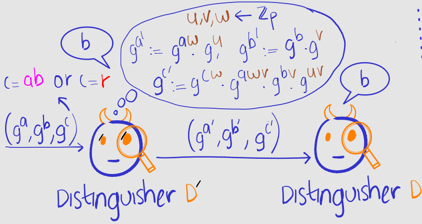

Analysis

- case I) $c = ab$ (real DDH) $\Rightarrow$ real DDH $\Uparrow$
  $c' = abw + awv + bv + uv = (aw+u)(b+v) = a'b'$
- case II) $c = r$ (random DDH) $\Rightarrow$
  $c' = rw + awv + bv + uv \Rightarrow$ random DDH
- Since D' mimics D, it distinguishes with same probability as D

$\square$

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- Random self-reducibility for DDH over $(\mathbb{G}, p, g)$:
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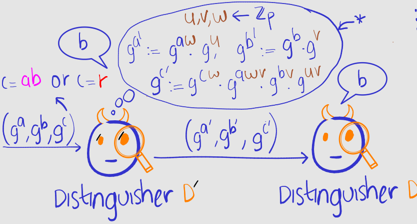

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Exercise 6

Is the DLog problem random self-reducible? What about CDH?

But We Can Do Better! Random Self-Reducibility...

Theorem 3 ← no loss in distinguishing advantage!

Multiple instances of Diffie-Hellman key-exchange is secret against eavesdroppers under the DDH assumption in $\mathbb{G}$ w.r.to S.

Proof, using random self-reducibility.

$\square$

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□

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DDH

Key exchange

Distinguisher D'

Eve*

$\square$

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DDH

Key exchange

$c = ab$ or $c = r$

$(g^a, g^b, g^c) \longrightarrow$

Distinguisher D'

Eve*

$\square$

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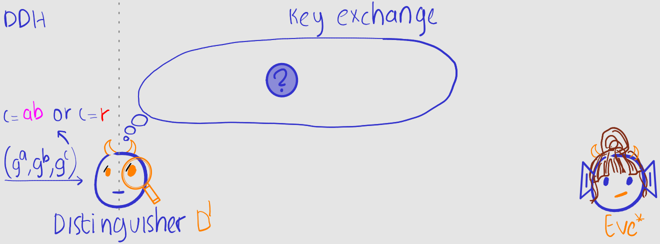

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DDH

Key exchange

$\{u_i, v_i, w_i\}_{i\in[1,t]} \leftarrow \mathbb{Z}_p$

$(g^a, g^b, g^c) \xrightarrow[*]{u_i, v_i, w_i} (g^{a_i'}, g^{b_i'}, g^{c_i'})$

$c = ab$ or $c = r$

(g^a, g^b, g^c)

Distinguisher D'

Eve*

□

Theorem 3 ← no Loss in distinguishing advantage!

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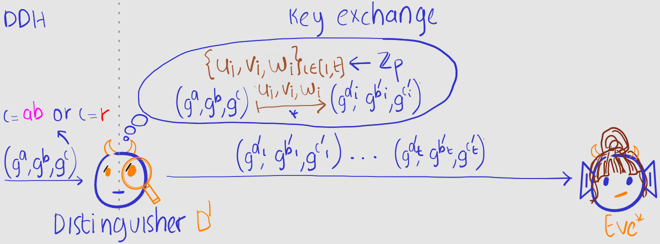

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But We Can Do Better! Random Self-Reducibility...

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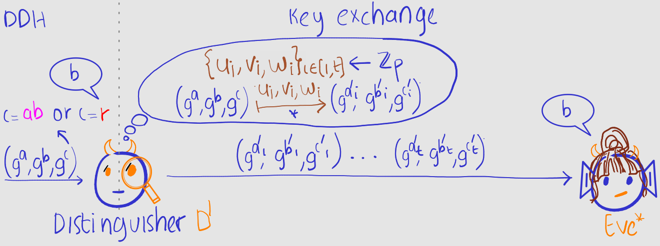

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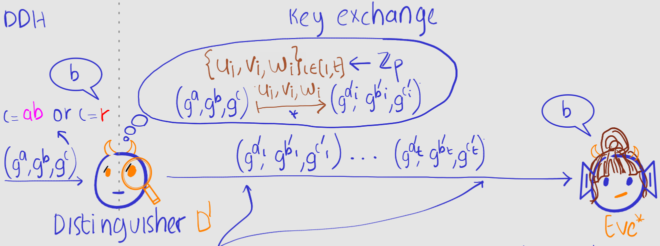
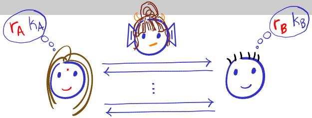


Analysis: If $c=ab$, then all t instances real; If $c=r$, then all t instances random □

To Recap Today's Lecture



- Task 3: sharing key in presence of eavesdropper
  - Modelled key exchange setting and security

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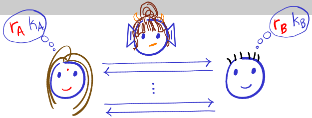

- Task 3: sharing key in presence of eavesdropper
  - Modelled key exchange setting and security
  - Diffie-Hellman key exchange protocol
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    - First proof using hybrid argument
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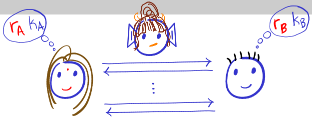

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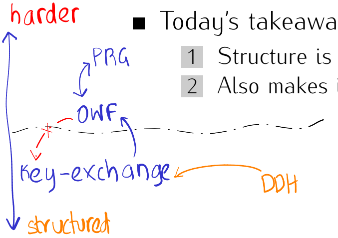

- Today's takeaway: structure vs hardness
  1. Structure is useful for protocol design and proofs
  2. Also makes it susceptible to algorithms

Next Lecture

- Task 4: public-key encryption (PKE)
  - Syntax and security
  - Relationship with key-exchange
  - Basic number theory
  - Goldwasser-Micali PKE

References

1. [KL14, Chapter 11] for details on this lecture.
2. Read the seminal paper by Diffie and Hellman [DH76] for a description of the namesake key-exchange.
3. Boneh's survey [Bon98] is an excellent source on the DDH problem.
4. Random self-reducibility was first studied in [BM84]. Refer to [FF93] to read more. RSR of the DDH problem were studied in [Sta96, NR04].

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