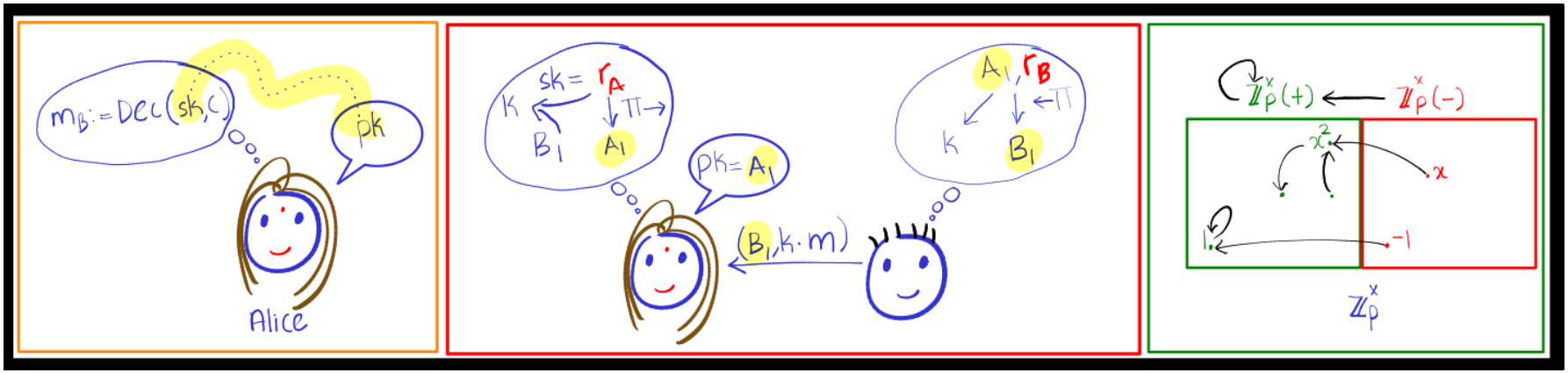


# CS783: Theoretical Foundations of Cryptography

Lecture 9 (27/Aug/24)

Instructor: Chethan Kamath

# Recall from Last Lecture

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  - Modelled key exchange setting and security

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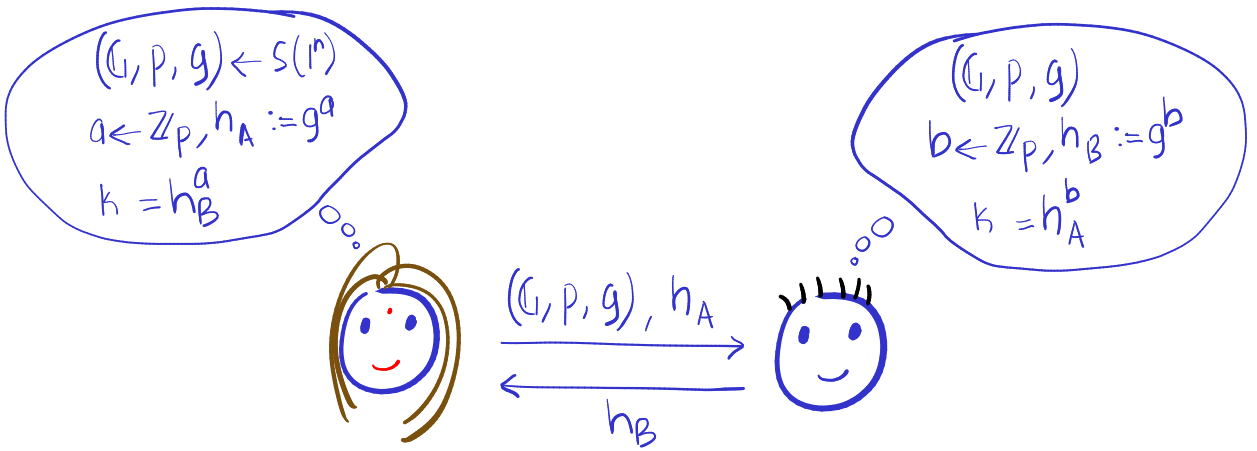


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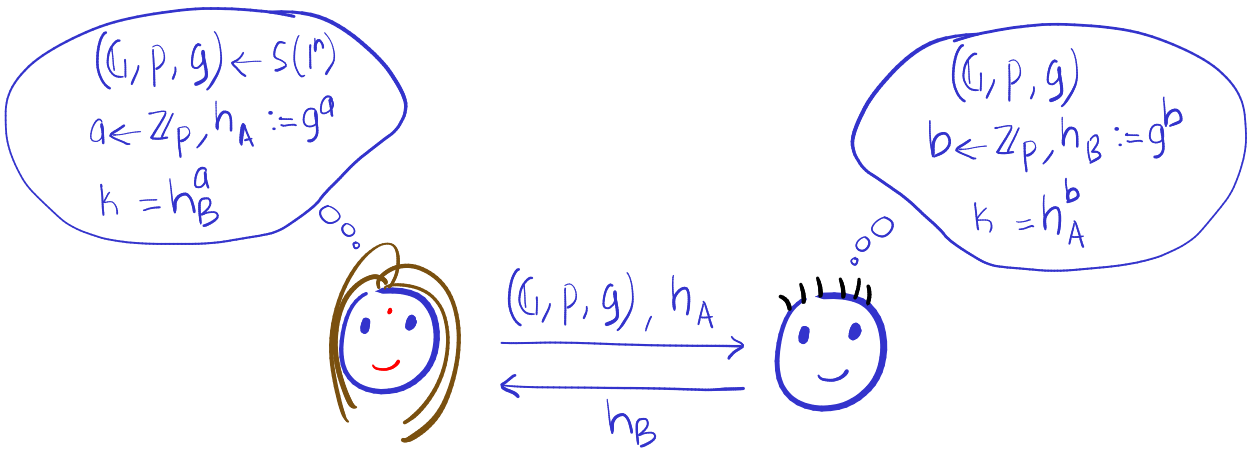


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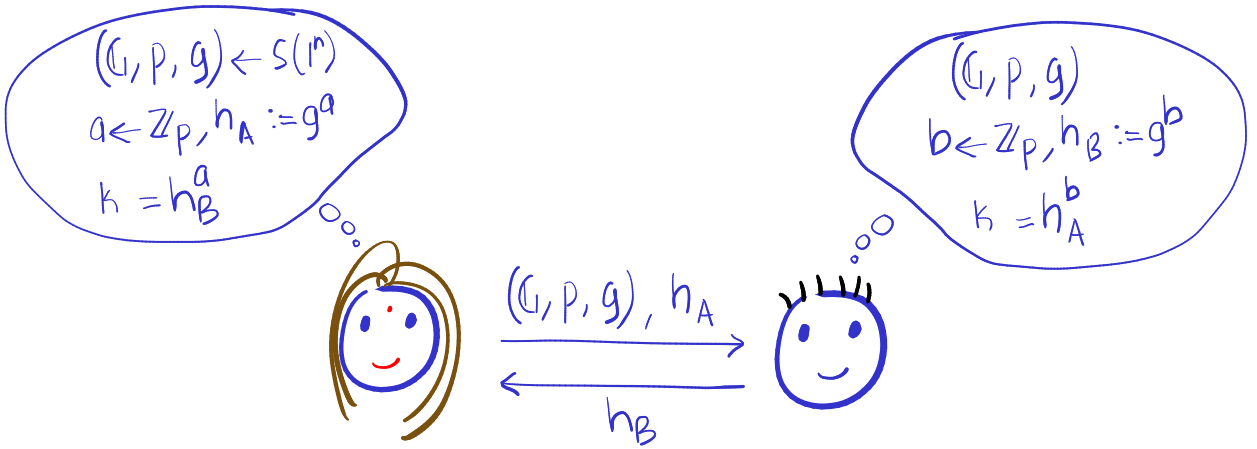


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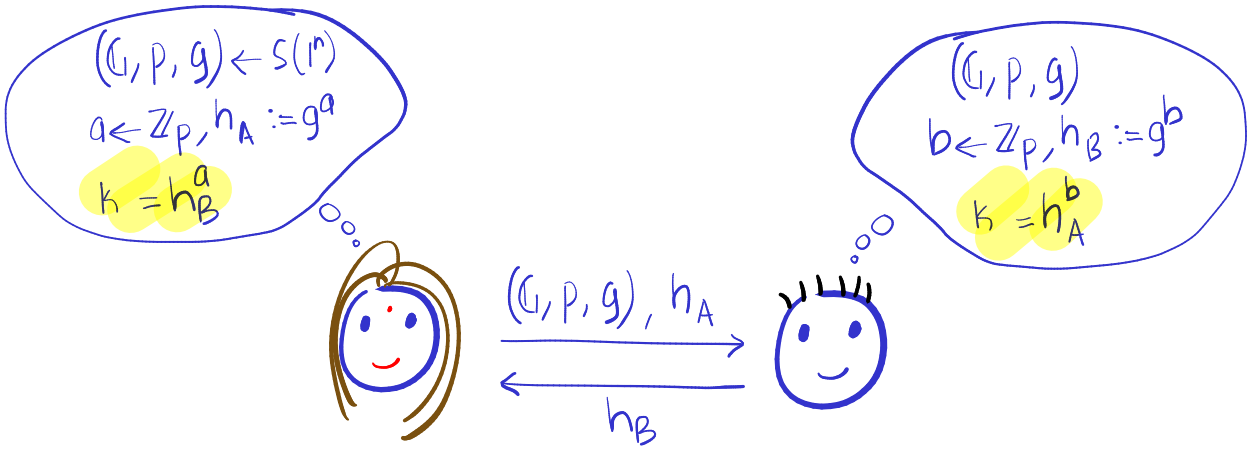


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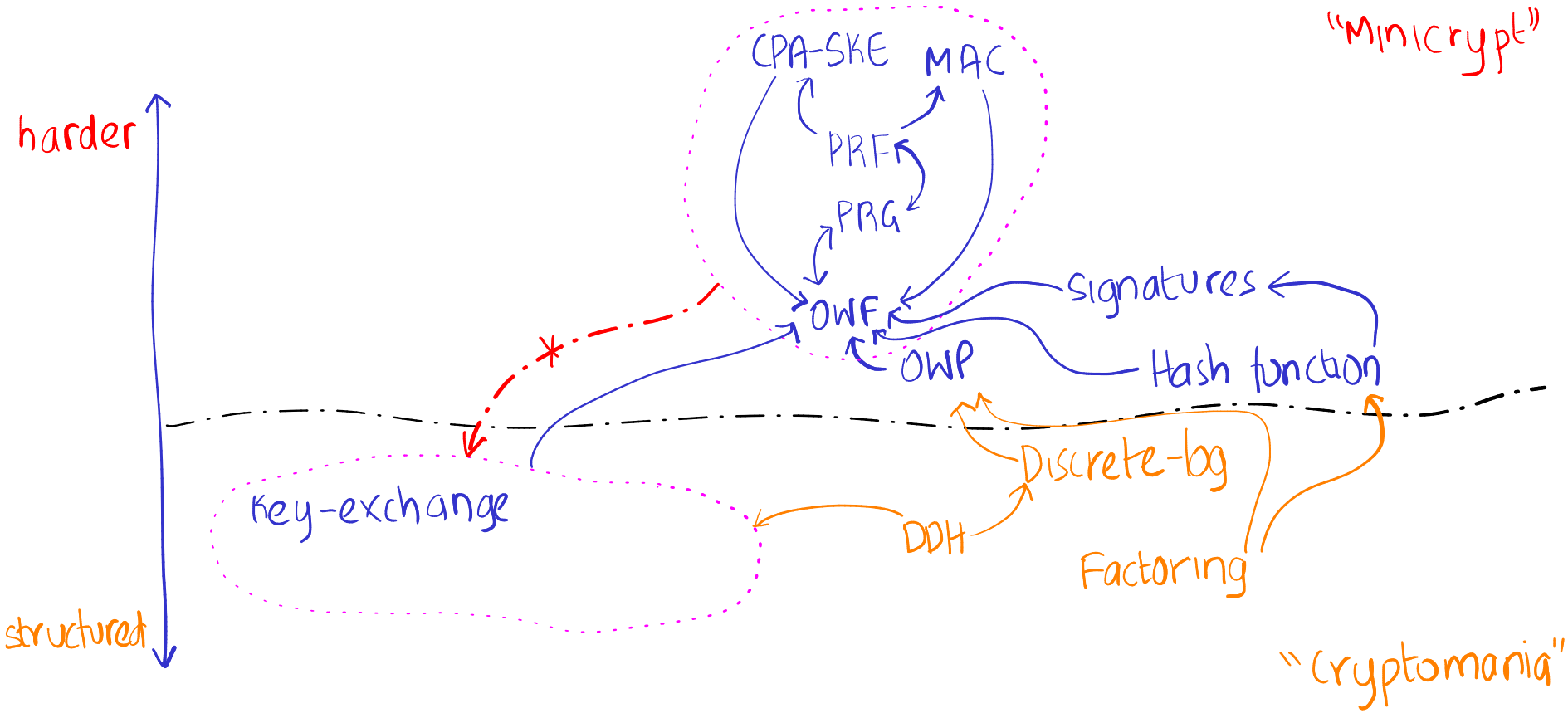

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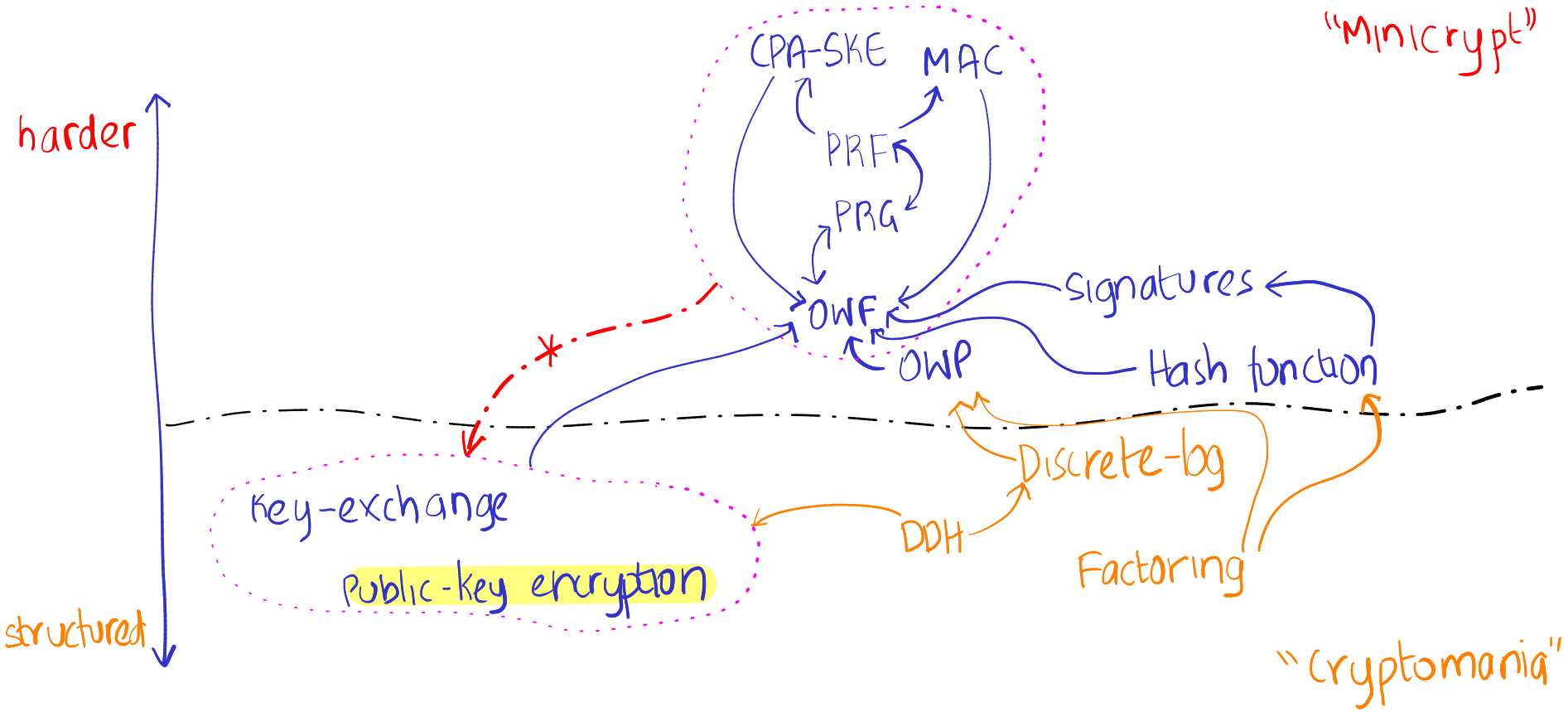


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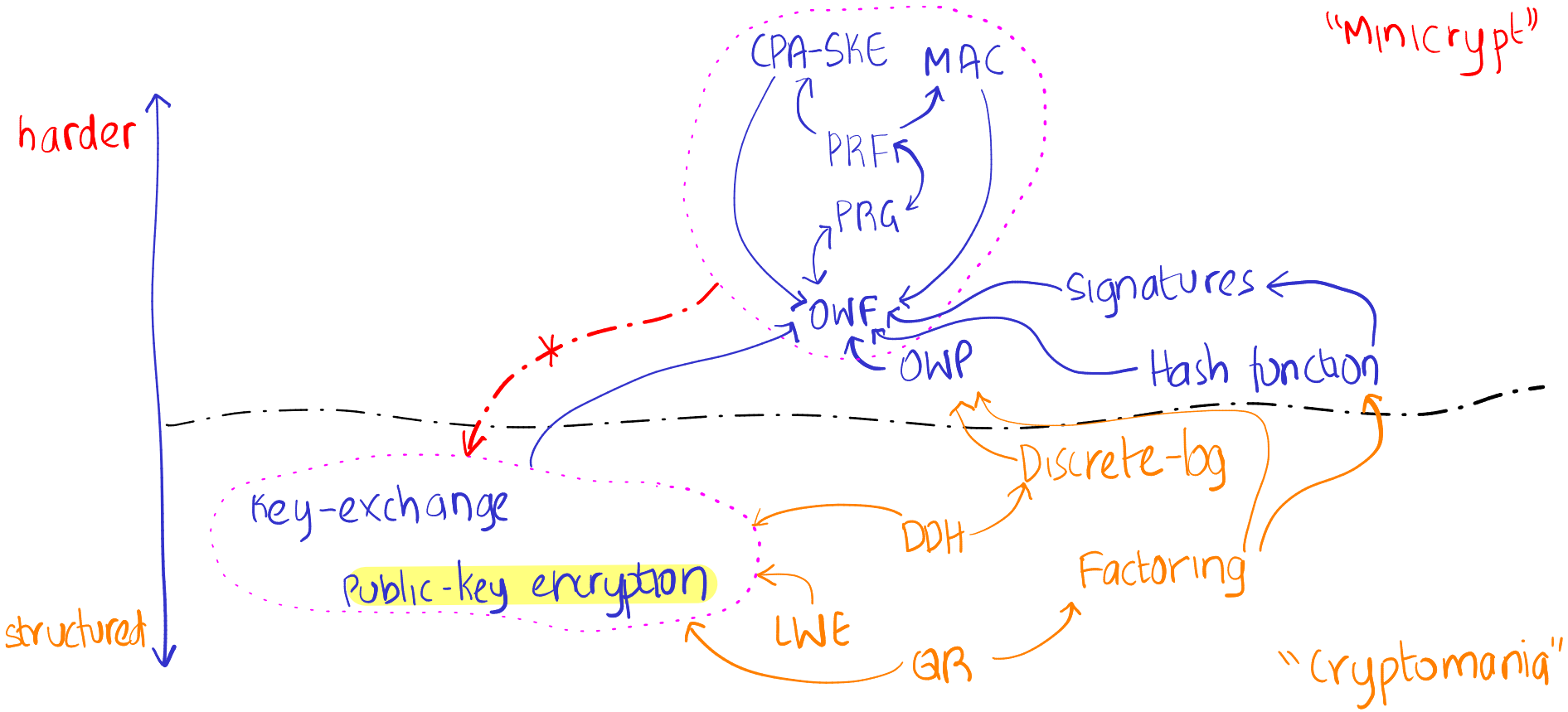


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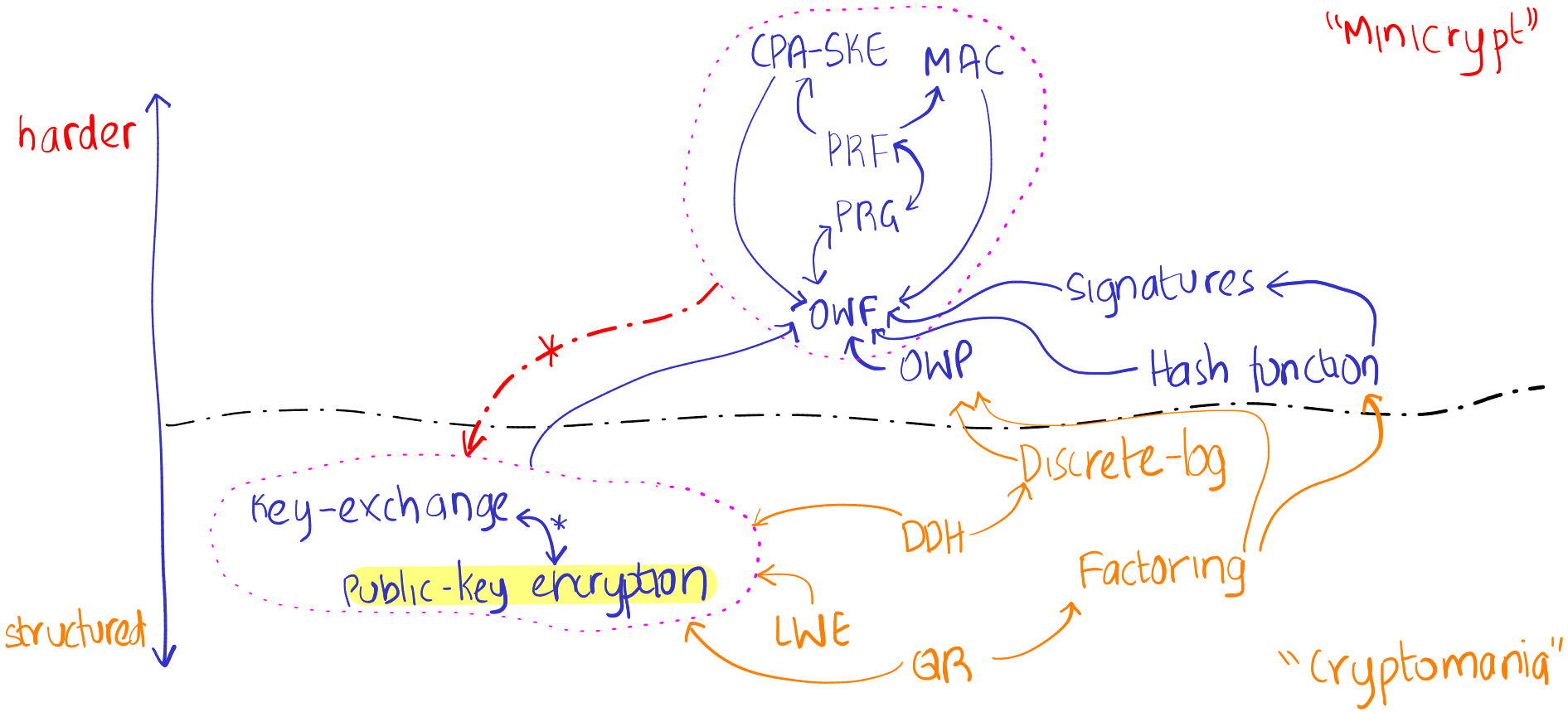


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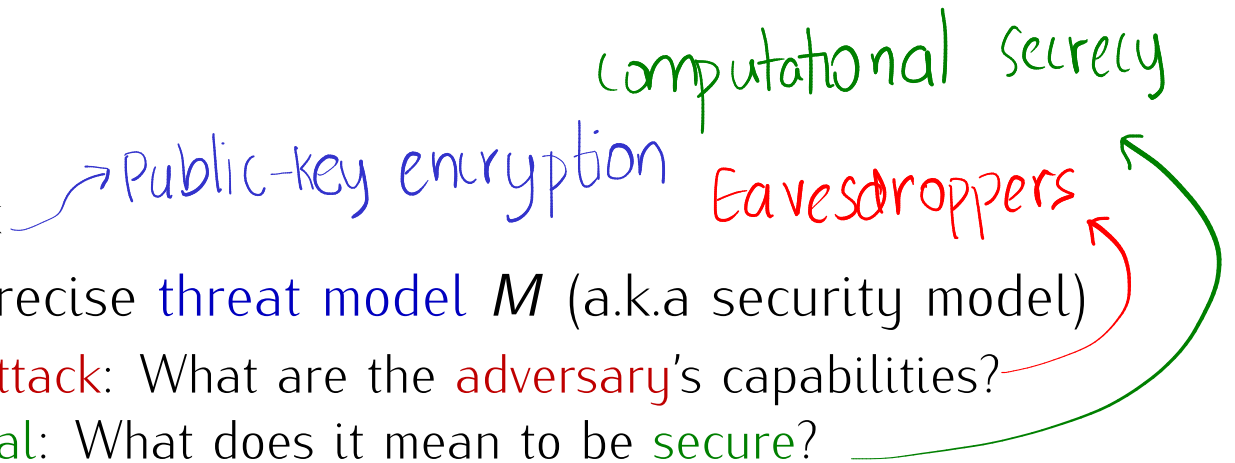

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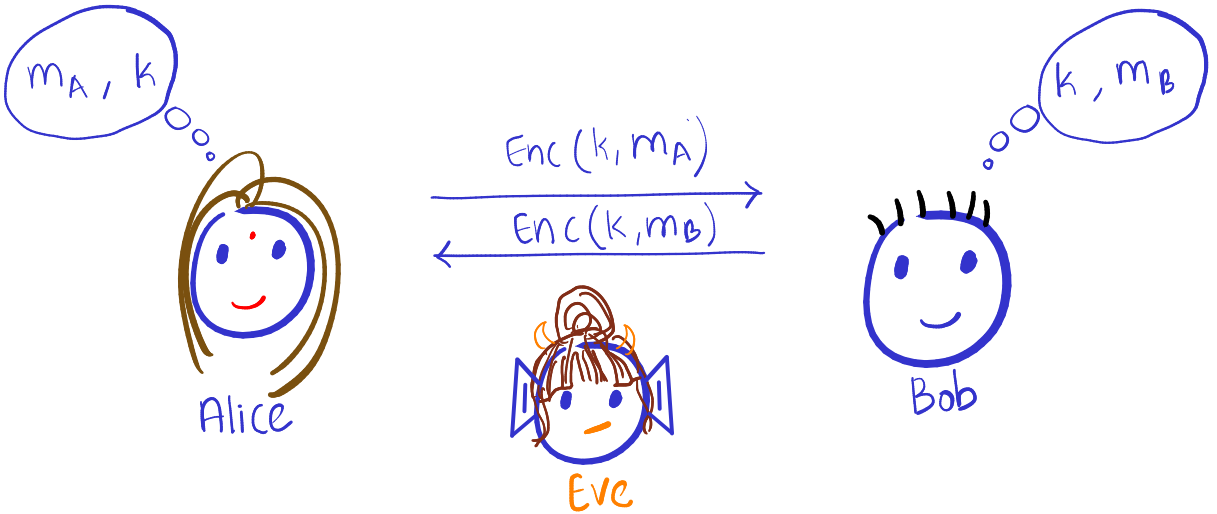


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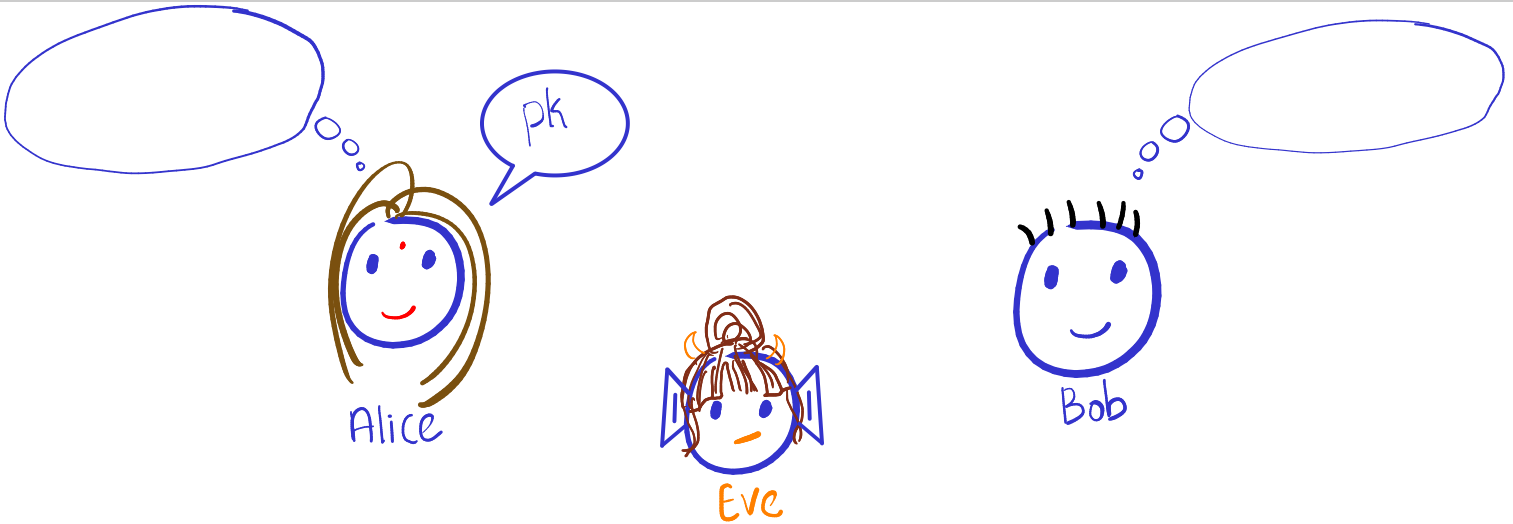


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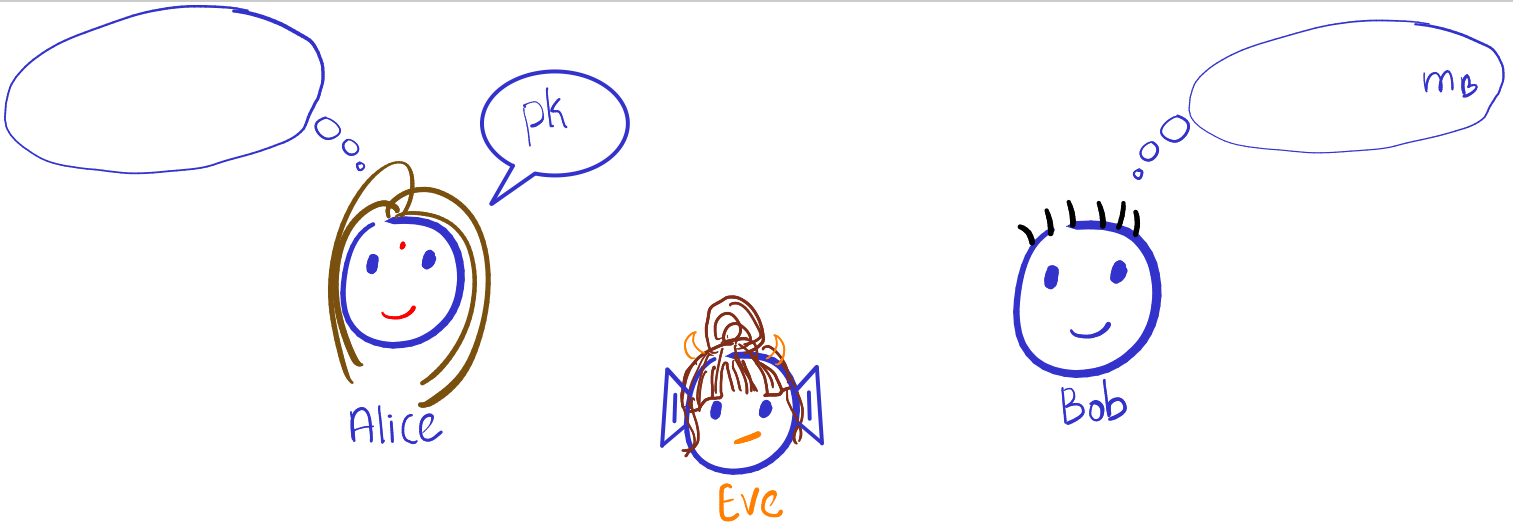


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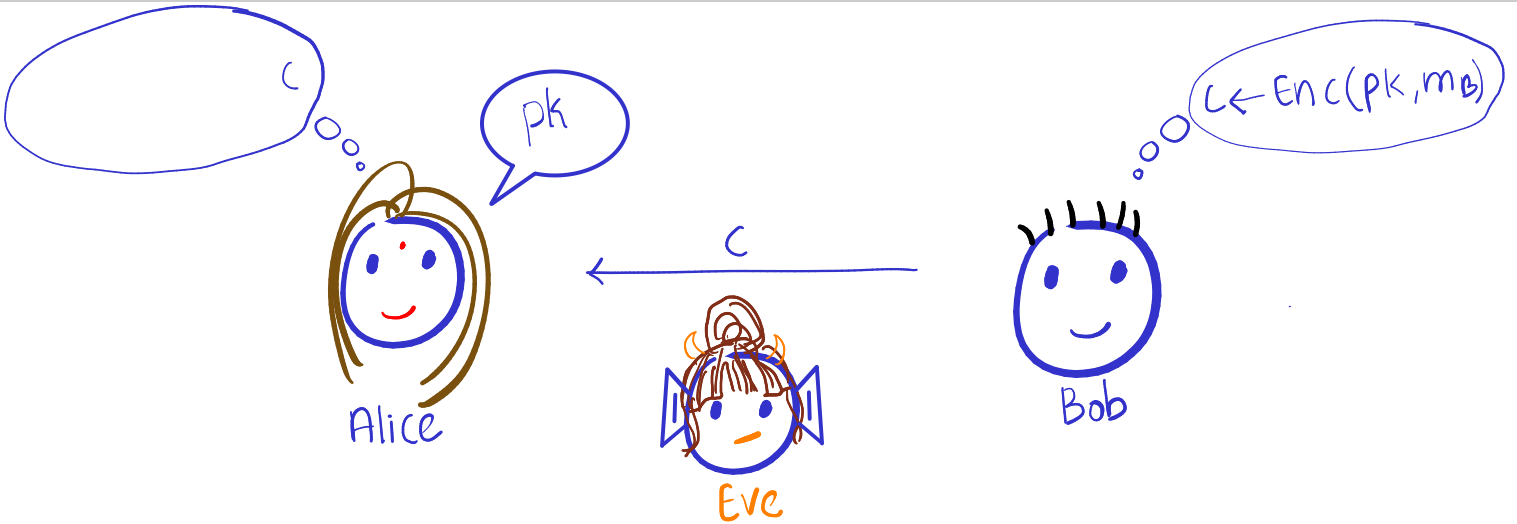


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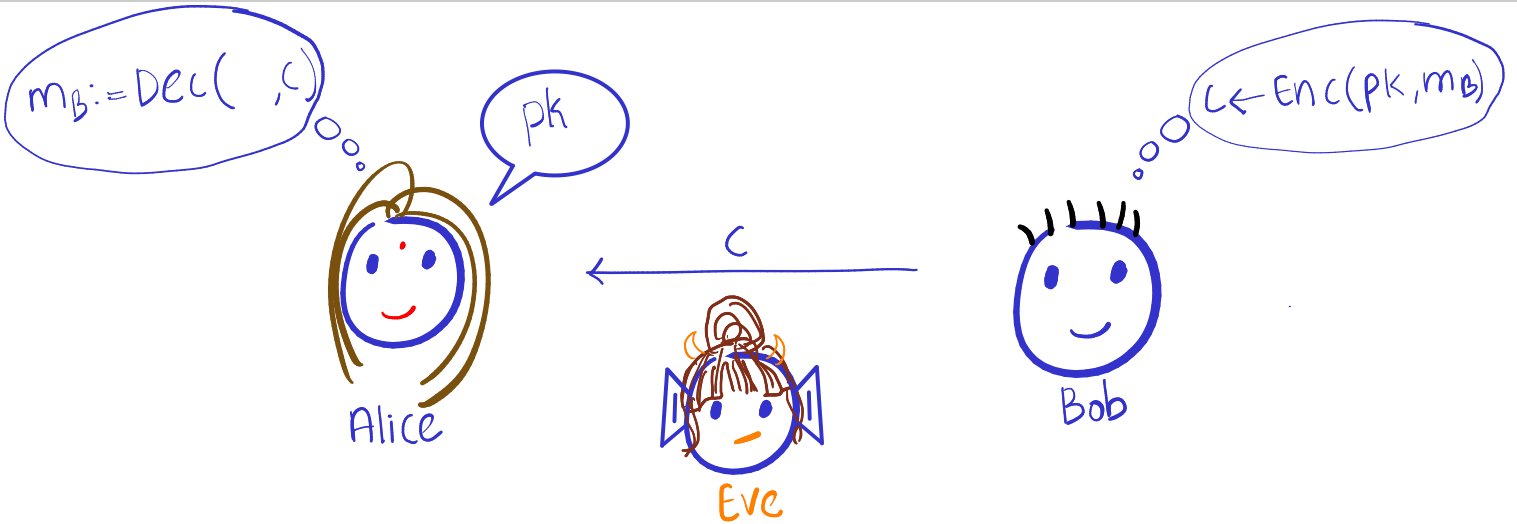


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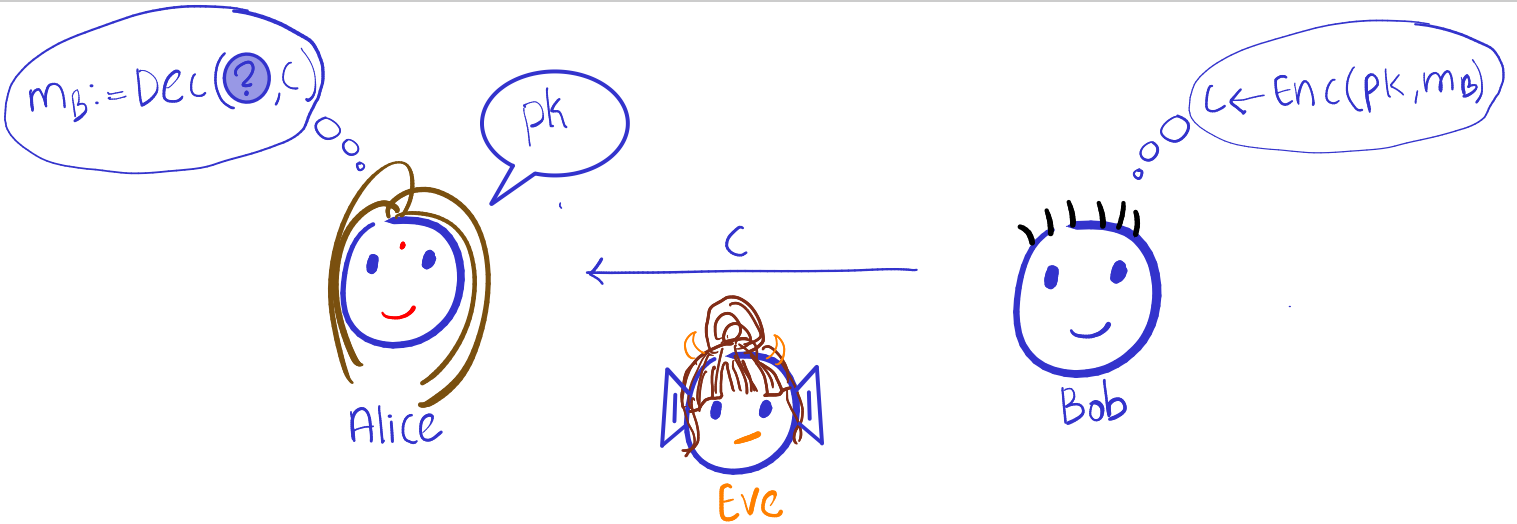


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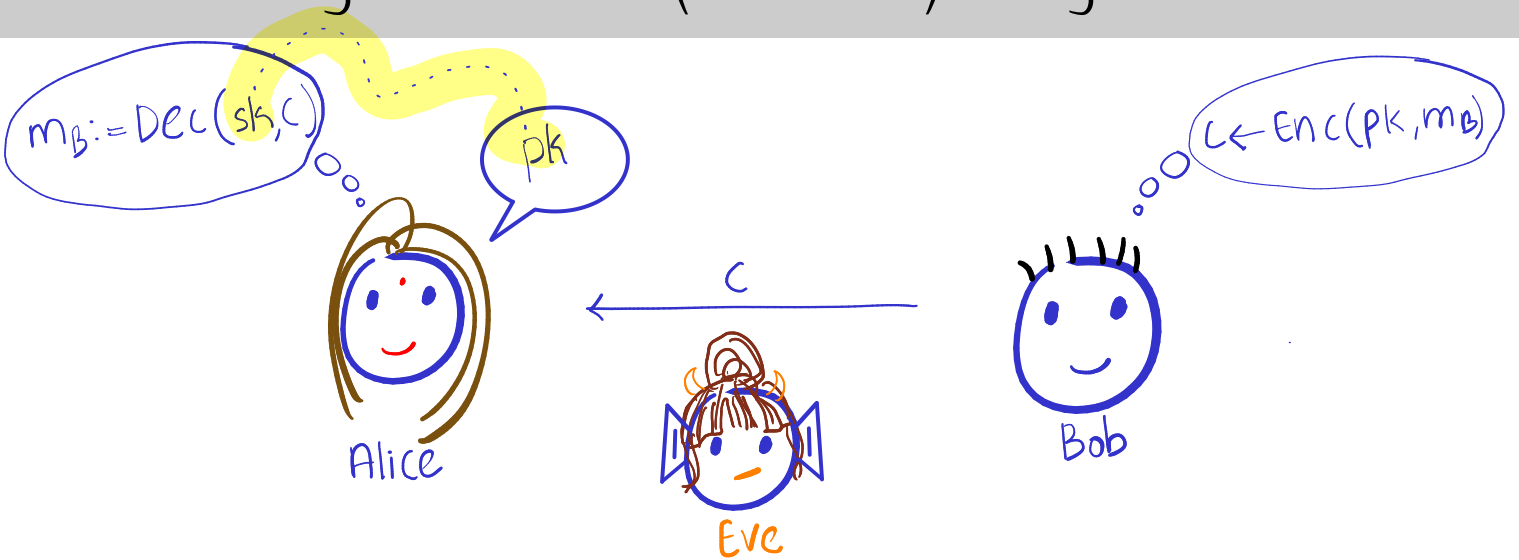


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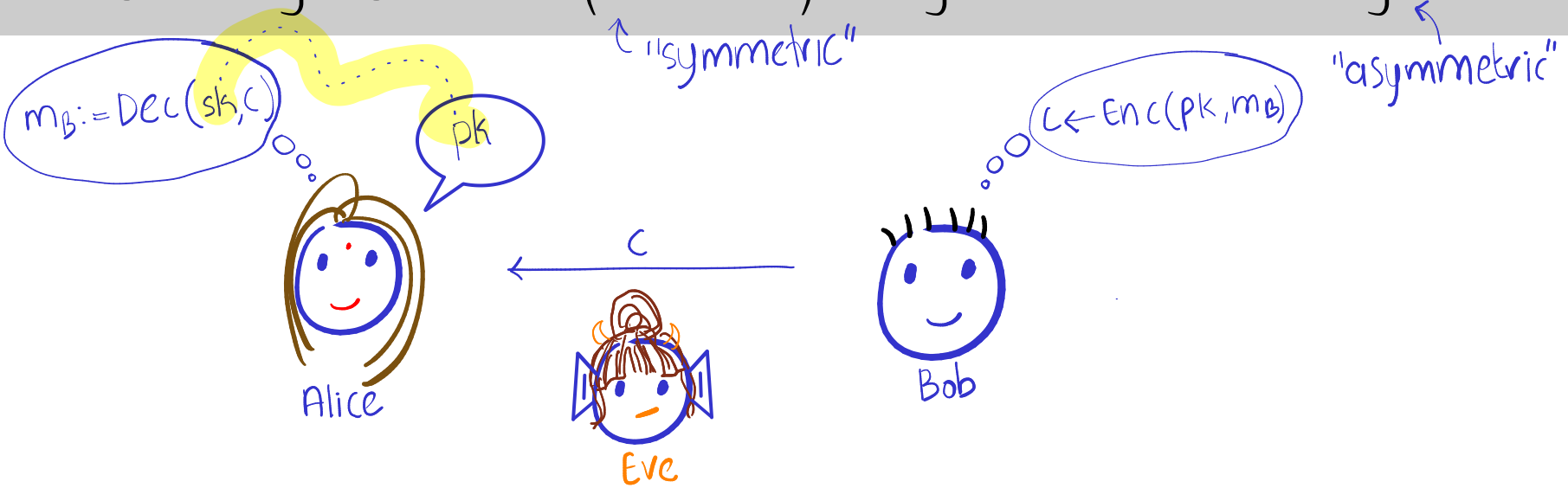


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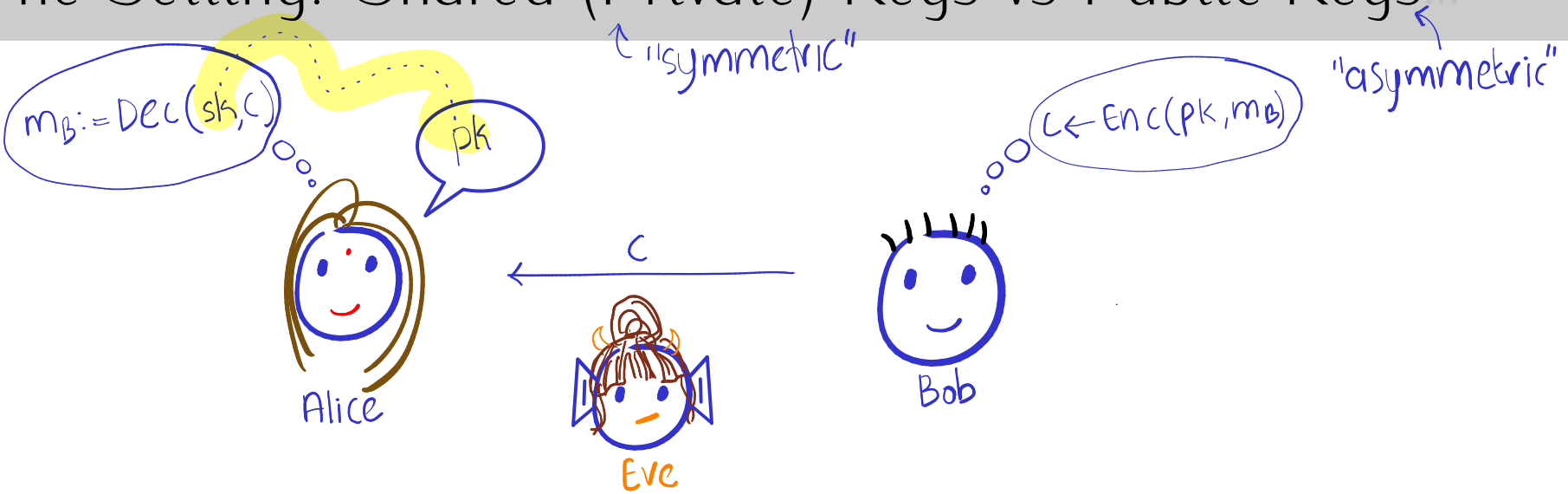


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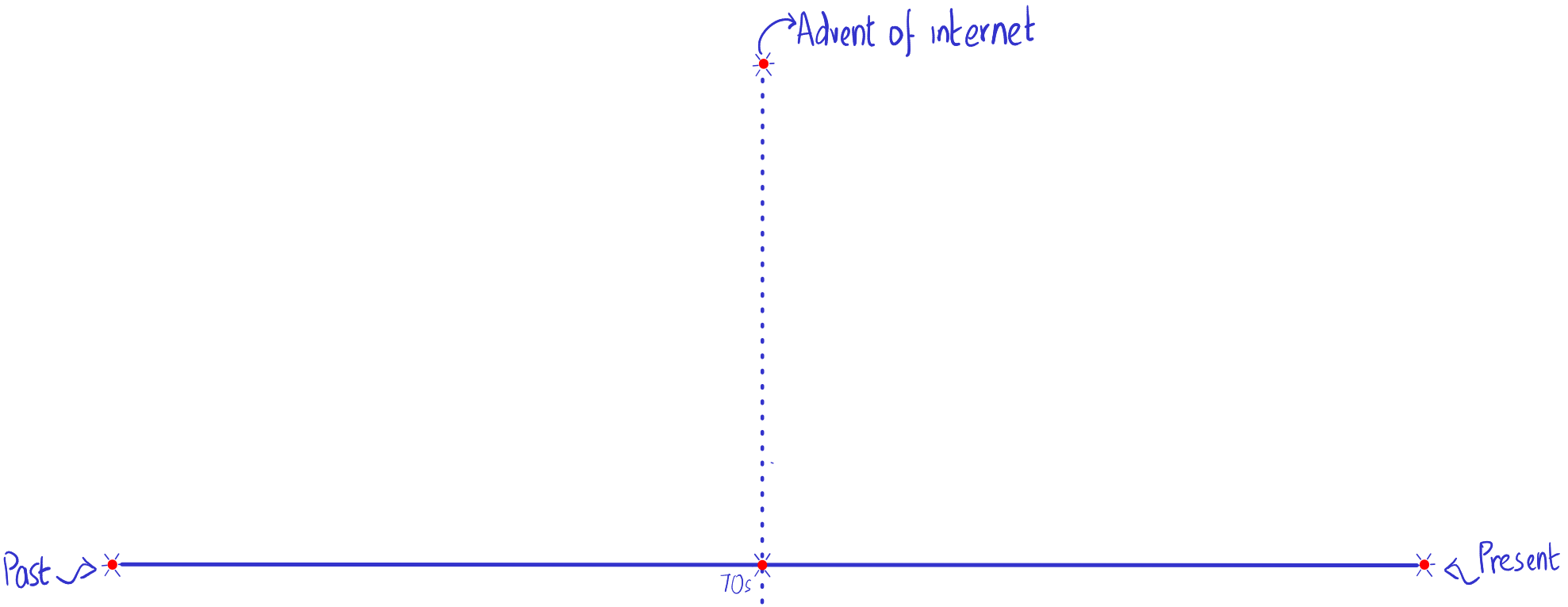


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Programming Techniques — S. L. Graham, R. L. Rivest Editors

**Secure Communications Over Insecure Channels**

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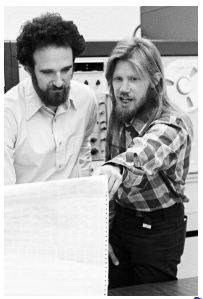

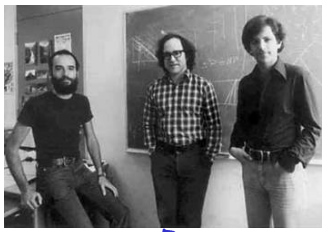




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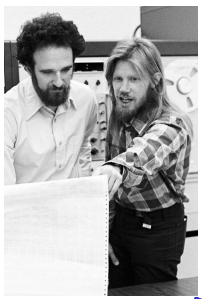

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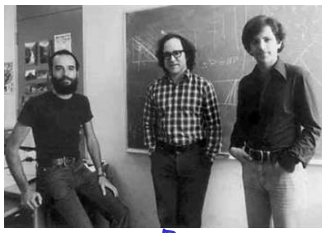

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RSA cryptosystem

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Past → 70s 74 76 78 82 ← Present

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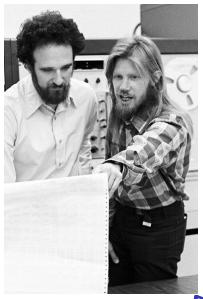

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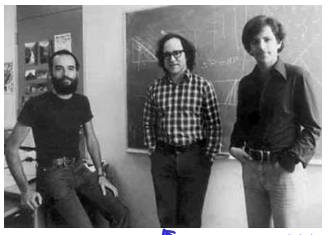

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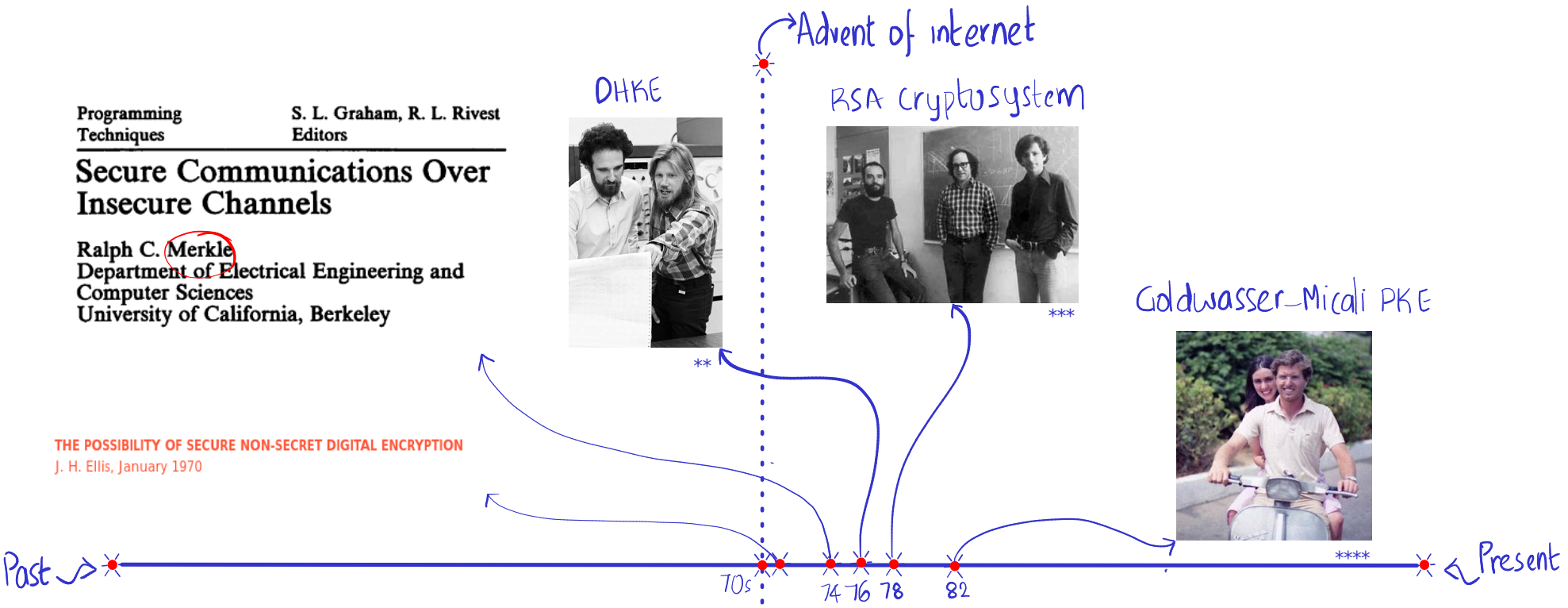


- PKE IRL: PGP, hybrid encryption

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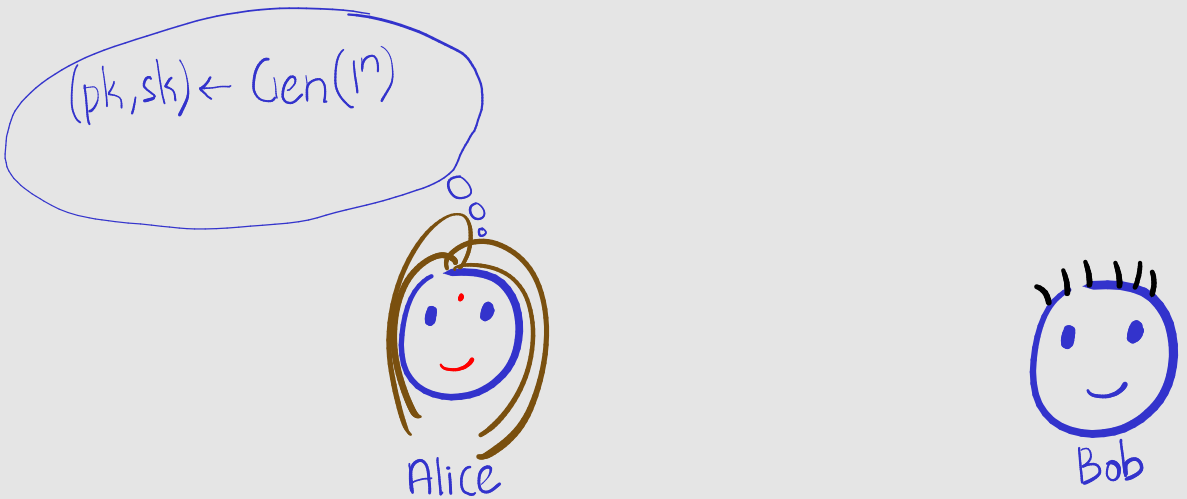

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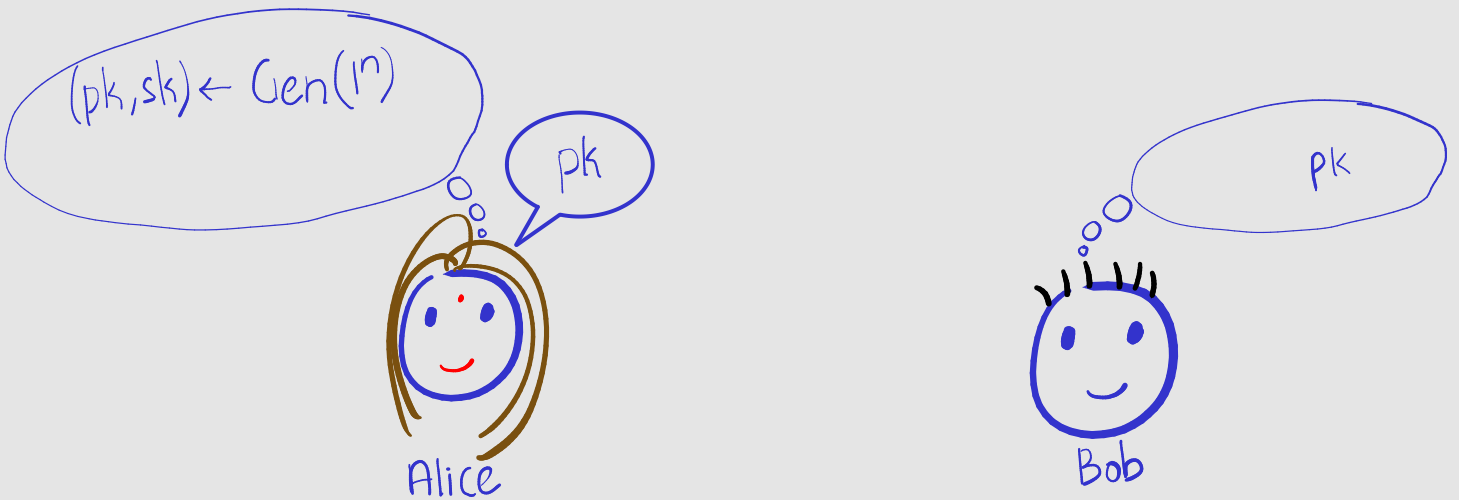

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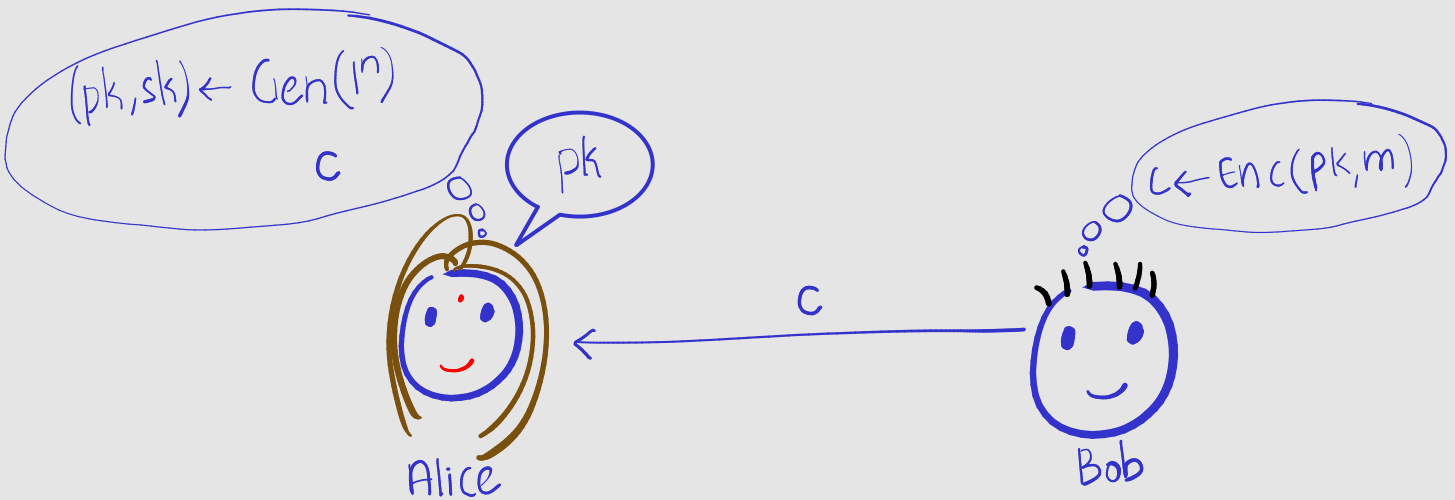

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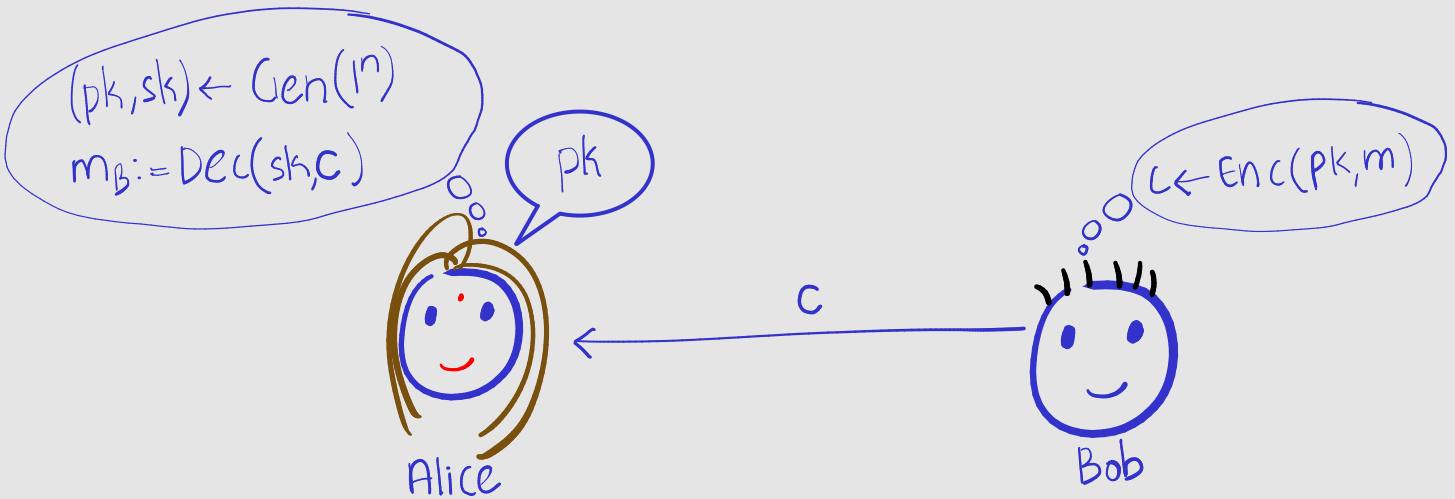

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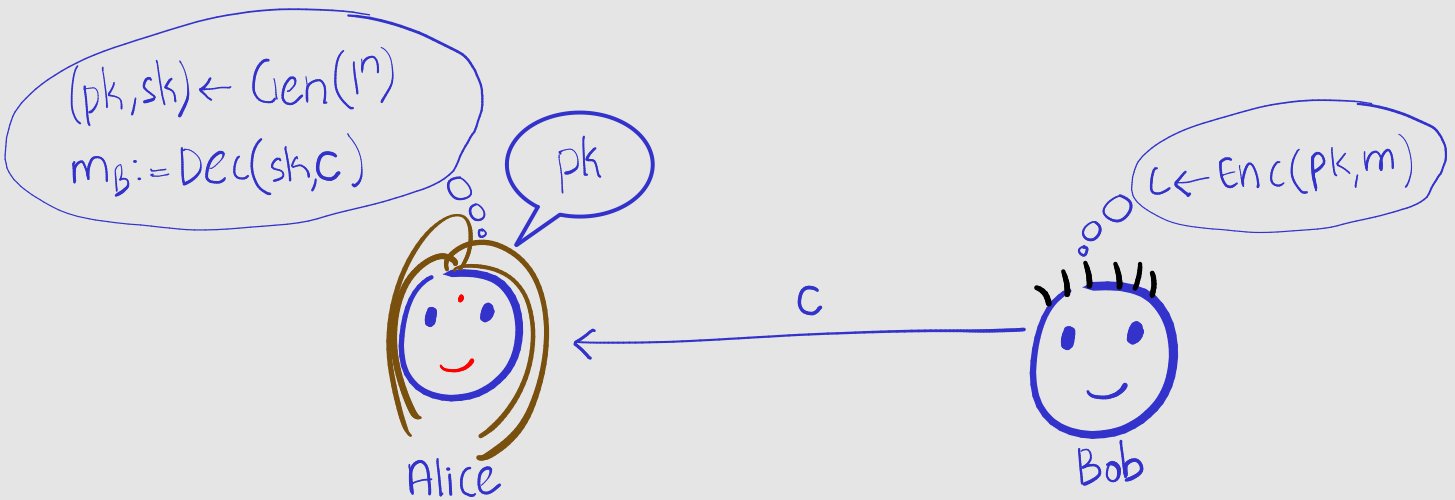


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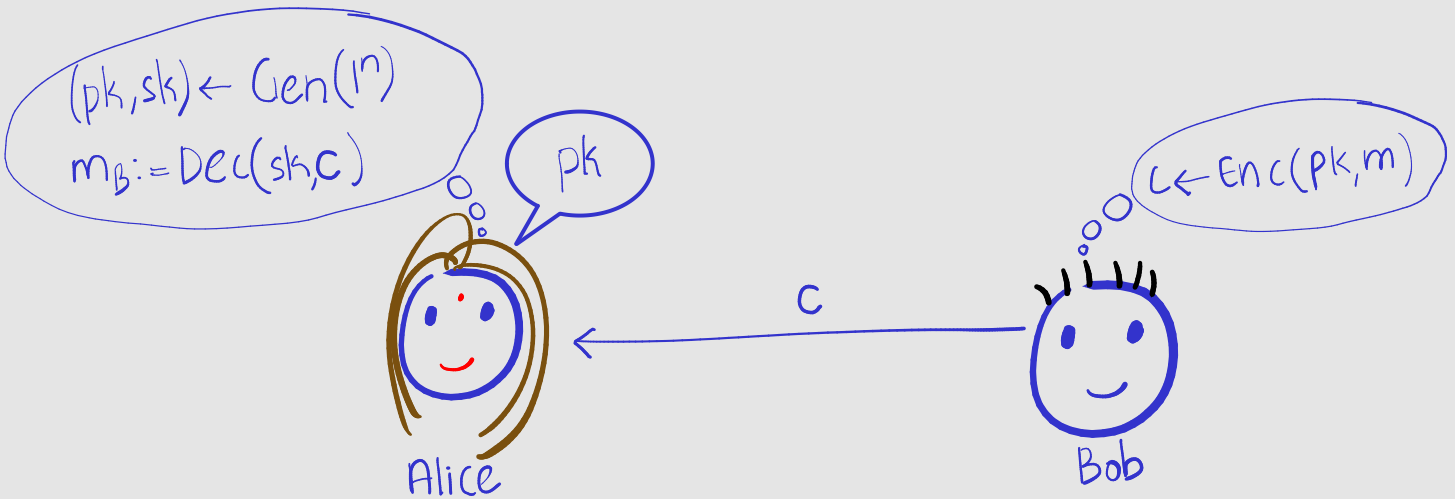


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How can an unbounded eavesdropper Eve break PKE?

# How to Define Security?...

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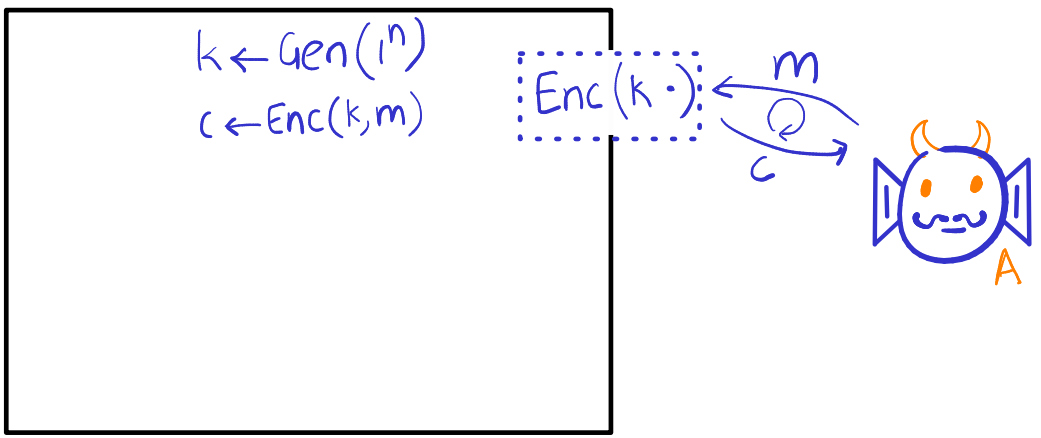

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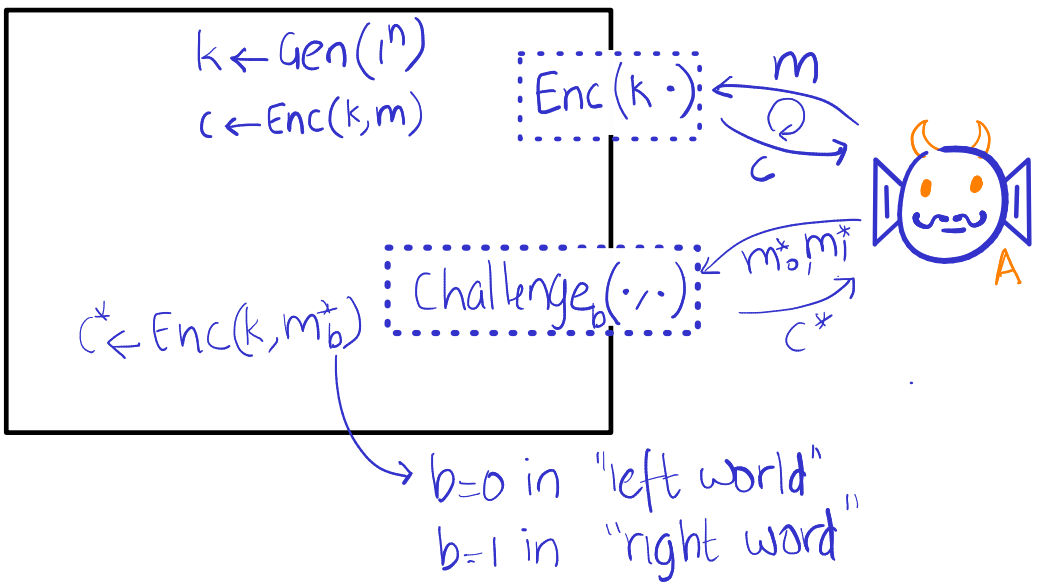

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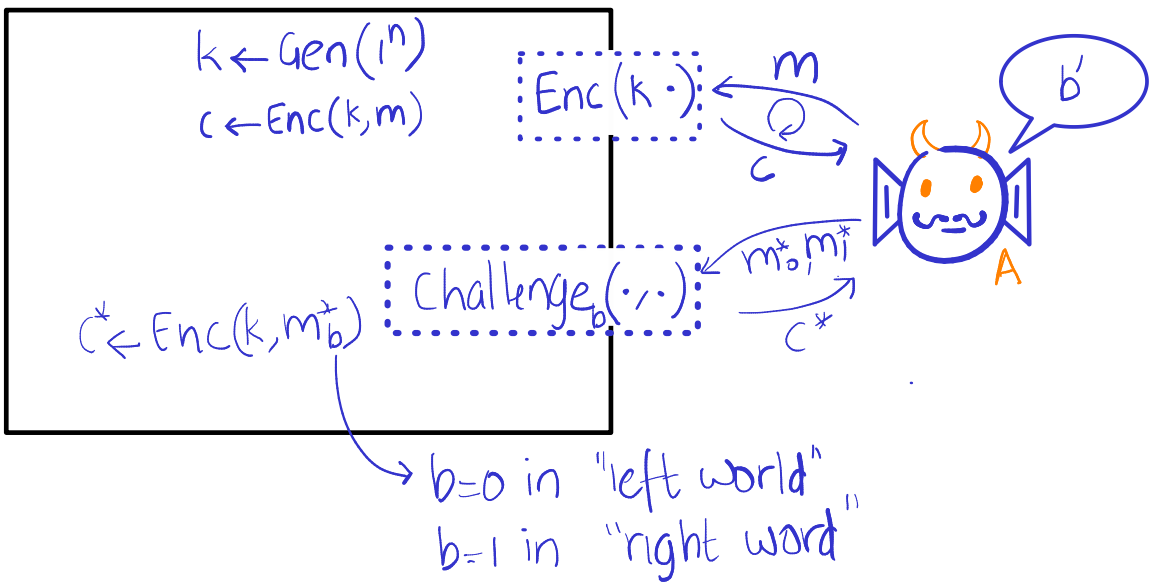

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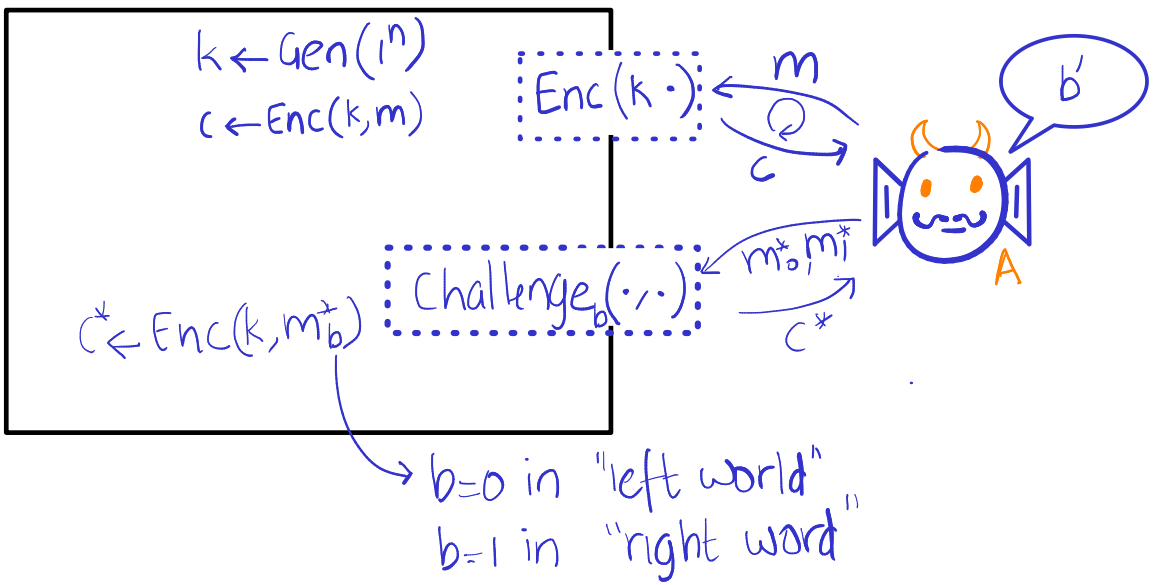

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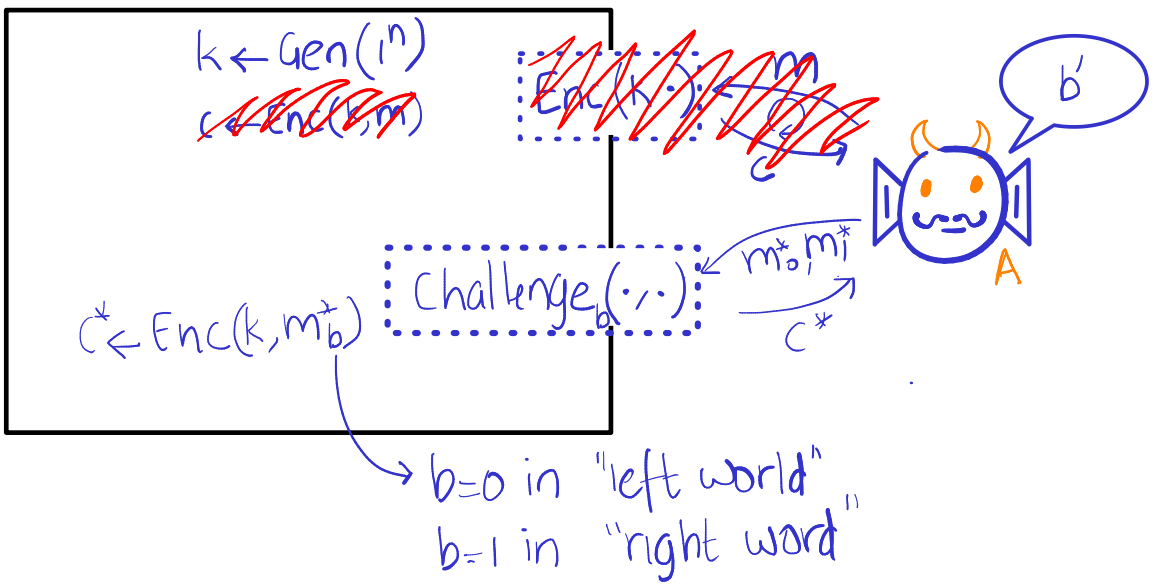

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+1 Stronger notion: secrecy against chosen-ciphertext attacker

# Plan for this Lecture

1. Public-Key Encryption (PKE)
2. ElGamal PKE ⟵ DDH
3. Goldwasser-Micali PKE ⟵ QR

# Recall Group Theory from Last Lecture...

**Defintion 3 (Group axioms)**

*A group $\mathbb{G}$ is a set $\mathcal{G}$ with a binary operation $\cdot$ satisfying: 1) closure 2) associativity, 3) existence of identity and 4) existence of inverse. $\mathbb{G}$ Abelian if it additionally satisfies 5) commutativity.*

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**Defintion 4 (Group terminology)**

- *Order of the group:* $|\mathcal{G}|$
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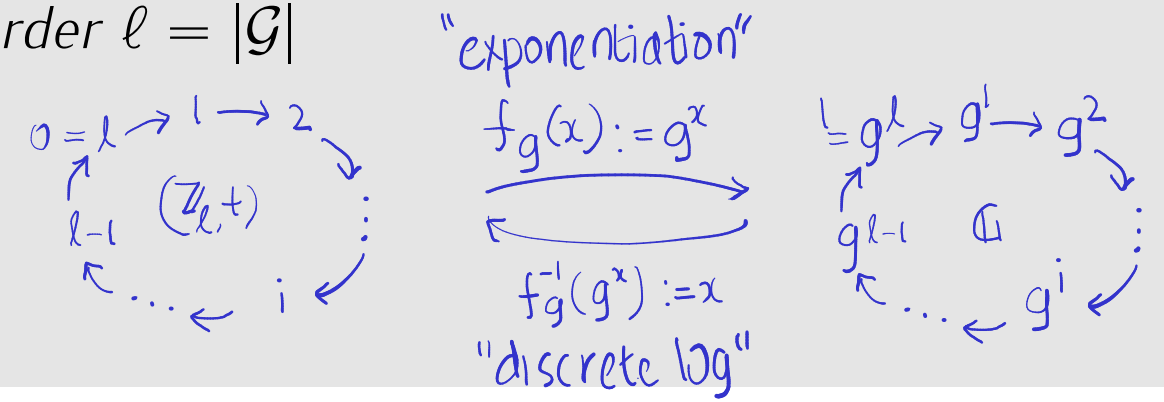

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**Assumption 1 (DLog assumption in $\mathbb{G}$ w.r.to $\mathsf{S}$)**

*The DLog assumption in cyclic group $\mathbb{G}$ w.r.to $\mathsf{S}$ holds if for all PPT inverters $\mathsf{Inv}$, the following is negligible:*

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## Assumption 2 (DDH assumption in in $\mathbb{G}$ w.r.to $\mathsf{S}$)

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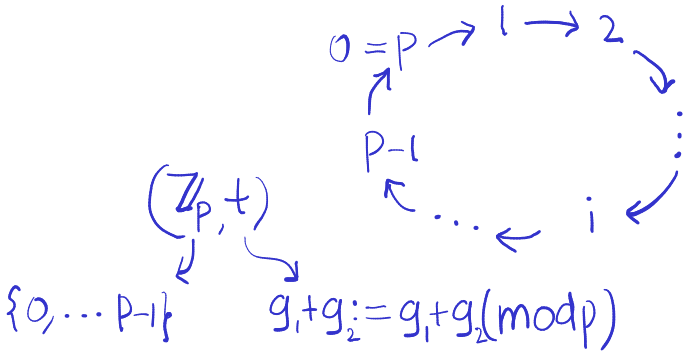
"real/pseudorandom" "random"

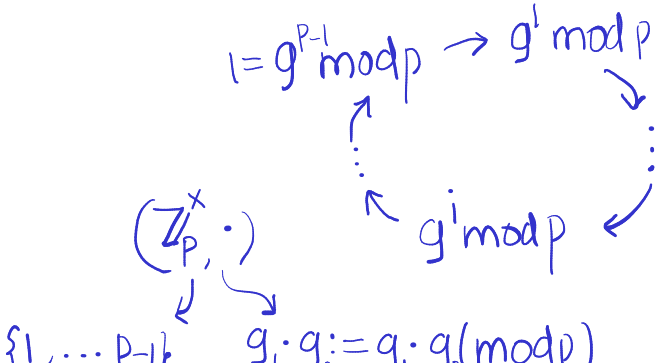
# Recall Group Theory from Last Lecture...

## Addition modulo prime $p$



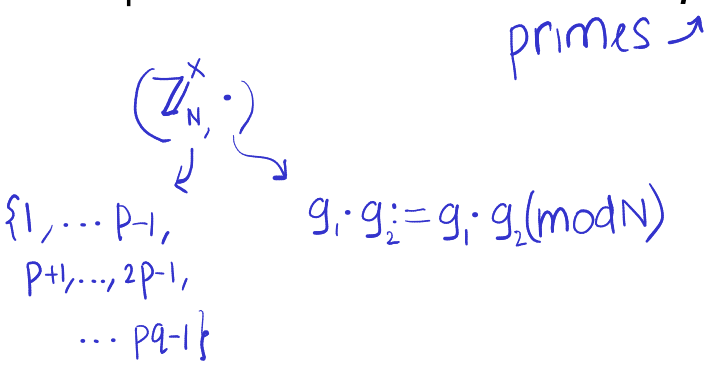

- order $p$
- cyclic

## Multiplication modulo prime $p$



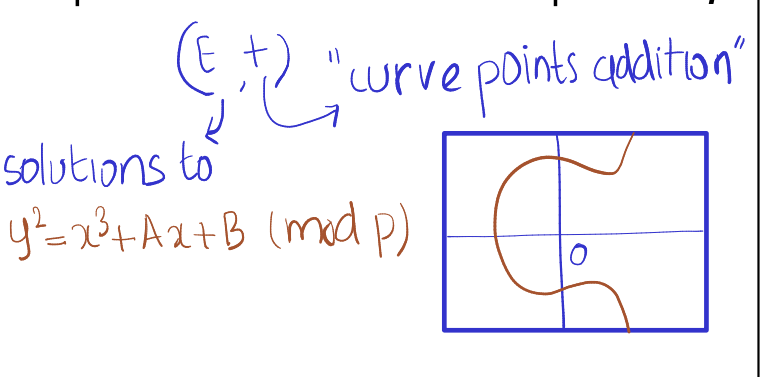

- order $p-1$
- cyclic

## Multiplication modulo $N = pq$




- order $(p-1)(q-1)$
- not cyclic

## Elliptic curves modulo prime $p$

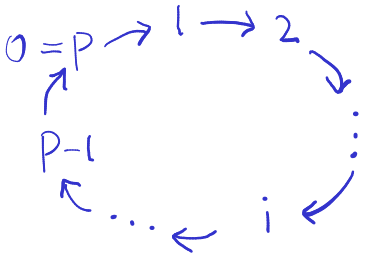



- $|p+1 - \text{order}| \leq 2\sqrt{p}$
- cyclic

# Recall Group Theory from Last Lecture...

DLog easy

## Addition modulo prime $p$




$(\mathbb{Z}_p, +)$

$\{0, \dots p-1\}$ $\quad g_1 + g_2 := g_1 + g_2 \pmod p$

- order $p$
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DLog hard, but DDH easy

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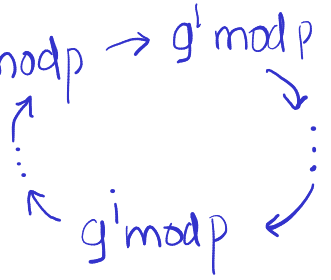


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## Multiplication modulo $N = pq$

primes

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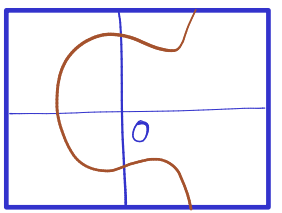
DDH hard in "subgroup"

## Elliptic curves modulo prime $p$

$(E, +)$ "curve points addition"

solutions to $y^2 = x^3 + Ax + B \pmod p$

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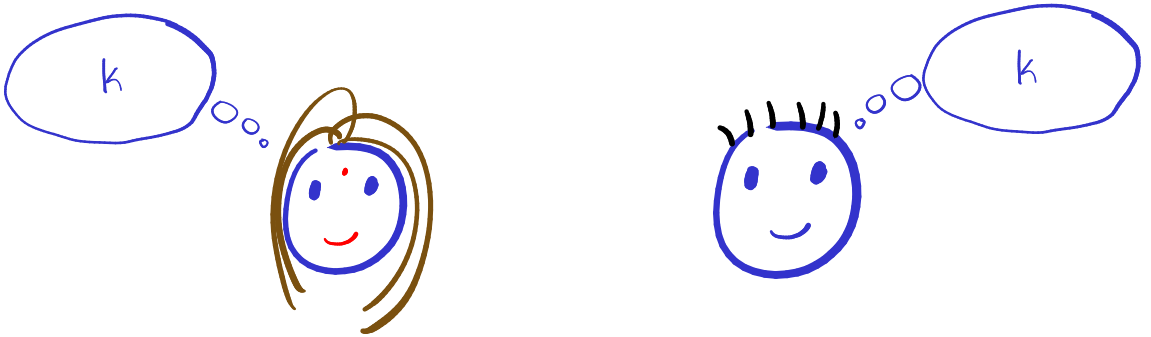
# One-Time Pad Can be Generalised Over Groups




**Pseudocode 1 (OTP over $(\{0,1\}^n, \oplus)$ with message space $\{0,1\}^n$)**

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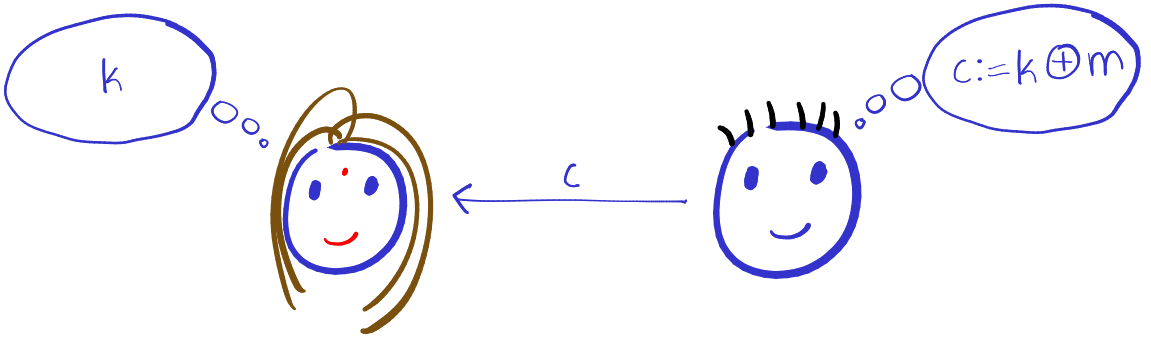


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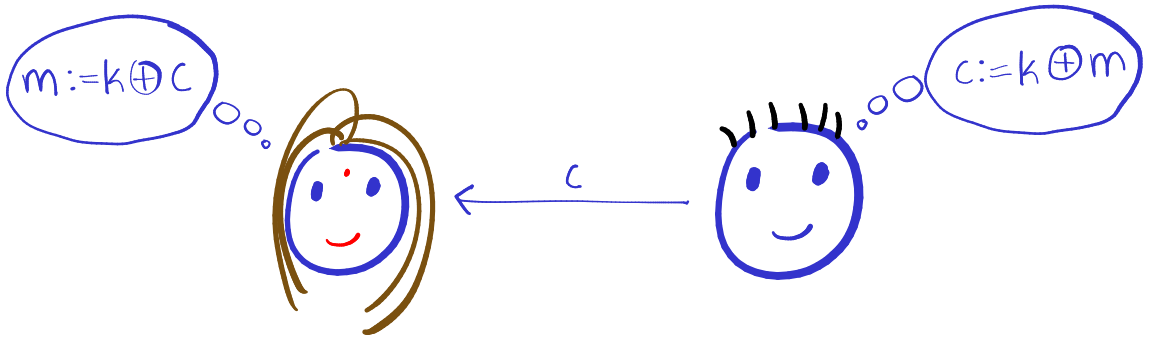


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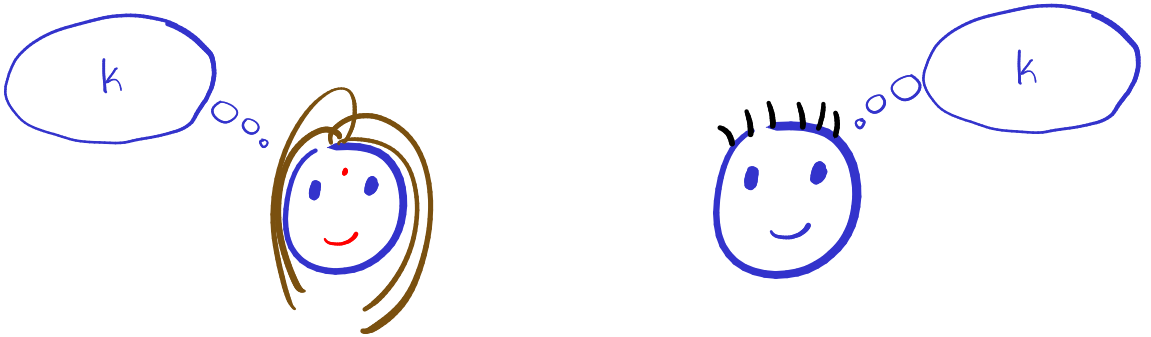


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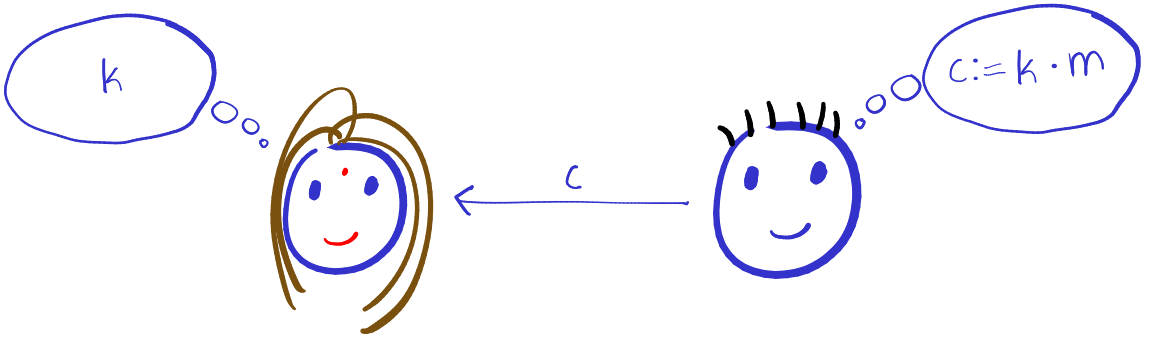


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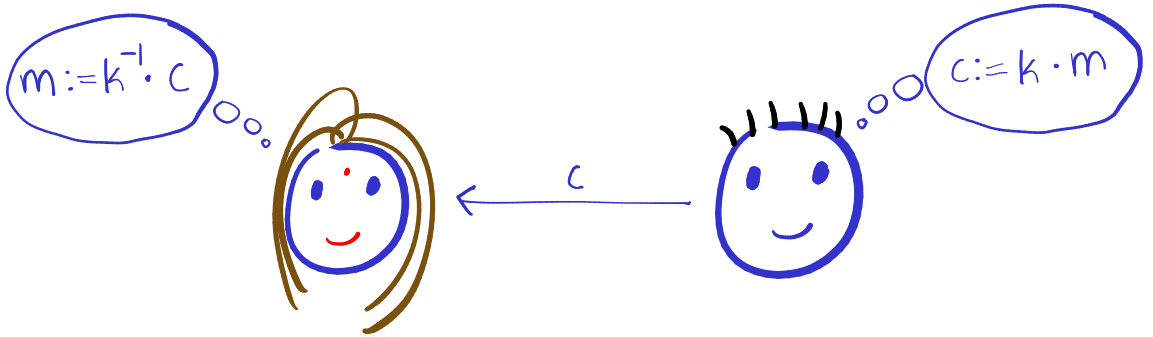


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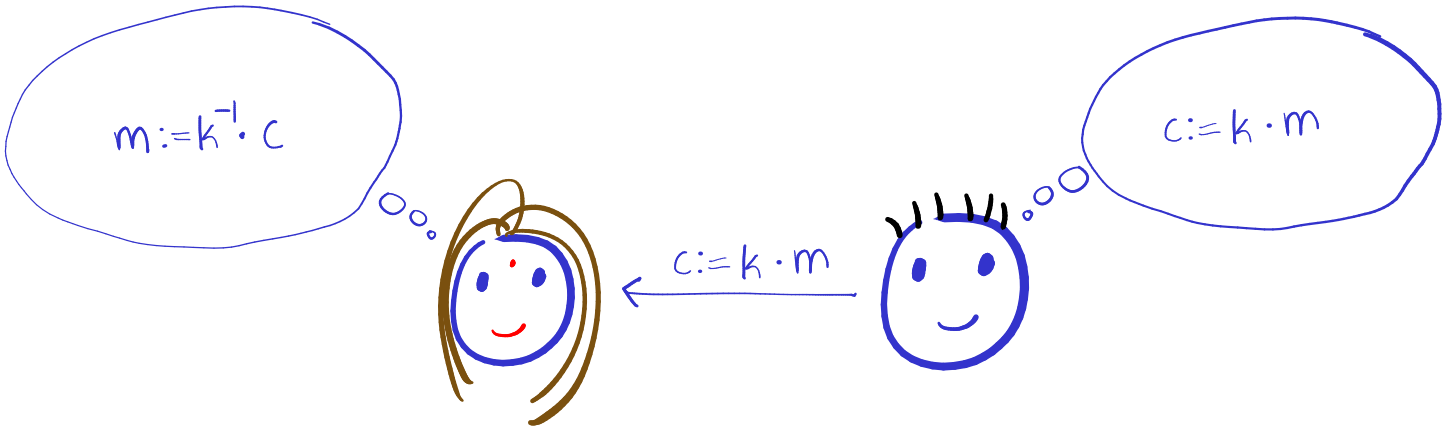

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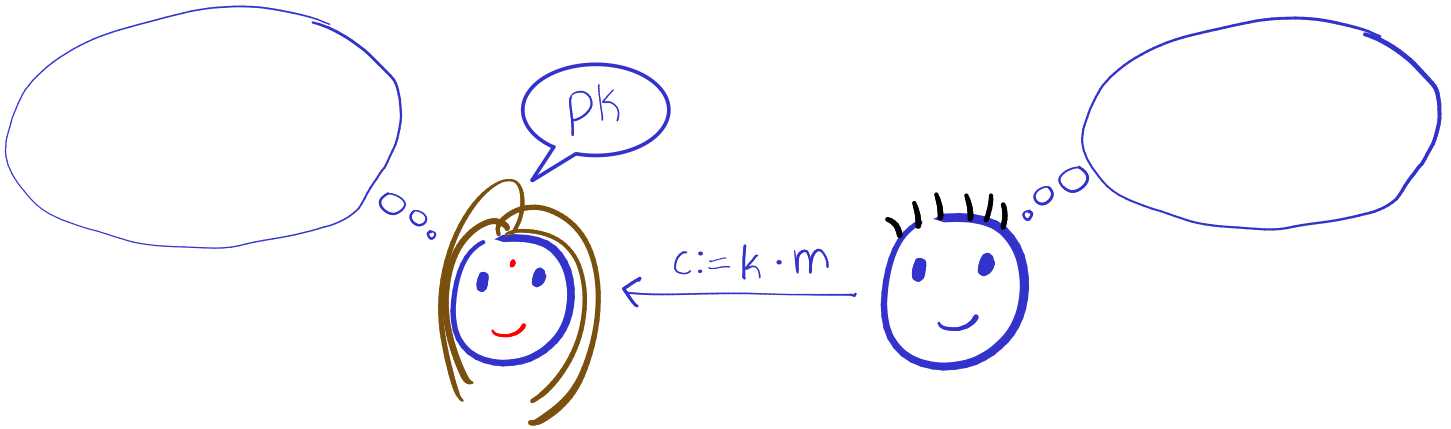


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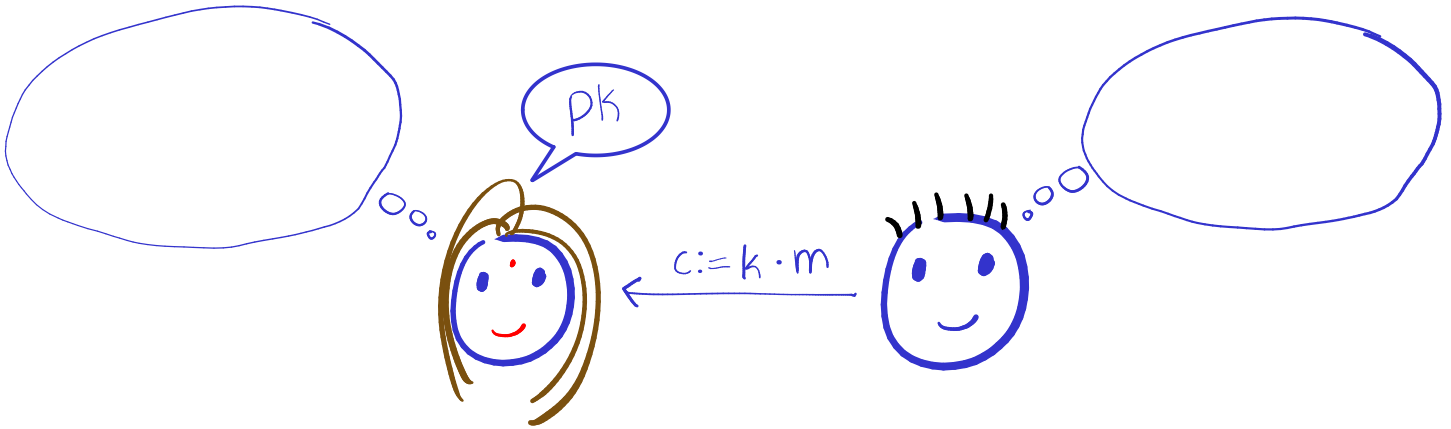


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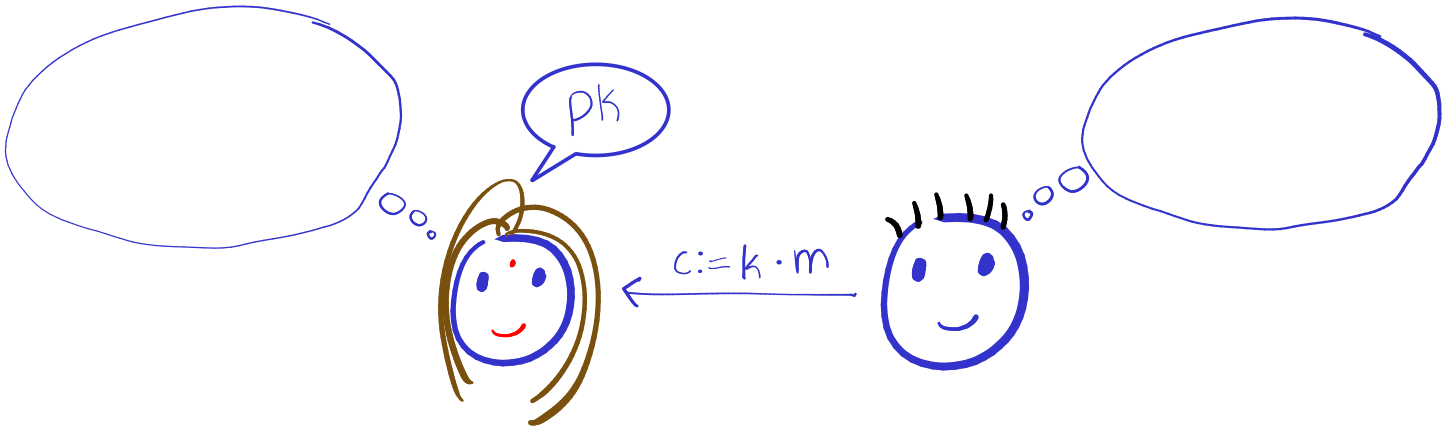


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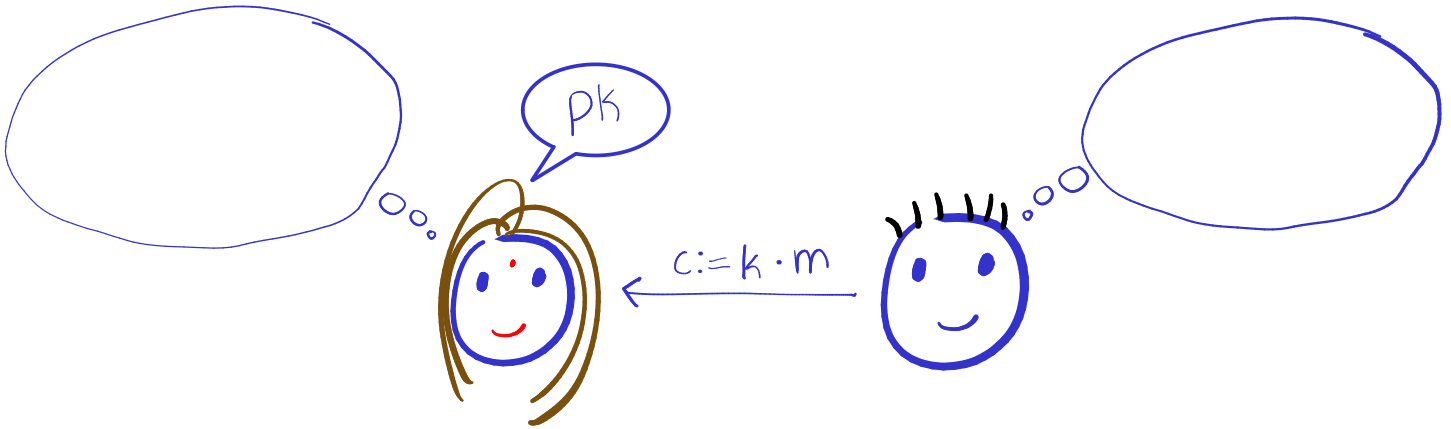


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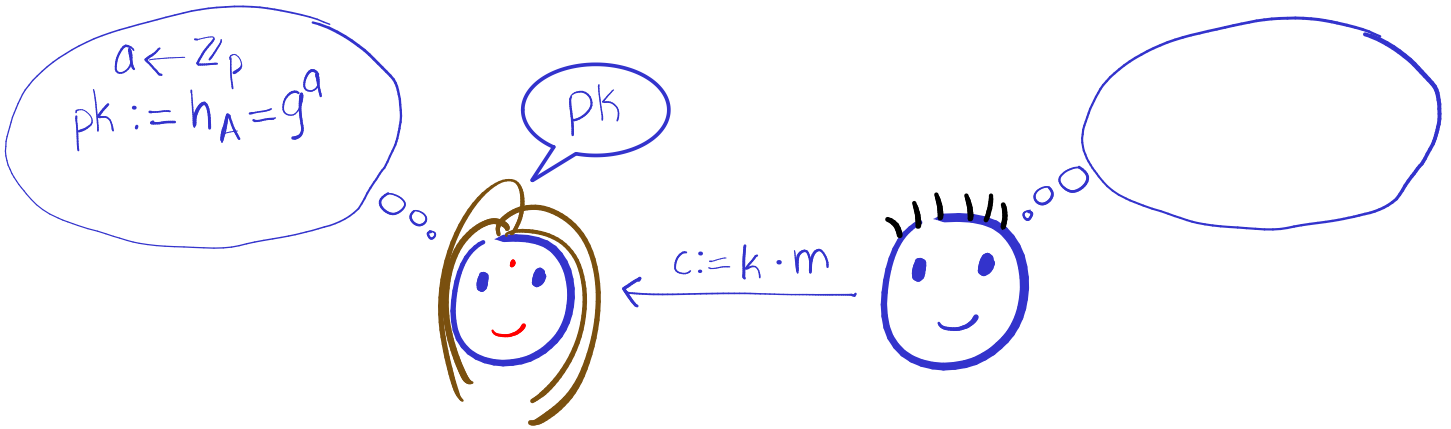


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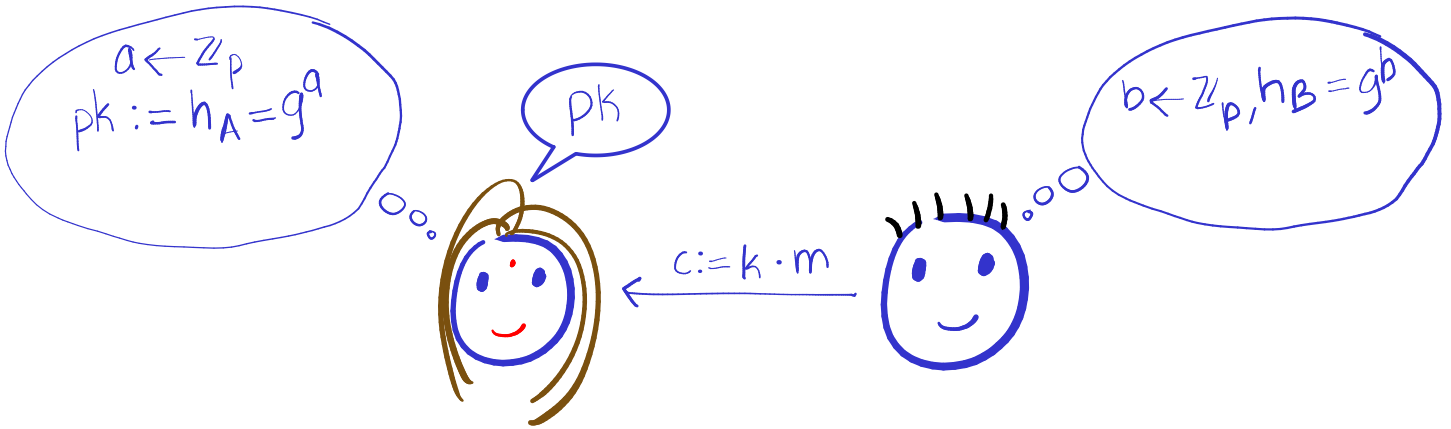


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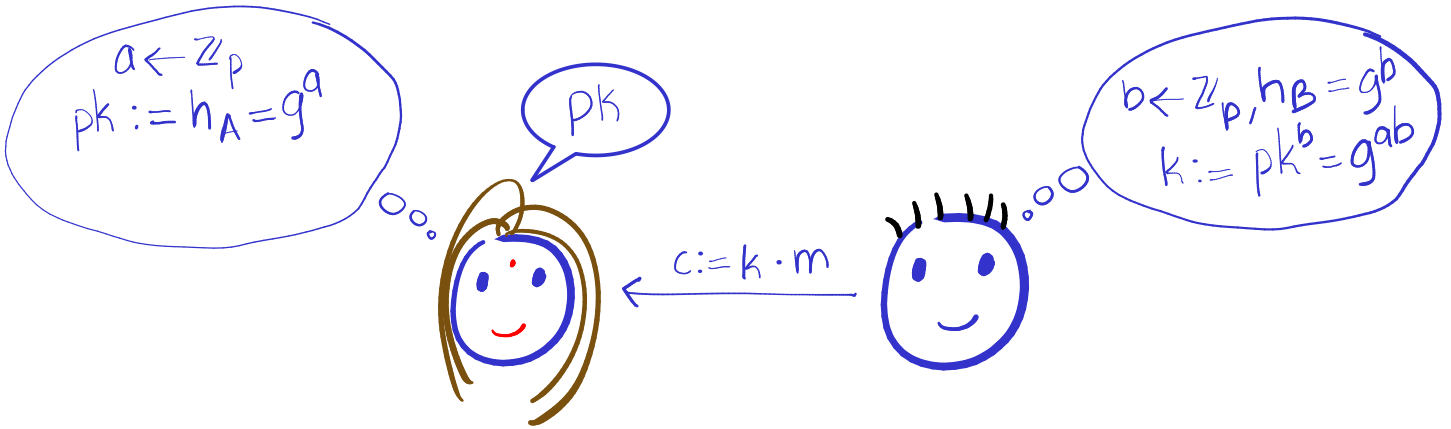


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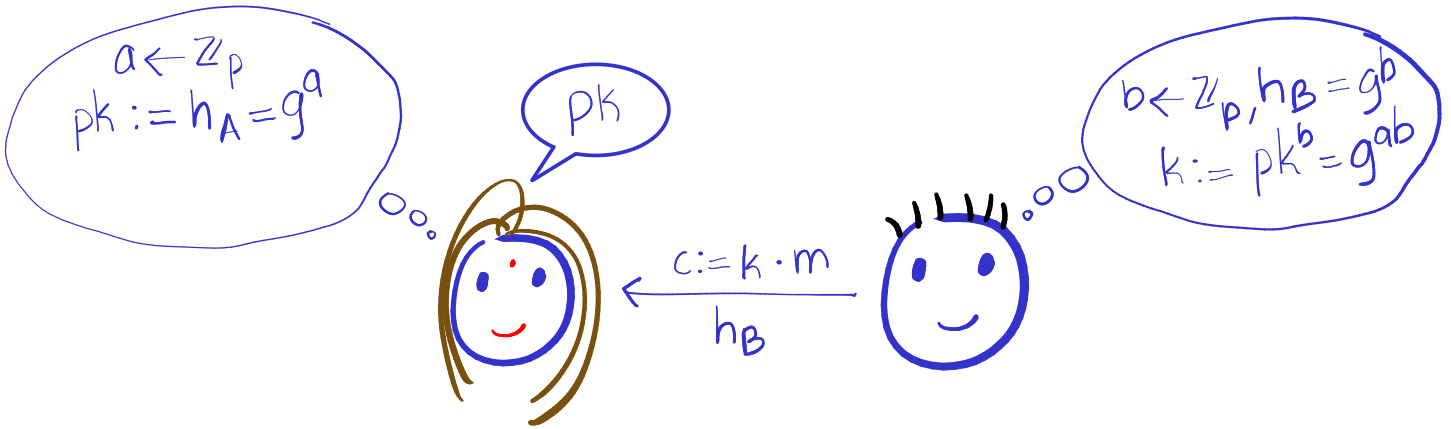


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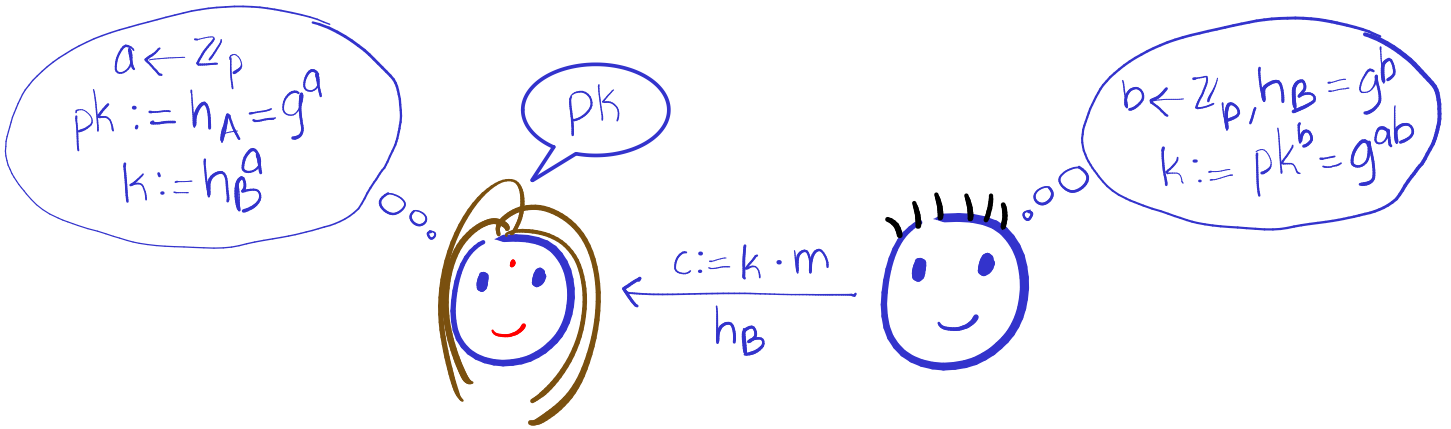


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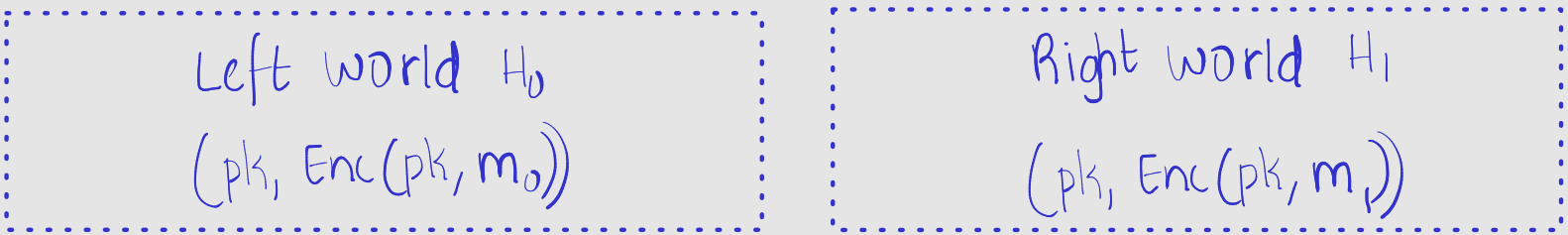

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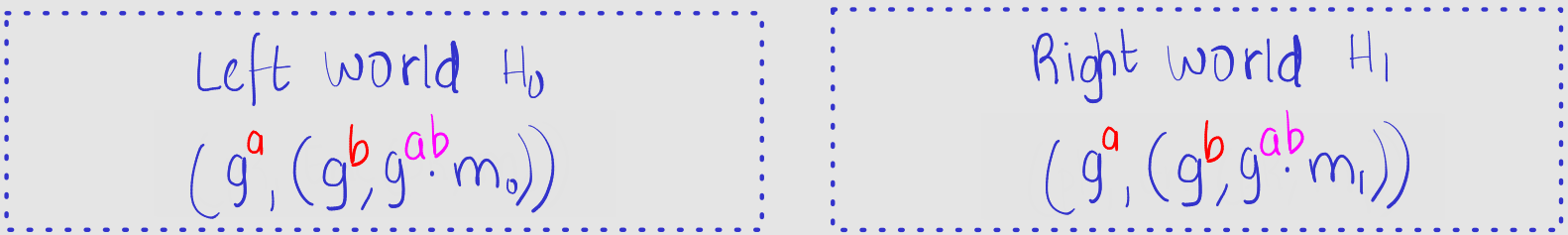

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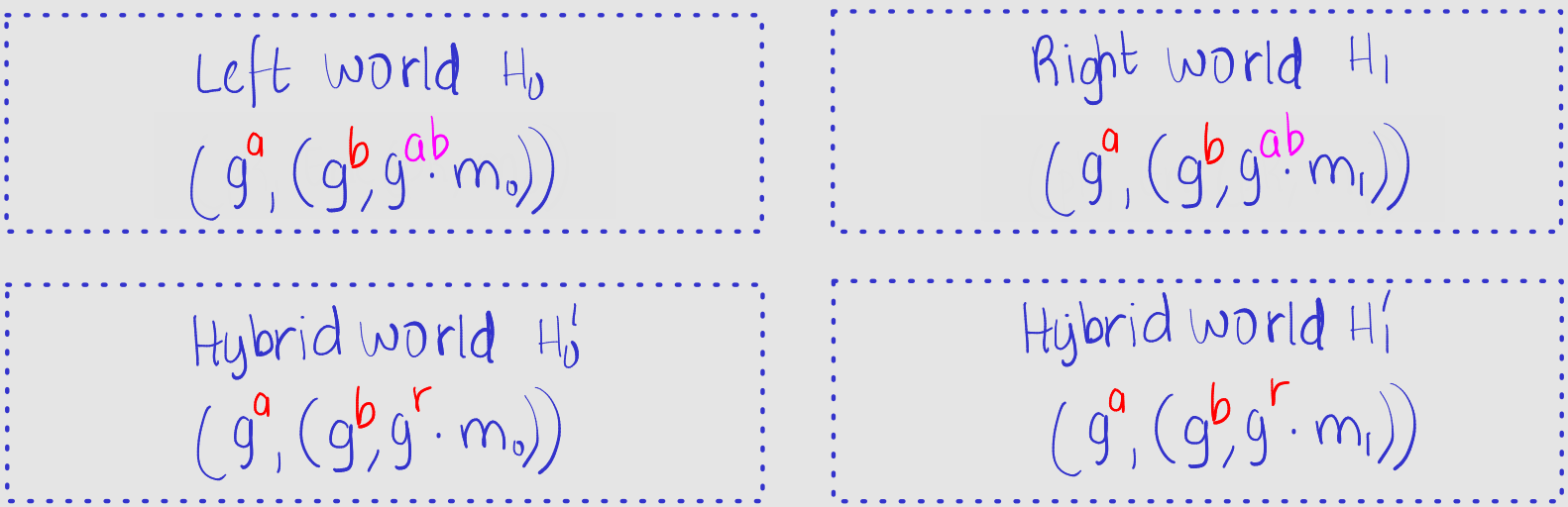

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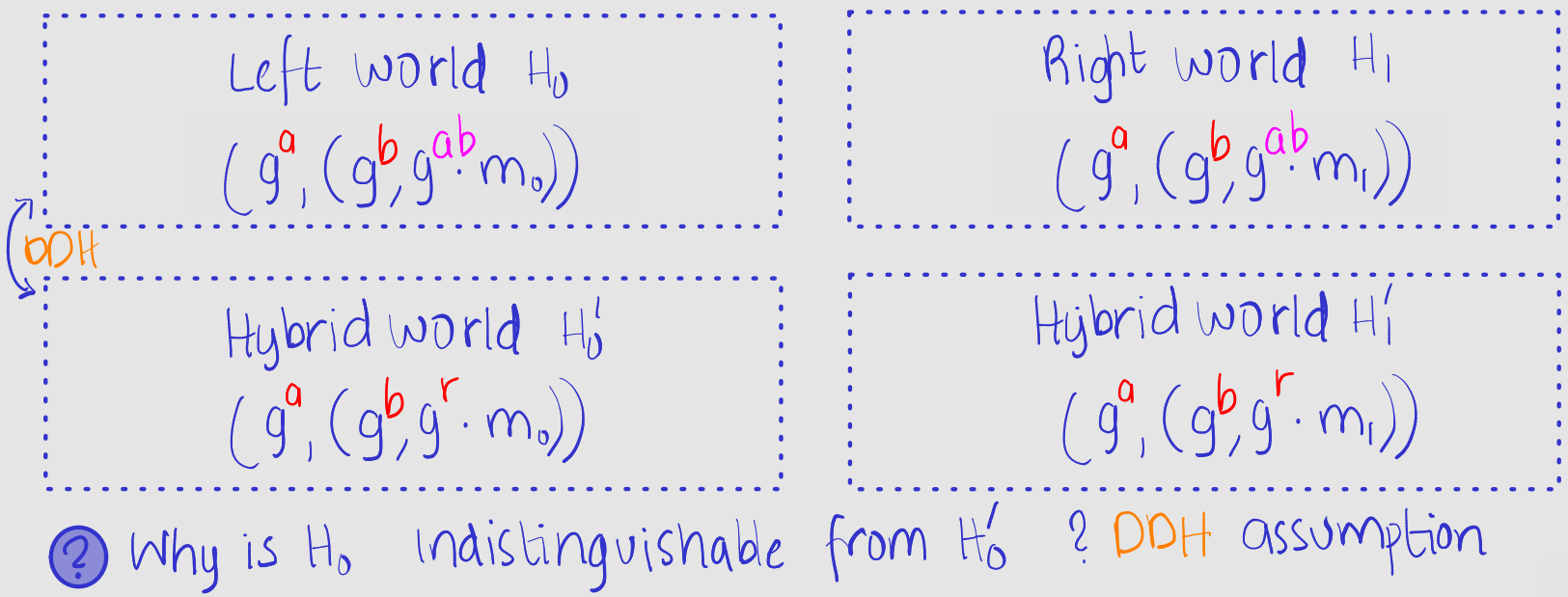


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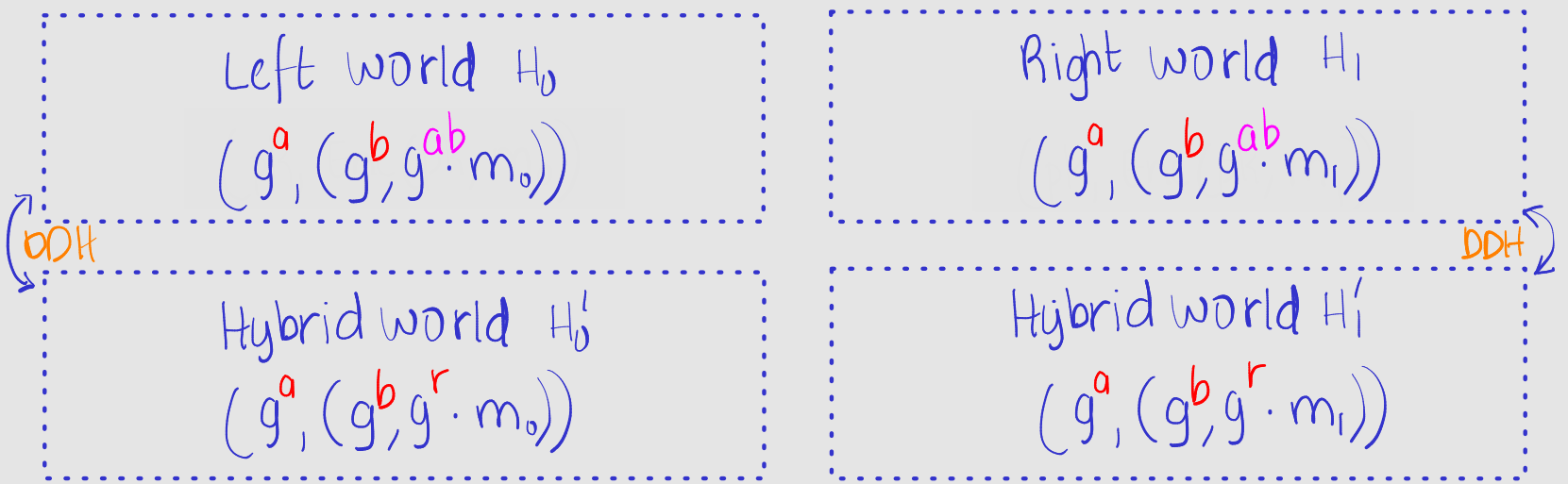


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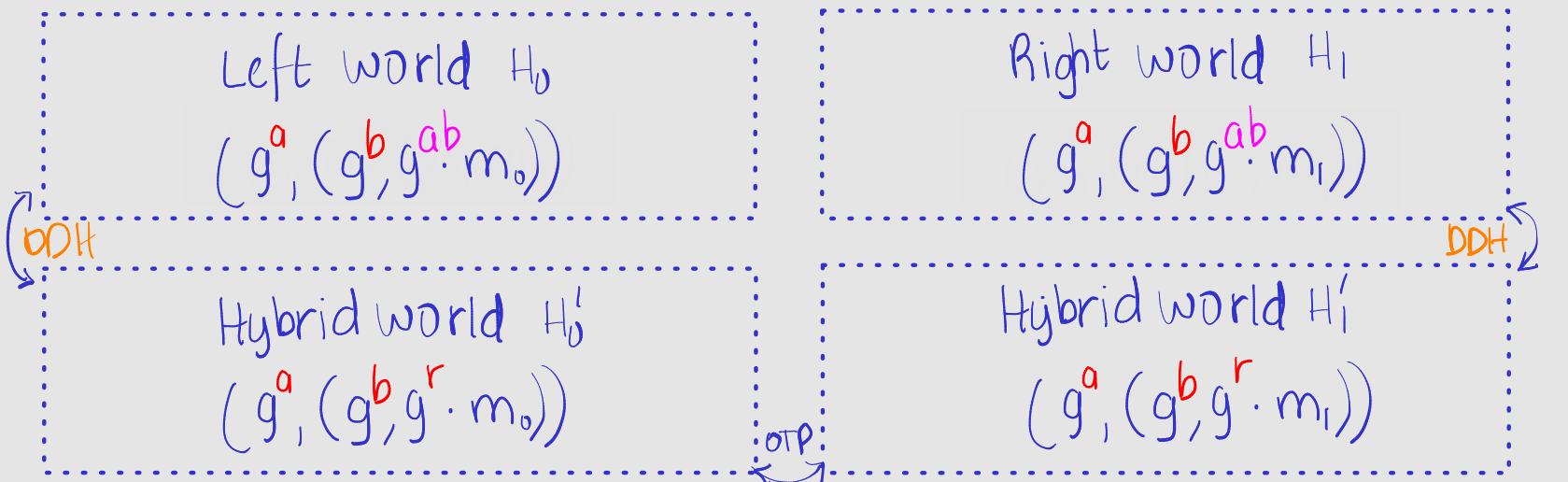


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**Claim 1 (Two-message KE $\rightarrow$ CPA-PKE)**

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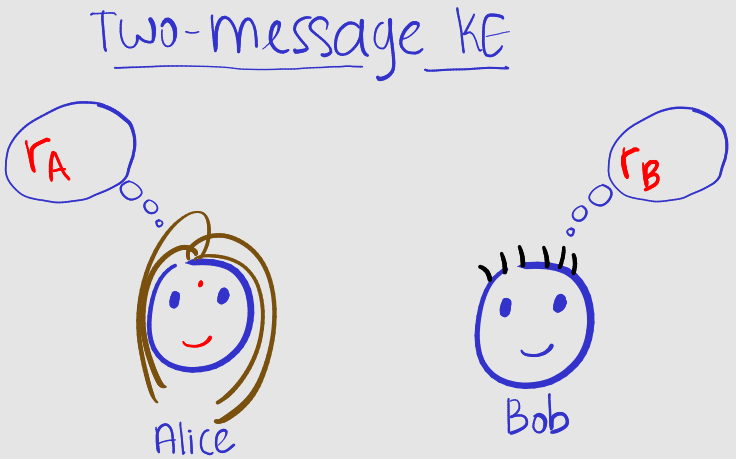

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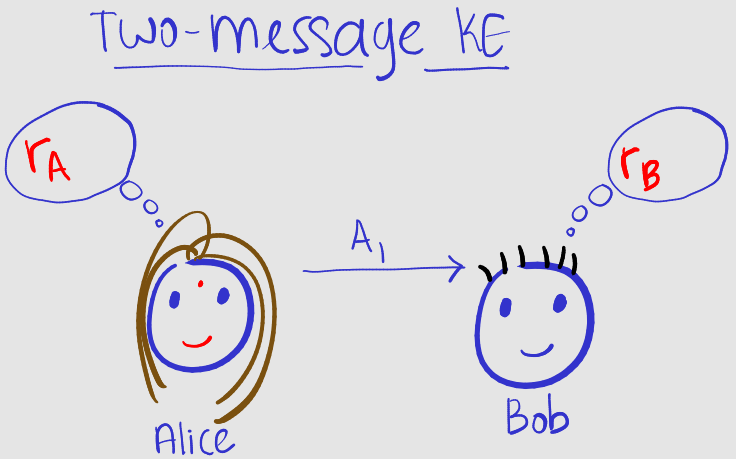

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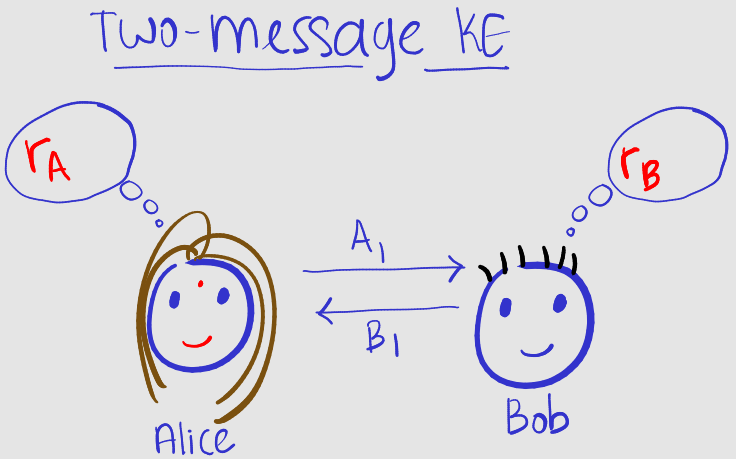

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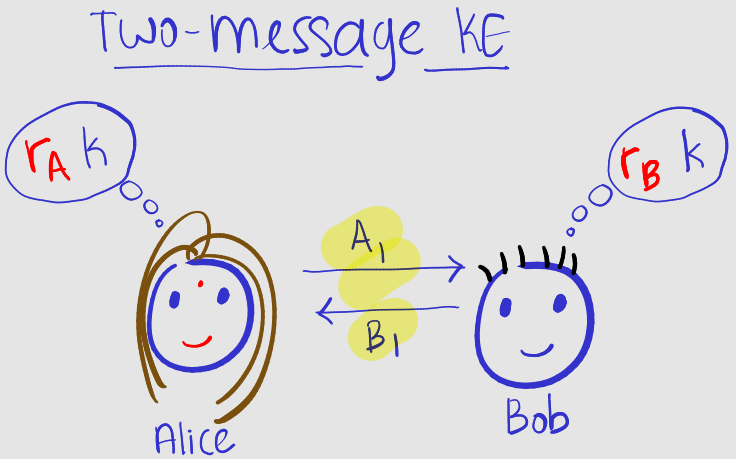

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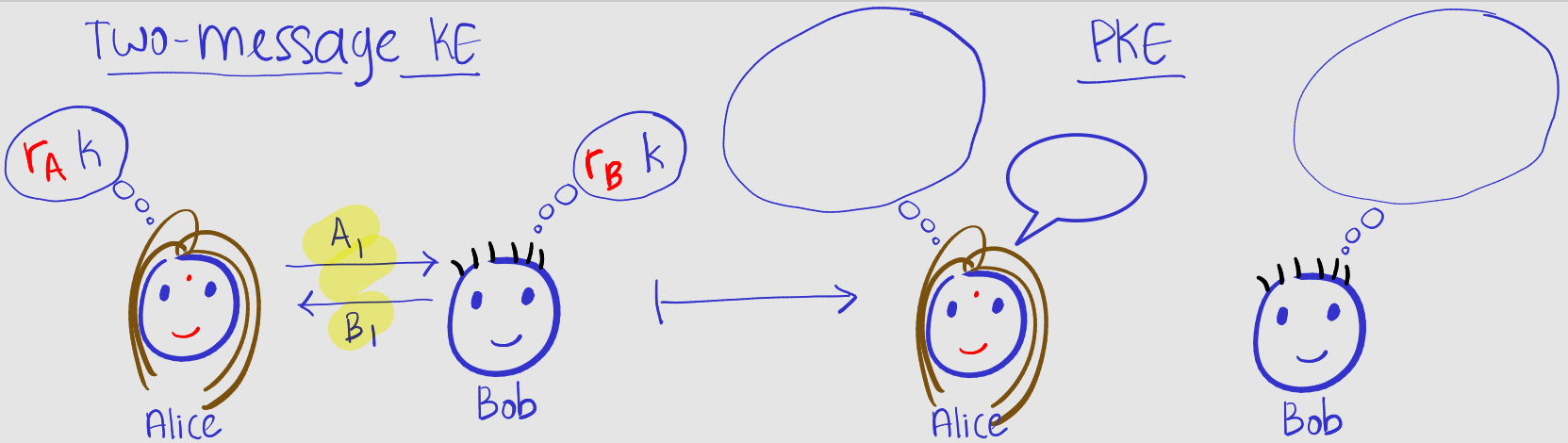

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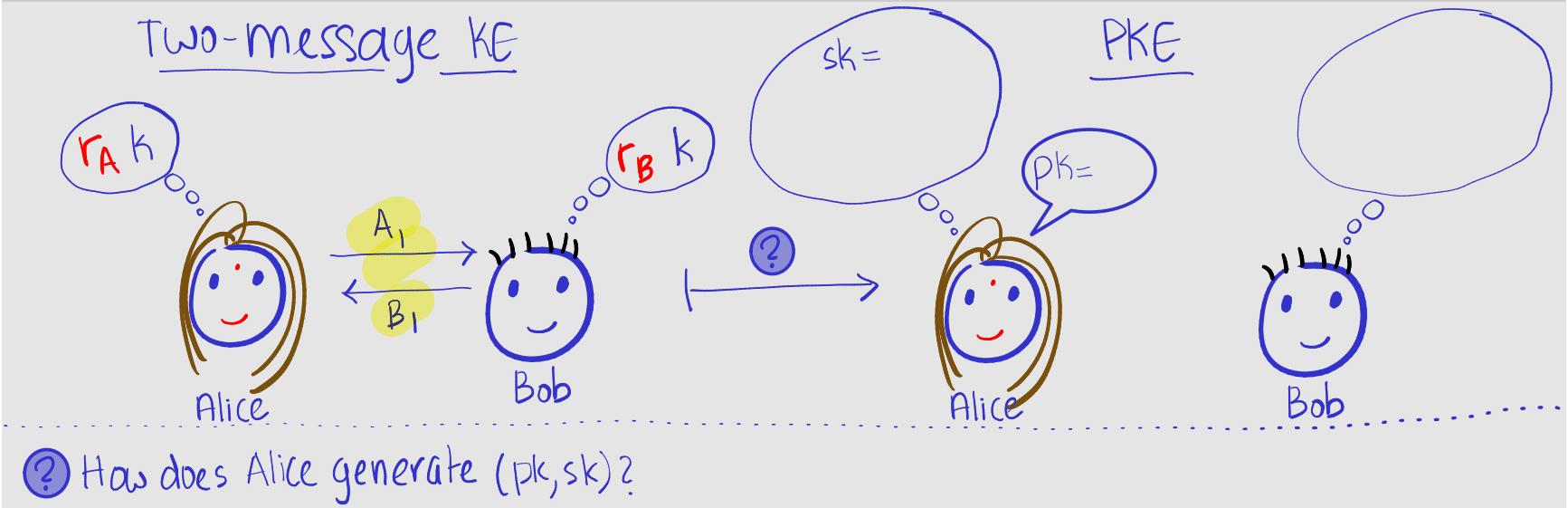

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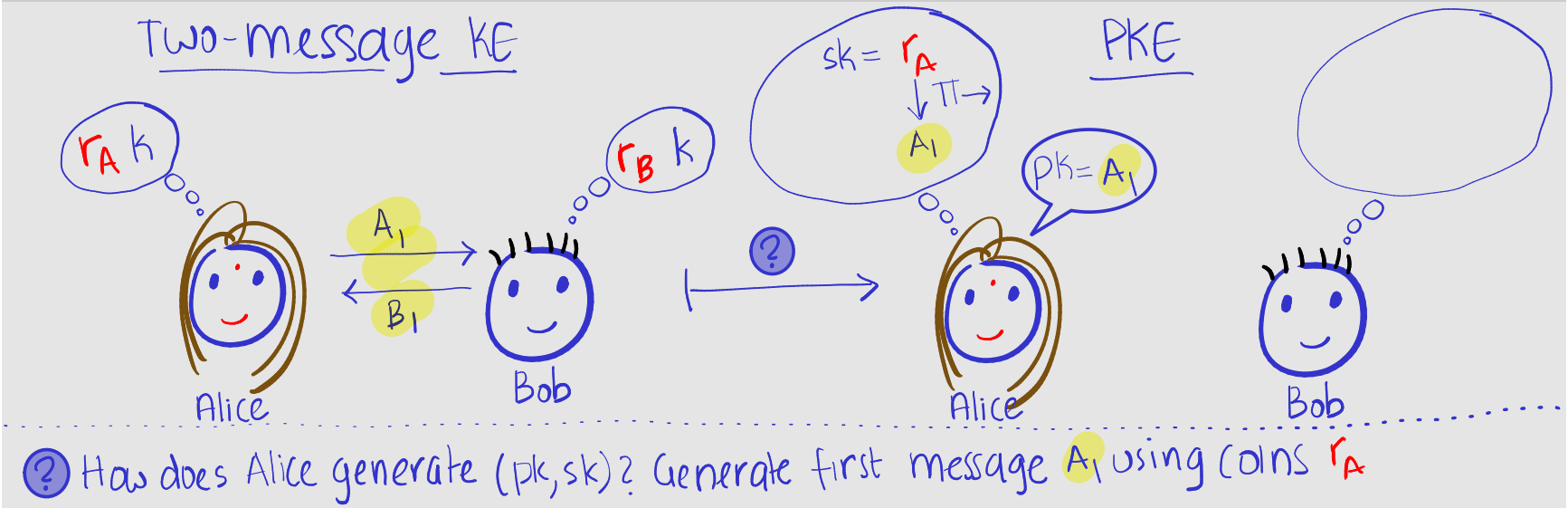

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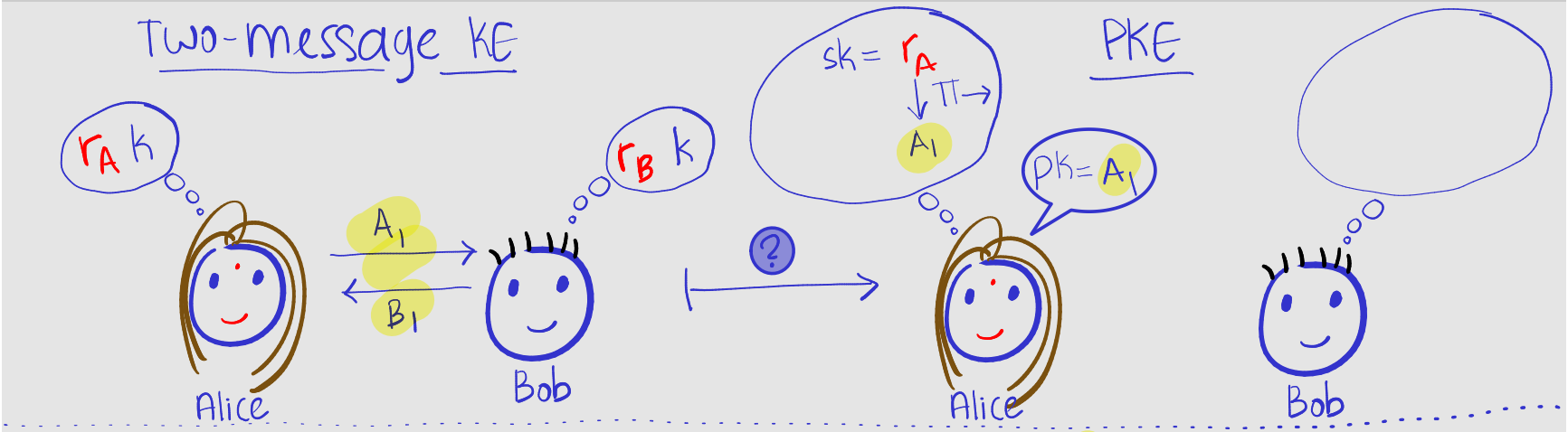


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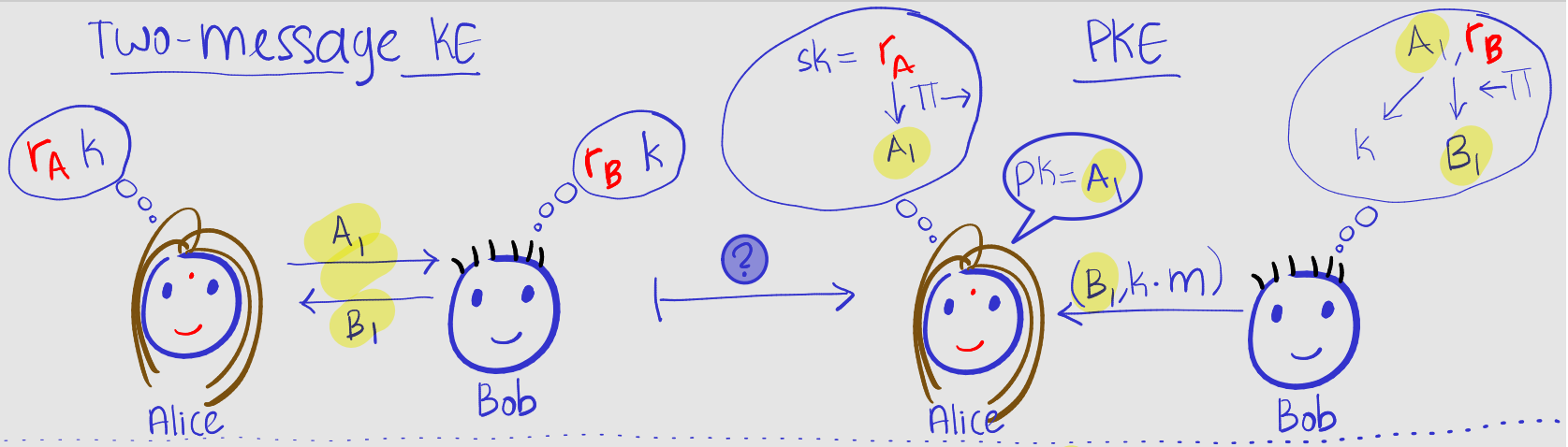


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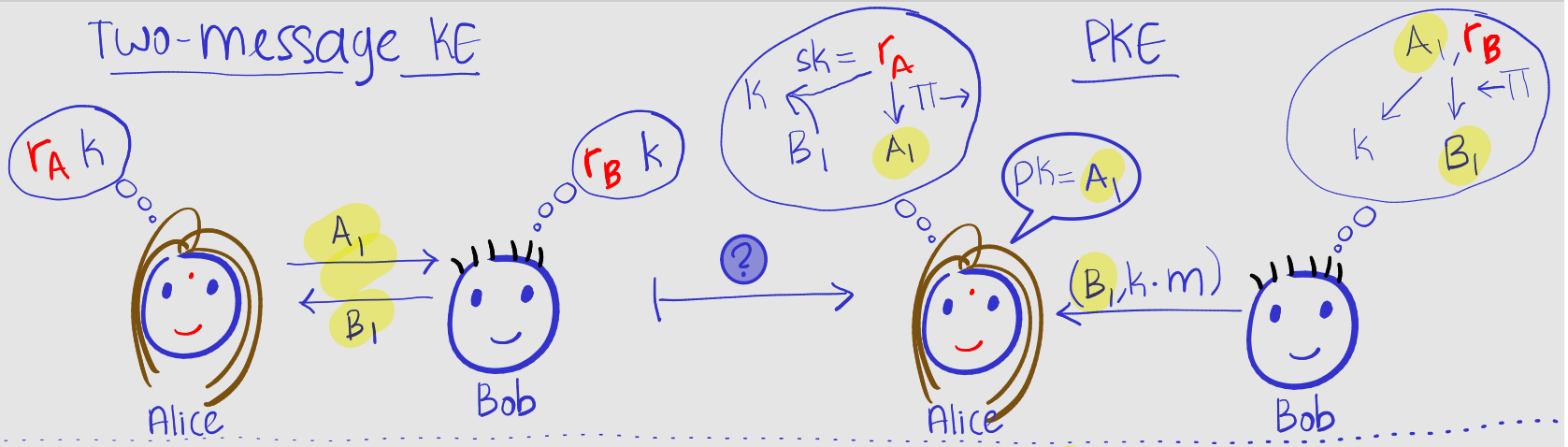


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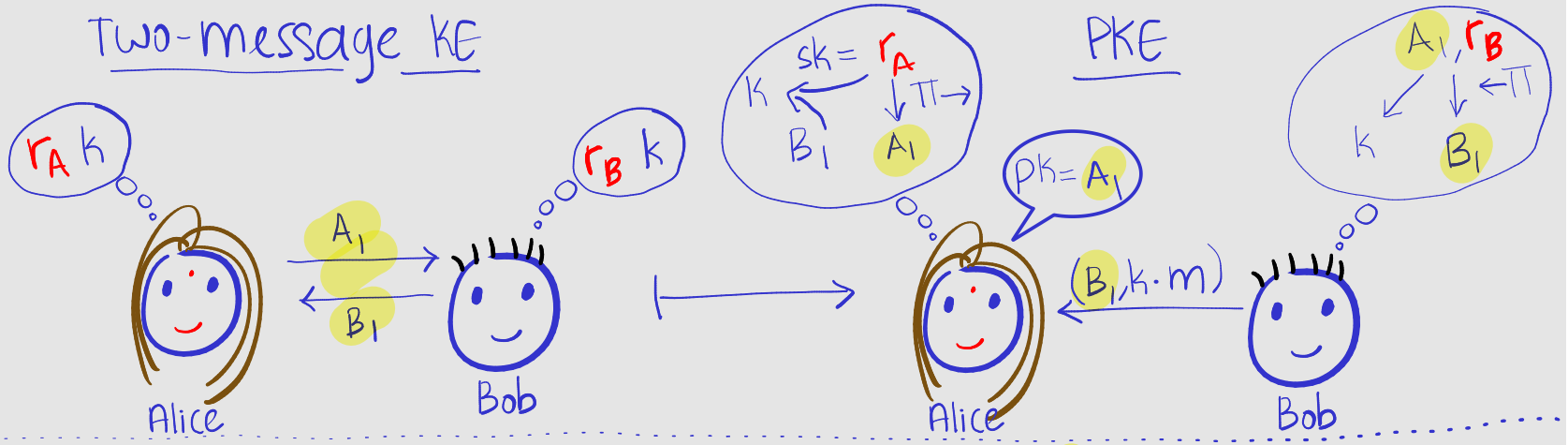


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## Exercise 1 (Converse to Claim 1: two-message KE ← CPA-PKE)

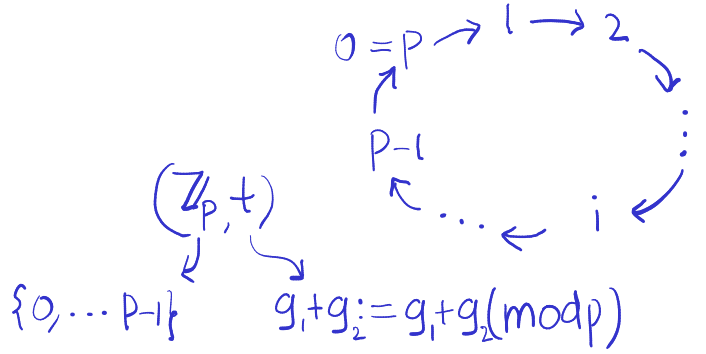
*If PKE exists then so does two-message key exchange.*

# Plan for this Lecture

1. Public-Key Encryption (PKE)
2. ElGamal PKE ⟵ DDH
3. Goldwasser-Micali PKE ⟵ QR

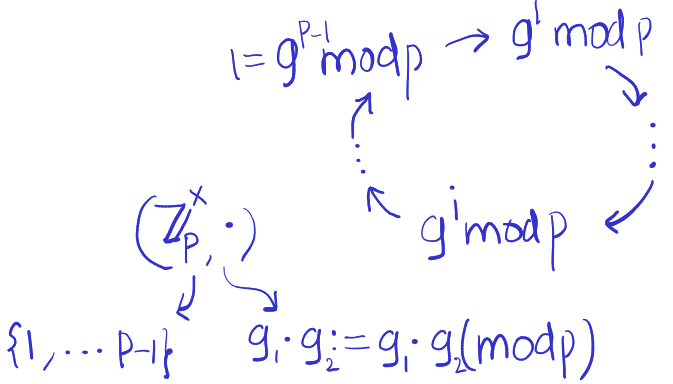
# But First, Some Number Theory

## Addition modulo prime $p$




- order $p$
- cyclic

## Multiplication modulo prime $p$




- order $p-1$
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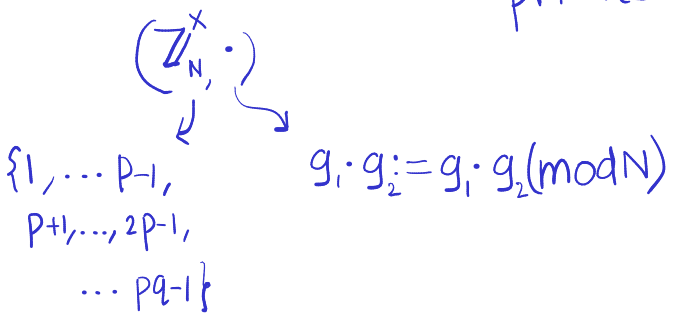
## Multiplication modulo $N = pq$

primes



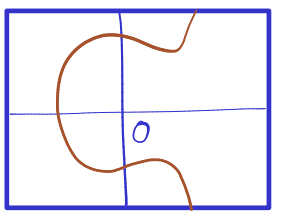

- order $(p-1)(q-1)$
- not cyclic

## Elliptic curves modulo prime $p$

$(E, +)$ "curve points addition"

solutions to

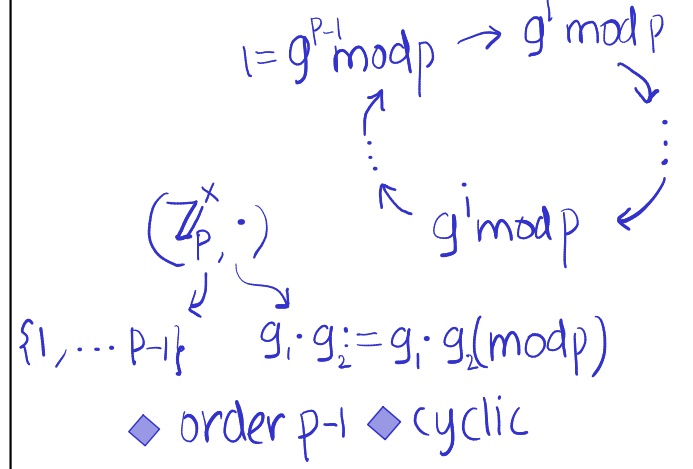
$y^2 = x^3 + Ax + B \pmod p$




- $|p+1 - \text{order}| \leq 2\sqrt{p}$
- cyclic

# But First, Some Number Theory

**Multiplication modulo prime $p$** $= 2p'+1$ ($p'$ prime)




- order $p-1$
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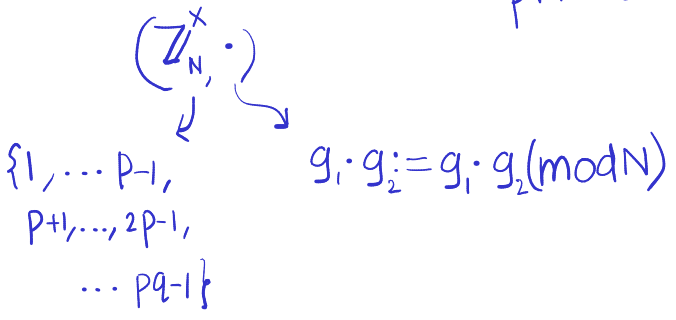
**Multiplication modulo $N = pq$** $= (2p'+1)\cdot(2q'+1)$ ($p', q'$ primes)

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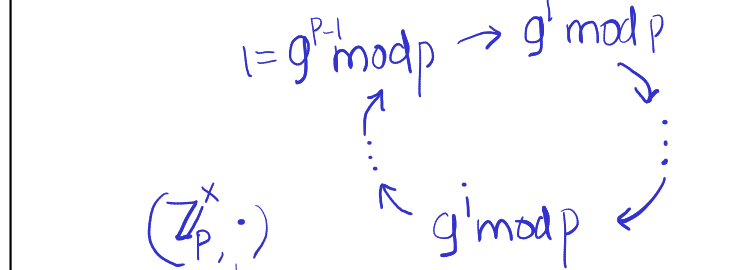


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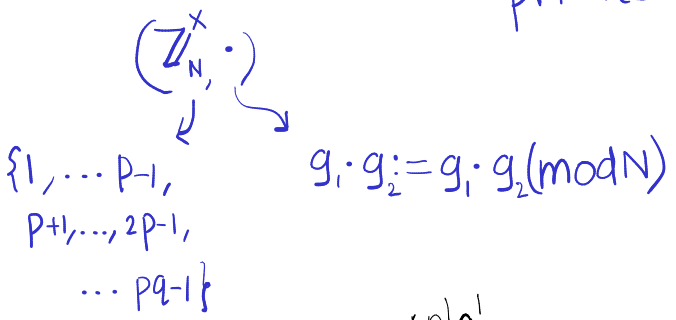
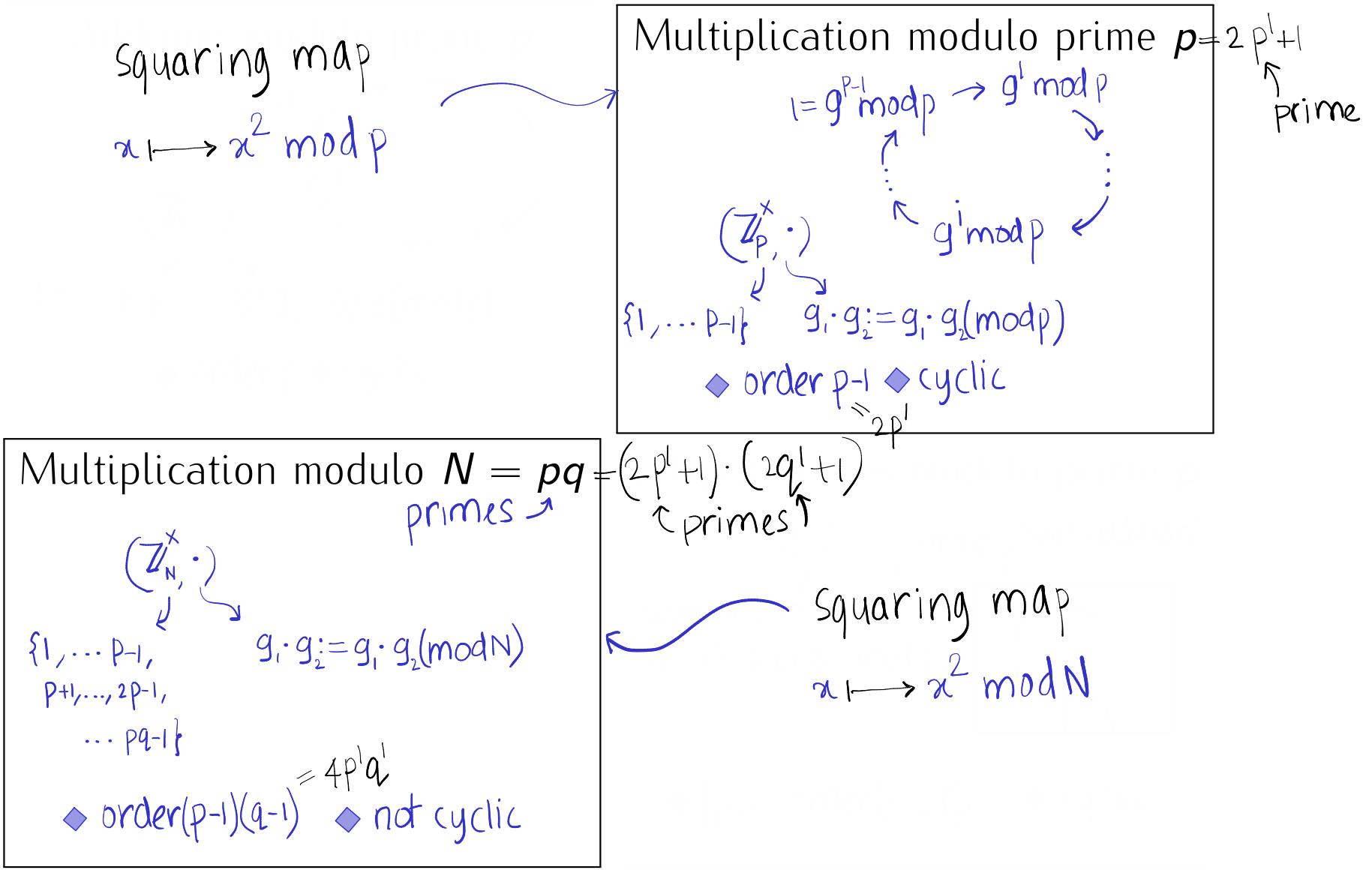
**Multiplication modulo $N = pq$** $= (2p'+1)\cdot(2q'+1)$ (primes)




- order $(p-1)(q-1)$ $= 4p'q'$
- not cyclic

# But First, Some Number Theory



# But First, Some Number Theory

Squaring map

$x \mapsto x^2 \bmod p$

Multiplication modulo prime $p = 2p'+1$ ($p'$ prime)

$1 = g^{p-1} \bmod p \to g^1 \bmod p$

$g^i \bmod p$

$(\mathbb{Z}_p^*, \cdot)$

$\{1, \dots p-1\}$ $\quad g_1 \cdot g_2 := g_1 \cdot g_2 \pmod p$

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Recall: this is a OWF, inverting as hard as factoring $N$

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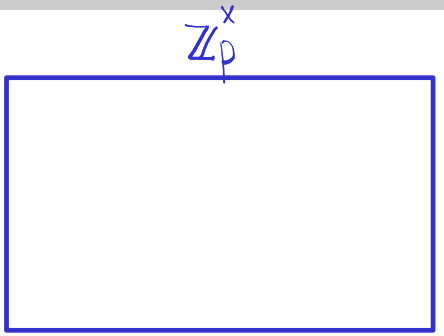
Looking ahead: our hard problem will be of form "is $y$ a square?"

# Squaring Map in $(\mathbb{Z}_p^\times, \cdot)$




- 2-1 map since 2 is not invertible modulo $2p'$

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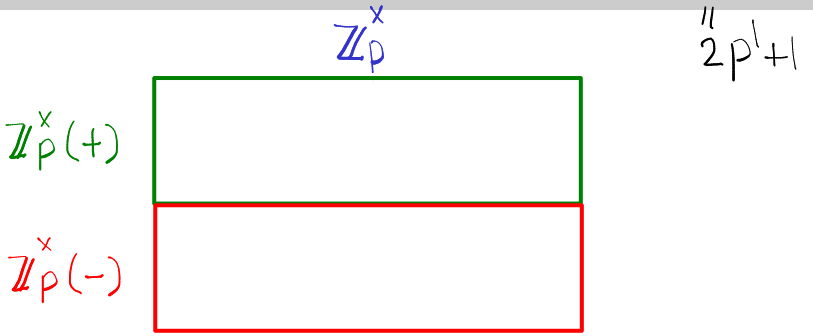


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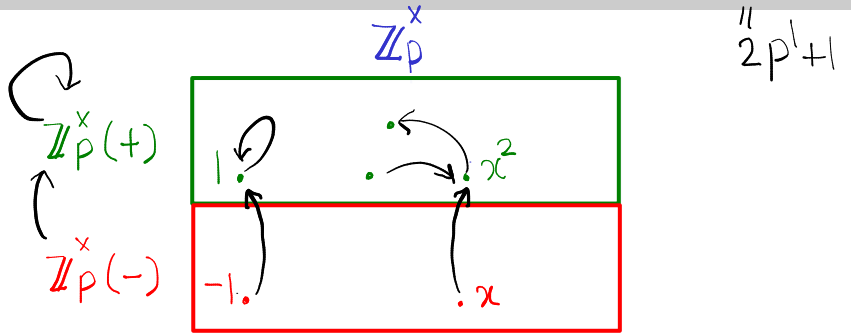


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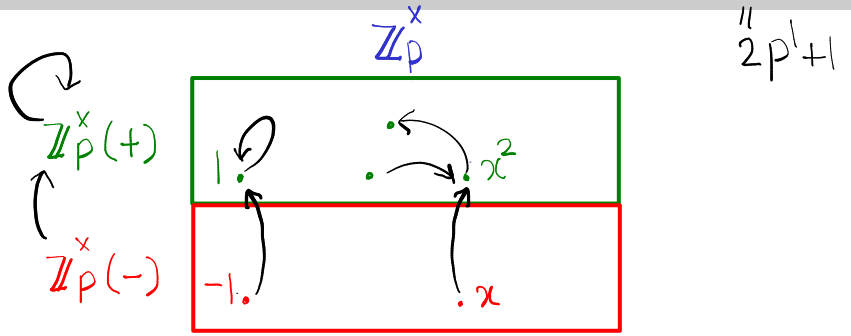


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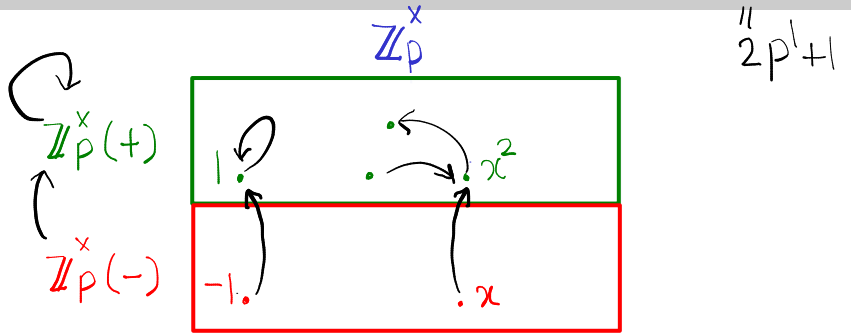


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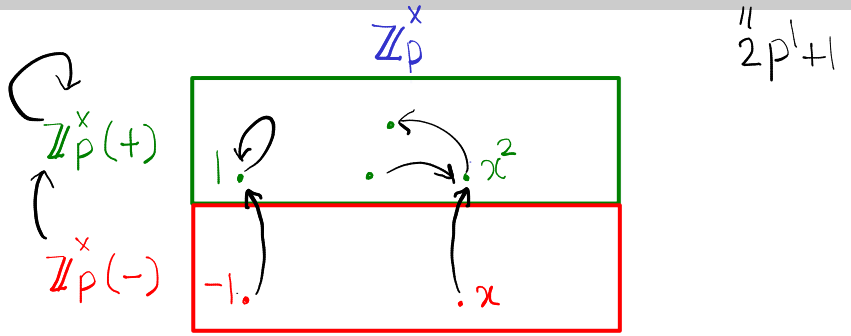


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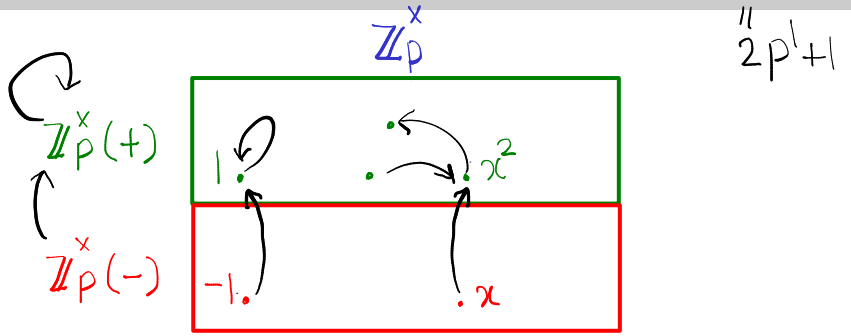


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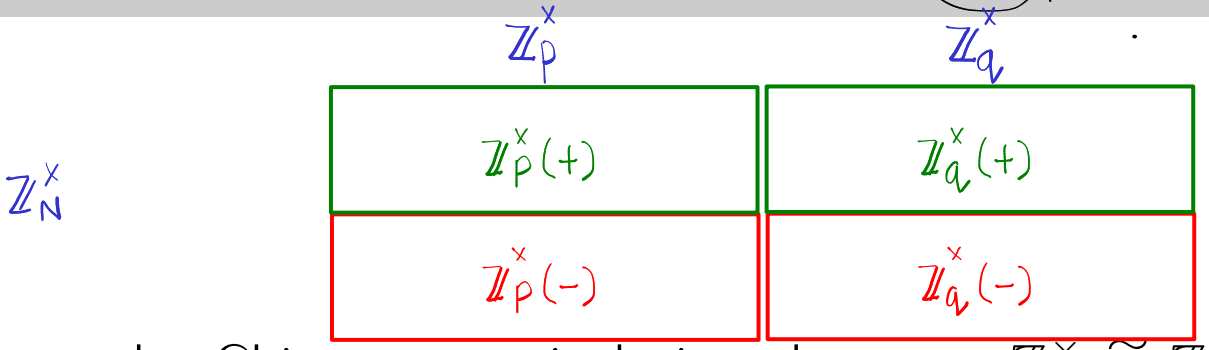
**Exercise 2 (Hint: Legendre symbol is multiplicative)**

*Show that DDH assumption doesn't hold in* $(\mathbb{Z}_p^\times, \cdot)$

# Squaring Map in $(\mathbb{Z}_N^\times, \cdot)$




- 4–1 map by Chinese remaindering theorem: $\mathbb{Z}_N^\times \cong \mathbb{Z}_p^\times \times \mathbb{Z}_q^\times$
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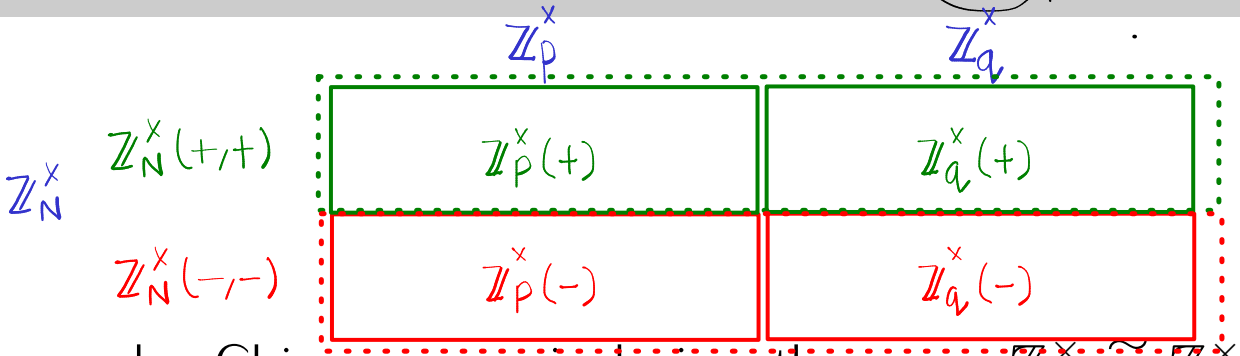


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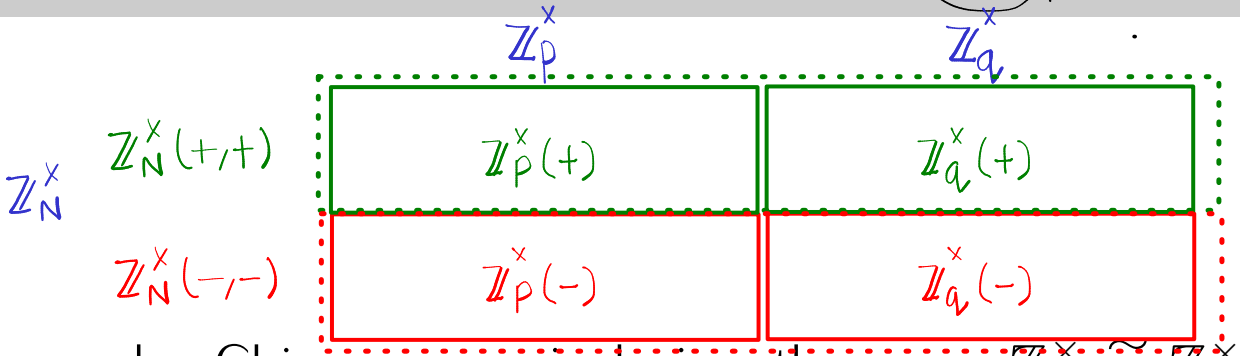


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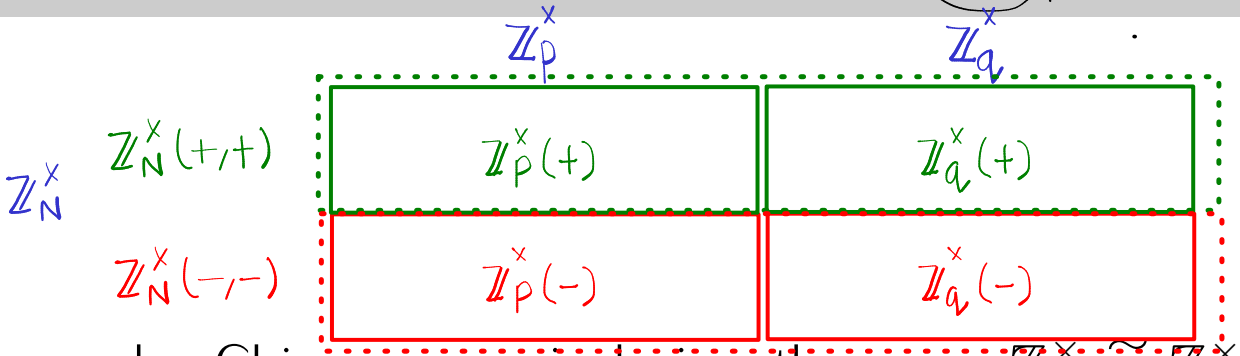


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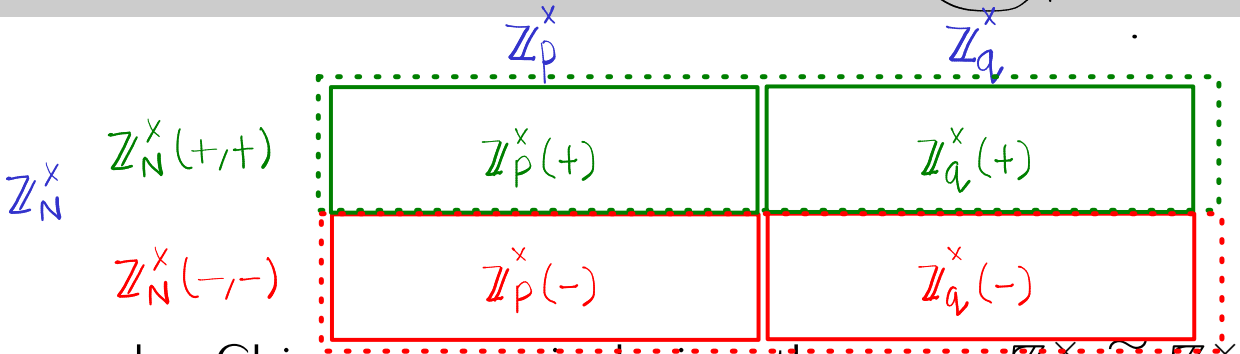


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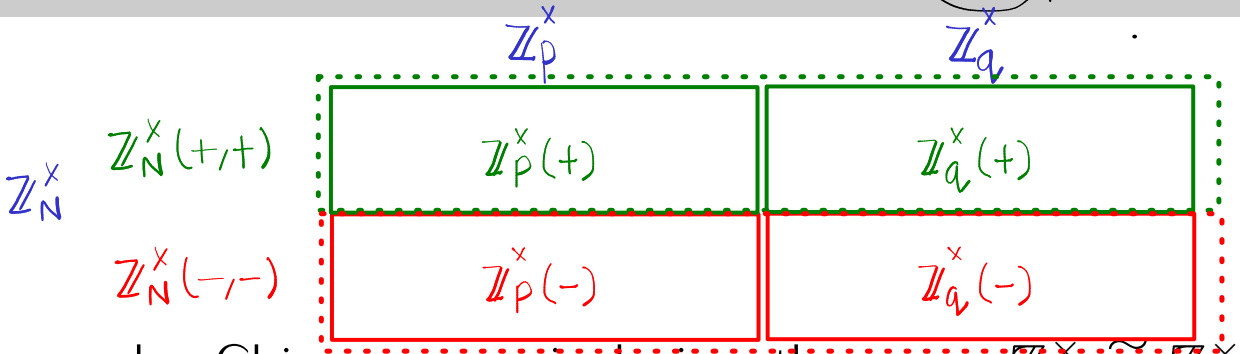


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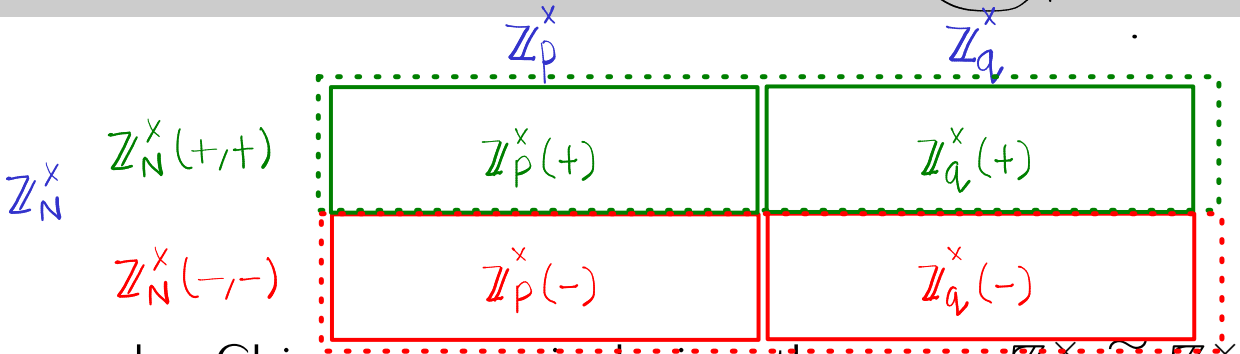


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**Assumption 3 (Quadratic residuosity (QR) assumption in $(\mathbb{Z}_N^\times, \cdot)$)**

*The QR assumption w.r.to* $\mathsf{S}$ *holds if for all distinguishers* $\mathsf{D}$*, the following is negligible:*

$$\delta(n) := \Pr_{\substack{N \leftarrow \mathsf{S}(1^n) \\ y \leftarrow \mathbb{Z}_N^\times(+,+)}}[\mathsf{D}(N,y) = 0] - \Pr_{\substack{N \leftarrow \mathsf{S}(1^n) \\ y \leftarrow \mathbb{Z}_N^\times(-,-)}}[\mathsf{D}(N,y) = 0]$$

# Goldwasser-Micali Public-Key Bit Encryption

- Key idea: encode message in the "sign"
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- *Correctness of decryption:* since $r^2 \in \mathbb{Z}_N^\times(+,+)$, $c \in \mathbb{Z}_N^\times(+,+)$ iff $m = 0$

# Goldwasser-Micali PKE is CPA-secret

**Theorem 2 (QR → CPA-PKE)**

*Goldwasser-Micali PKE is CPA-secret under QR assumption.*

**Proof.** ∃D against QR ⇐ ∃Eve against PKE.


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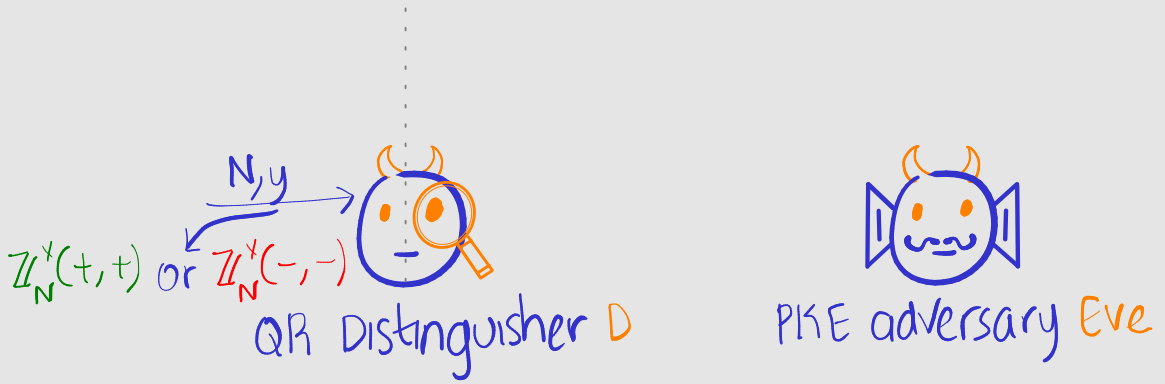

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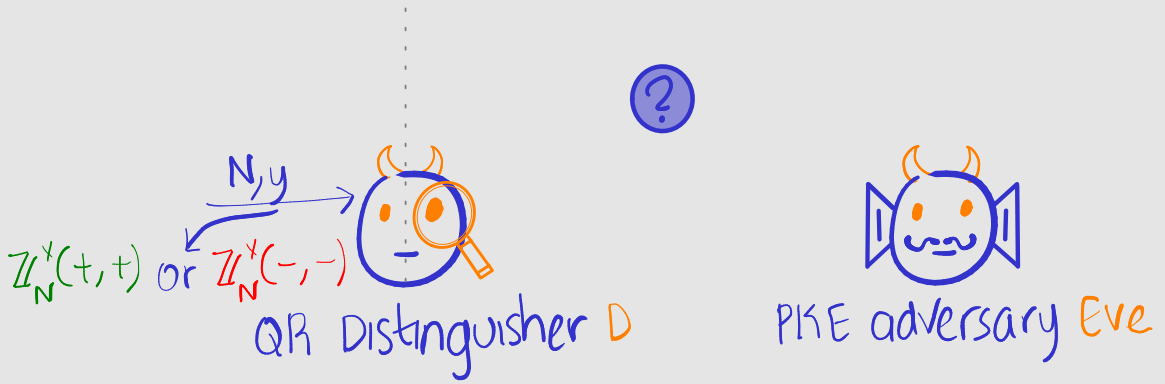

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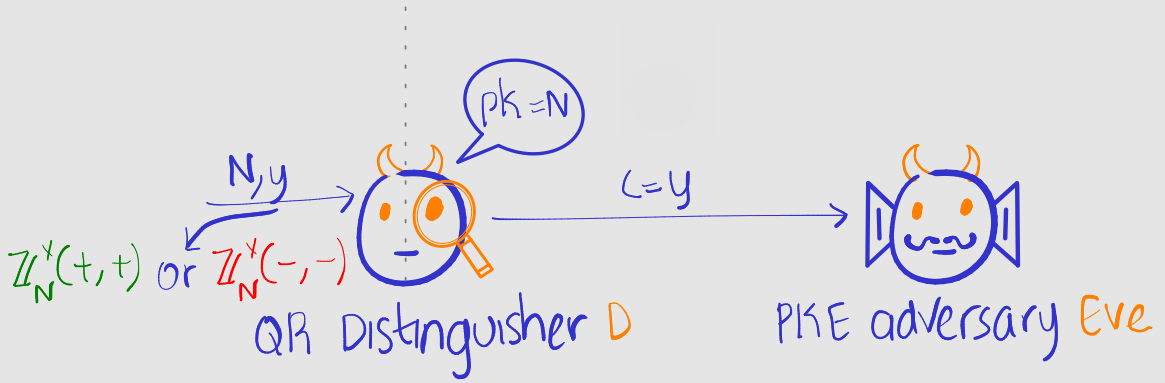

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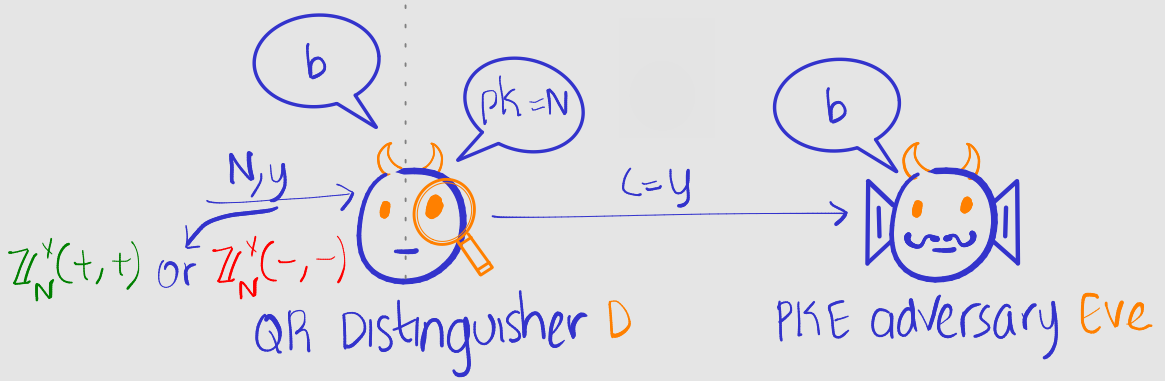

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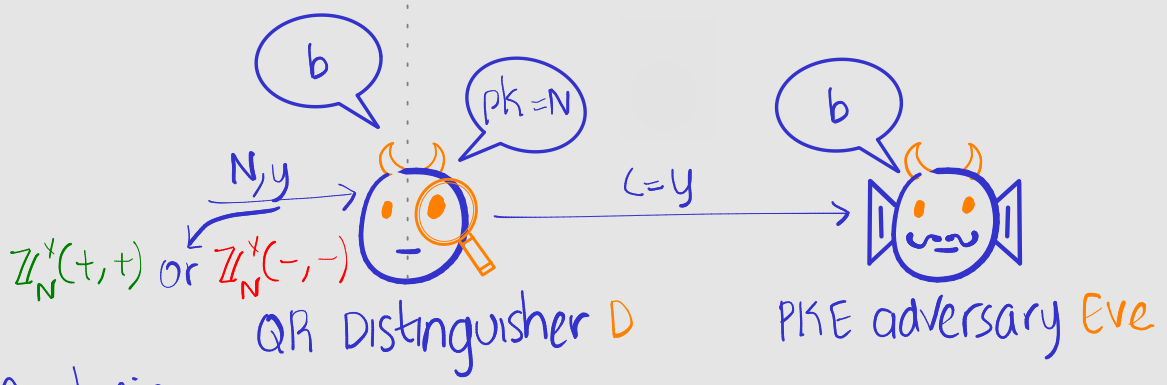


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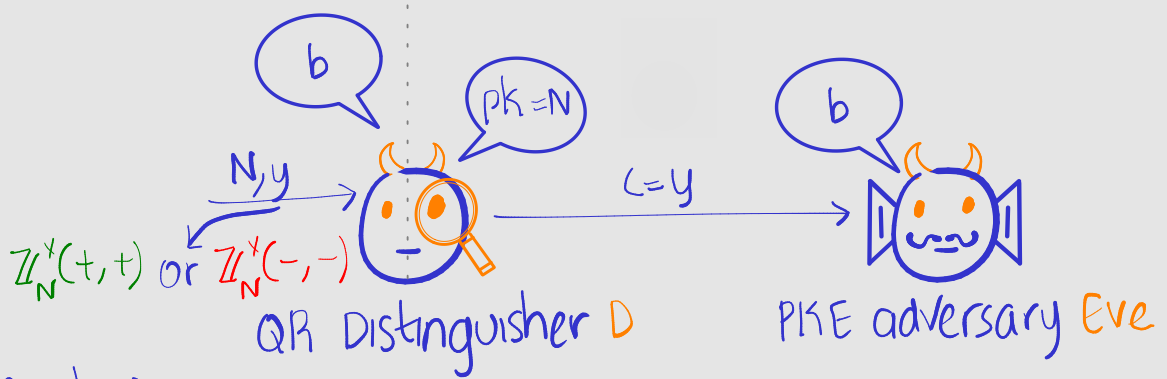


Analysis

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*Goldwasser-Micali PKE is CPA-secret under QR assumption.*

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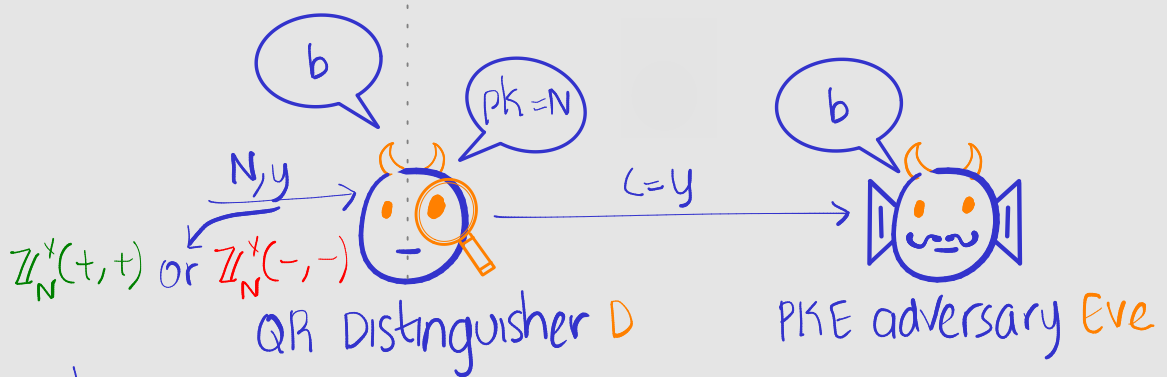


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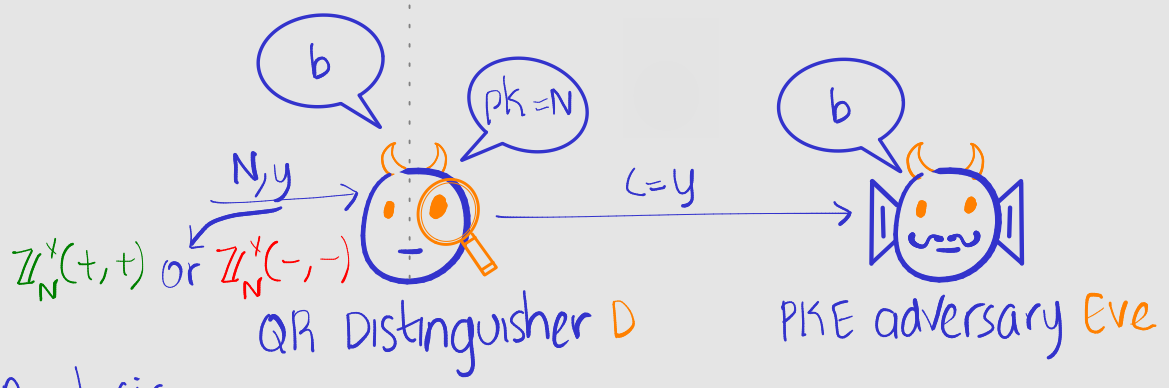


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⇒ non-negligible

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# What About Multiple Bits? Encrypt Bit by Bit

**Pseudocode 5 (Modified encryption algorithm)**

- *Encryption* $\mathsf{Enc}(pk, m =: (m_1, \ldots, m_\ell)$*:*
  1. *Sample random* $r_1, \ldots, r_\ell \leftarrow \mathbb{Z}_N^\times$
  2. *Output* $c_1, \ldots, c_\ell$*, where* $c_i := (-1)^{m_i} \cdot r_i^2 \bmod N$

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## Exercise 3

1. *Show that QR is random self-reducible*
2. *Can we exploit this to get a tight reduction for multiple bits?*

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- Some open questions:
  1. CPA-PKE $\xrightarrow{?}$ CCA-PKE
     - Recall that CPA-SKE $\rightarrow$ CCA-SKE!
  2. DLog $\xrightarrow{?}$ CPA-PKE
     - We know CDH $\rightarrow$ CPA-PKE in the random-oracle model

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- Next Tuesday (03/Sep): public-key encryption from lattices
  - Lattices and learning with errors (LWE) problem
  - Believed to be secure against quantum eavesdroppers!
  - Regev's PKE

# References

1. [KL14, Chapter 12] for details on this lecture.
2. Goldwasser–Micali PKE was presented in [GM84]. ElGamal PKE was presented in [ElG84].
3. Barak's exposition [Bar17] is an excellent source to understand the interplay of structure and hardness in PKE (and OWF)


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