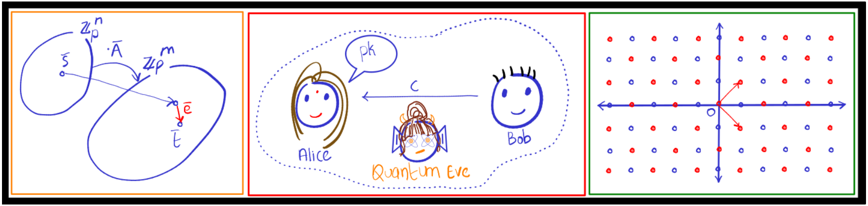

CS783: Theoretical Foundations of Cryptography

Lecture 10 (03/Sep/24)

Instructor: Chethan Kamath

Recall from Last Lecture

- Task 4: Public-key encryption
  - Modelled setting and security (CPA secrecy)

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  - Conceptual takeaways:
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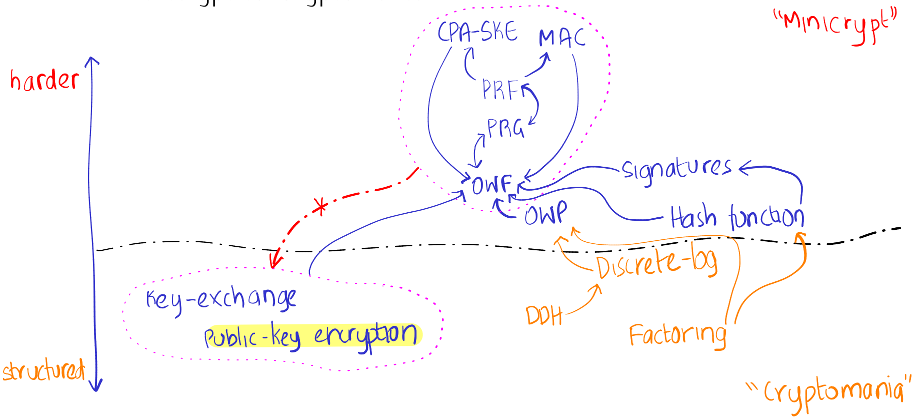
       ↖ Two ways to generate the same "OTP" $(g^a)^b \rightarrow g^{ab} \leftarrow (g^b)^a$
- Some open questions:
  1. CPA-PKE $\overset{?}{\rightarrow}$ CCA-PKE
     - Recall that CPA-SKE → CCA-SKE!
  2. DLog $\overset{?}{\rightarrow}$ CPA-PKE
     - We know CDH → CPA-PKE in the "random-oracle model"

- Minicrypt to Cryptomania



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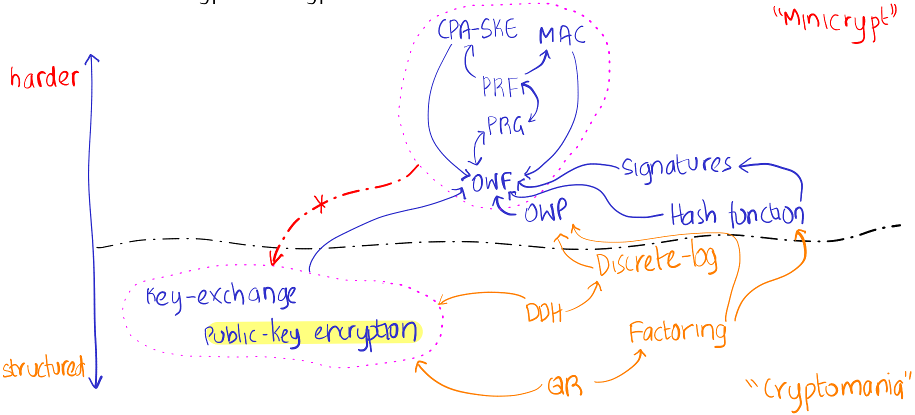

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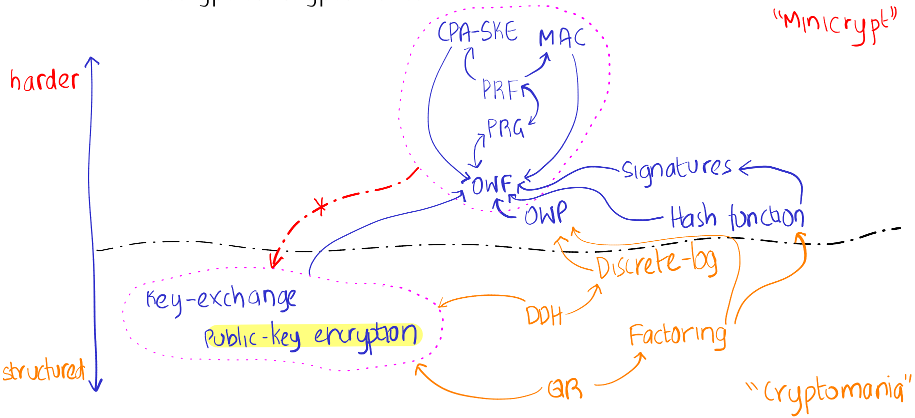

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Plan for This Lecture...

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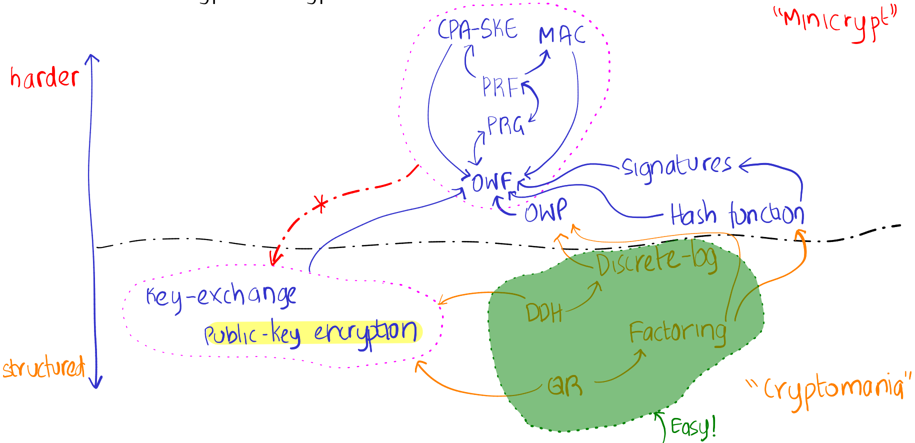

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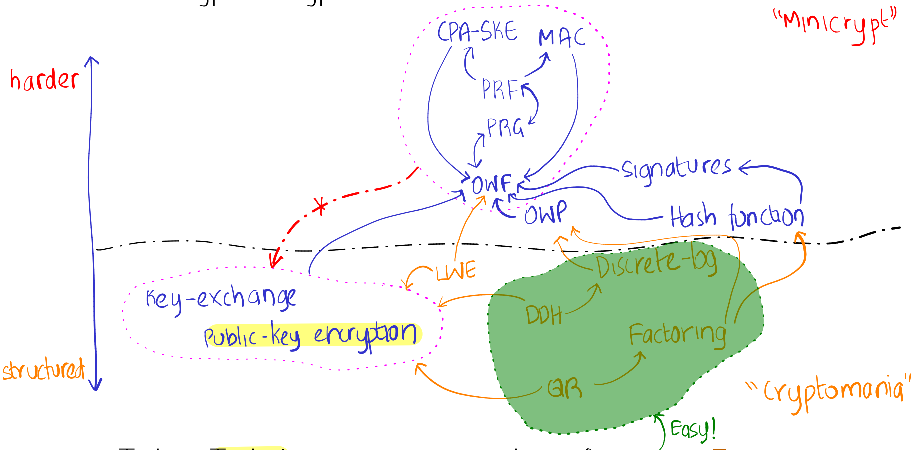

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Plan for This Lecture...

computational secrecy

General *template*:

1. Identify the task — Public-key encryption
2. Come up with precise threat model M (a.k.a security model) — Eavesdroppers
   - Adversary/Attack: What are the adversary's capabilities?
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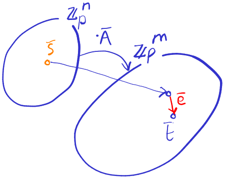
   Assuming quantum hardness of "learning with errors" (LWE)

Plan for This Lecture...

1 Motivation: Quantum Adversaries



2 Learning with Errors (LWE)

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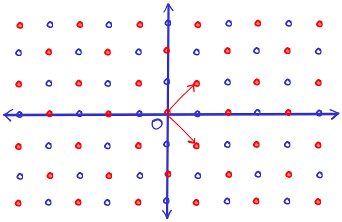

4 LWE and Lattices

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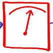
     $\psi \to$  $\to \bar{b}$

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  5. PPT adversary .... Quantum PT adversary

     

Modelling the Setting for Quantum Adversaries...

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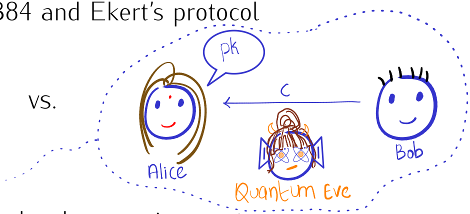

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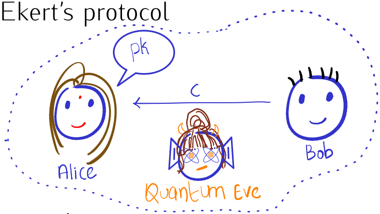

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  - Honest parties are classical; adversary is quantum
  - Possible attack scenario: "Harvest now, decrypt later"
    - Potential adversaries: Five Eyes, state actors...

Modelling the Setting for Quantum Adversaries...

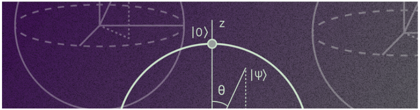
NEWS

NIST Releases First 3 Finalized Post-Quantum Encryption Standards

August 13, 2024

Quantum Resistance and the Signal Protocol

ehrenkret on 19 Sep 2023



 Security Research

February 21, 2024

iMessage with PQ3: The new state of the art in quantum-secure messaging at scale

Posted by Apple Security Engineering and Architecture (SEAR)

विज्ञान एवं प्रौद्योगिकी विभाग
DEPARTMENT OF
SCIENCE & TECHNOLOGY

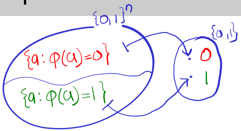
सत्यमेव जयते

National Quantum Mission (NQM)

The Union Cabinet, approved the National Quantum Mission (NQM) on 19th April 2023 at a total cost of Rs.6003.65 crore from 2023-24 to 2030-31, aiming to seed, nurture and scale up scientific and industrial R&D and create a vibrant & innovative ecosystem in Quantum Technology (QT). This will accelerate QT

- Recent effort to research/deploy post-quantum cryptography

What is Easier for Quantum Computers?...



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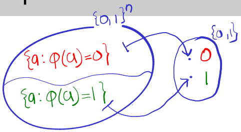

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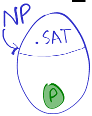

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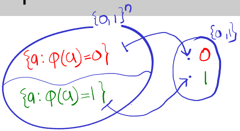

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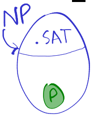

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Theorem 1 (Grover's algorithm)

There is a quantum algorithm that given φ (represented as a classical circuit) finds a satisfying assignment in time $2^{O(n/2)}$

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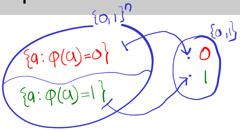

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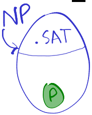

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- Impact on cryptography: SKEs broken in *quantum* time $2^{O(n/2)}$
  - Solution: double key-size (use 256-bit AES instead of 128-bit)

What is Easier for Quantum Computers?...

Addition modulo prime p

$0 = p \to 1 \to 2 \to \cdots \to i \to \cdots \to p-1 \to 0$

$(\mathbb{Z}_p, +)$

$\{0, \dots, p-1\}$ $\quad$ $g_1 + g_2 := g_1 + g_2 \pmod p$

- order p
- cyclic

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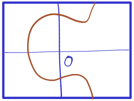
- not cyclic

Elliptic curves modulo prime p

$(E, +)$ "curve points addition"

solutions to
$y^2 = x^3 + Ax + B \pmod p$



- $|p+1 - \text{order}| \leq 2\sqrt{p}$
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DLog easy

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DDH hard in "subgroup"

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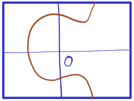

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What is Easier for Quantum Computers?...

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  - Input: $f : (\mathbb{Z}_\ell, +) \to \mathbb{G}$ that is "periodic"
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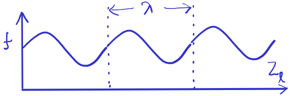

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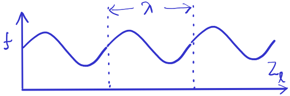

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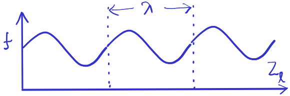

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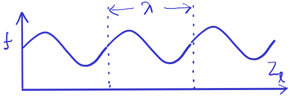

- Classical setting: *PPT* algorithms believed not to exist for certain fs. E.g.:
  1. $f_{a,N}(x) := a^x \bmod N$, where $\mathbb{G} = (\mathbb{Z}_N^\times, \cdot)$ and $a \leftarrow \mathbb{Z}_N^\times$
     - ? What is the period of $f_{a,N}$?

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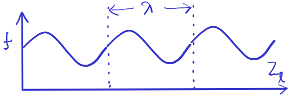

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  1. $\mathrm{f}_{a,N}(x) := a^x \bmod N$, where $\mathbb{G} = (\mathbb{Z}_N^\times, \cdot)$ and $a \leftarrow \mathbb{Z}_N^\times$
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What is Easier for Quantum Computers?...

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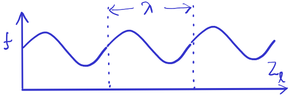

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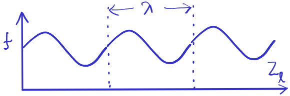

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$$f_{g,h}(x+\log_g h, y+1) = g^{x+\log_g h} \cdot h^{-y-1} = g^x \cdot \cancel{h} \cdot h^{-y} \cdot \cancel{h^{-1}} = g^x h^{-y} = f_{g,h}(x,y)$$

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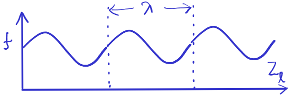

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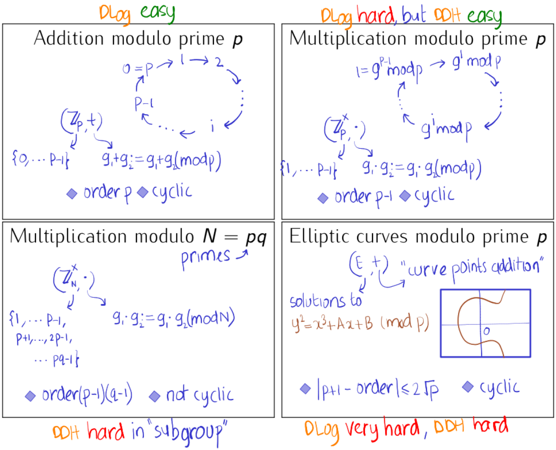
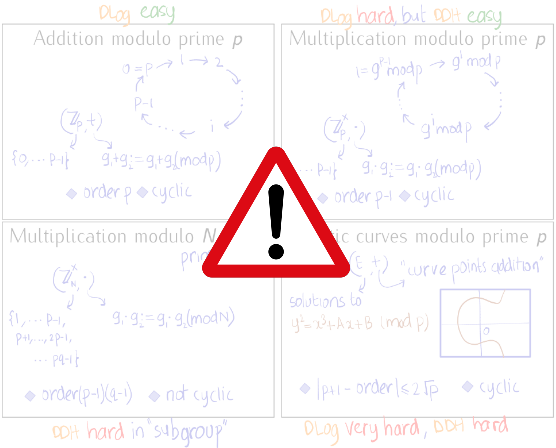
Theorem 2 (Shor's algorithm)

There is a quantum algorithm that finds the period λ of a periodic function f as above (represented as a classical circuit) in time polynomial in $|\mathbb{Z}_\ell| = \log(\ell)$.

What is Easier for Quantum Computers?...



- Corollary: factoring and discrete log are quantum *easy*!



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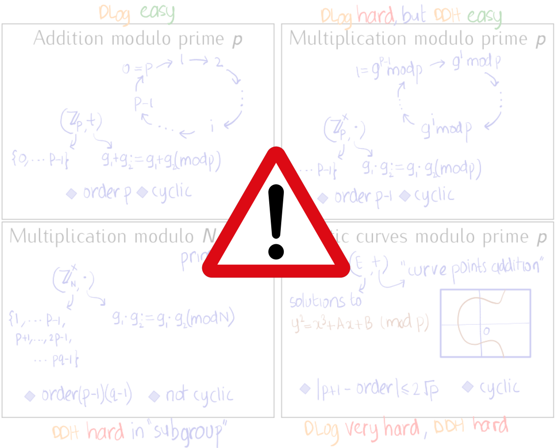

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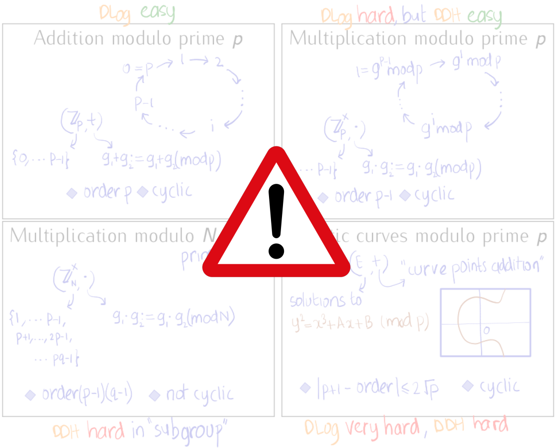

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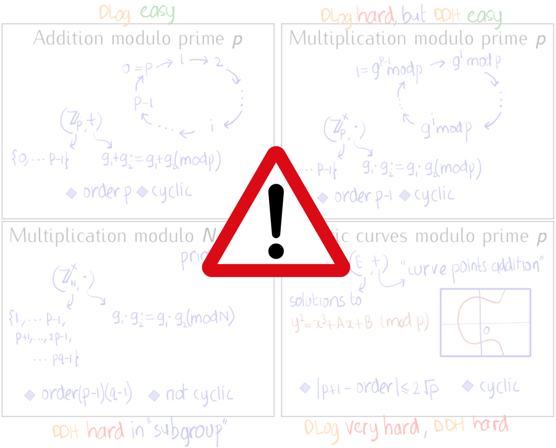


- Corollary: factoring and discrete log are quantum *easy*!

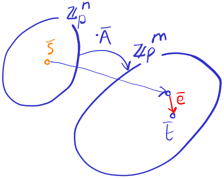
Impact on cryptography: PKEs from previous lecture insecure!

  - We need: hardness assumption that holds against QPT...
  - ...that has sufficient structure to allow PKE/key exchange

Plan for this Lecture

1 Motivation: Quantum Adversaries

2 Learning with Errors (LWE)



3 Cryptography from LWE

4 LWE and Lattices

Solving Linear Equations Over $(\mathbb{Z}_p, +, \cdot)$...

- Let's consider $(\mathbb{Z}_p, +, \cdot)$, i.e., $(\mathbb{Z}_p, +)$ with multiplication over $\mathbb{Z}_p^\times$

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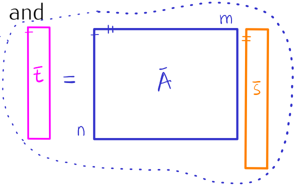

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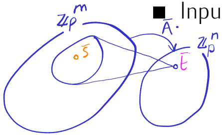

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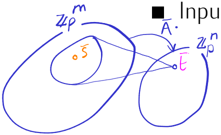

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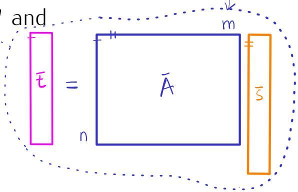

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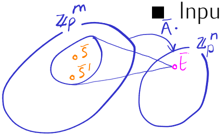

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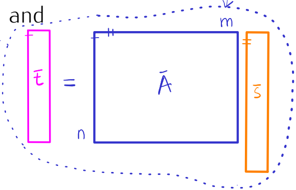

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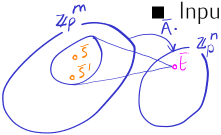

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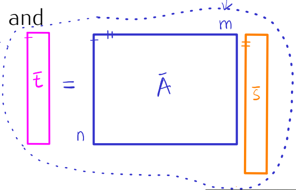

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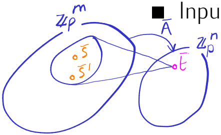

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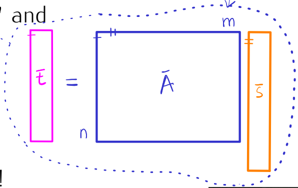

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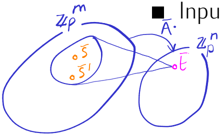

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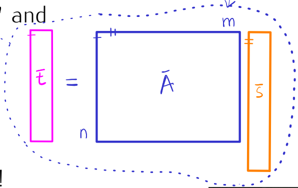

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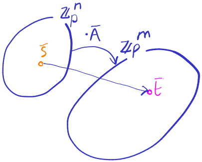

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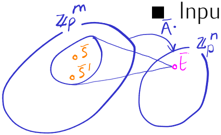

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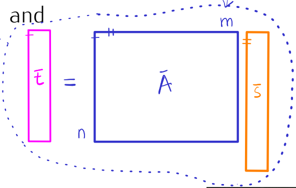

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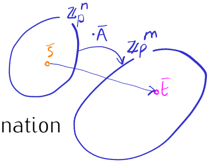

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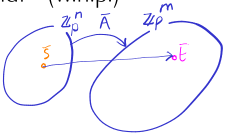
Solving Linear Equations Over $(\mathbb{Z}_p, +, \cdot)$...

- The map $\bar{s} \mapsto (\bar{s}^\top \bar{A})^\top$ is a "random linear code"
  - Two "codewords" $\bar{t}_1^\top := \bar{s}_1^\top \bar{A}$ and $\bar{t}_2^\top := \bar{s}_2^\top \bar{A}$ are "far" (w.h.p.)
  - However, efficient "decoding" algorithm to recover $\bar{s}$ from "noisy" $\bar{t}^\top \approx \bar{s}^\top \bar{A}$ *not known*



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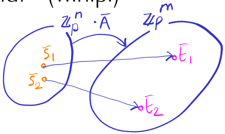

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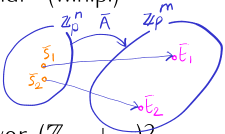

3 Potentially hard: solve "noisy" linear equations over $(\mathbb{Z}_p, +, \cdot)$?

- Input $(\bar{A}, \bar{t})$, where $\bar{A} \leftarrow \mathbb{Z}_p^{n\times m}$, $\bar{s} \leftarrow \mathbb{Z}_p^n$, $\bar{e} \leftarrow \mathrm{E}^m$ and

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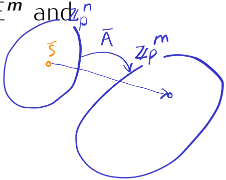

- Solution: $\bar{s}$

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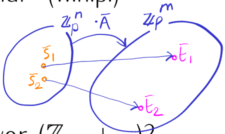

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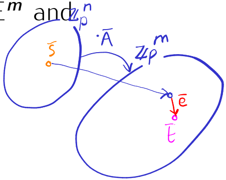

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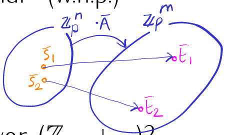

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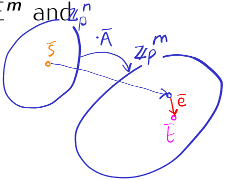

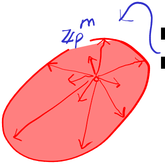

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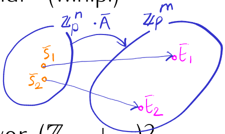

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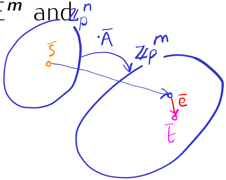

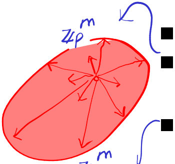

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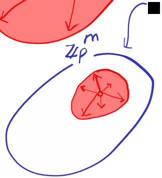

- Interesting: $\mathrm{E} = \mathrm{E}_\alpha$, the *discrete Gaussian distribution* over $\mathbb{Z}$
  - Centred at 0; parameter $\alpha < 1$ determines s.d. $\sigma := \alpha p$

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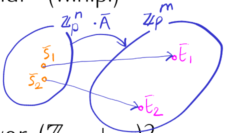

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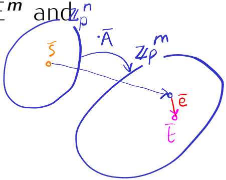

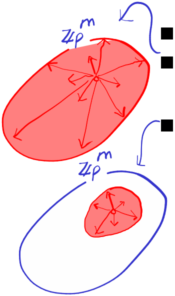

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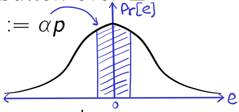

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Learning with Errors (LWE)

Assumption 1 (Search LWE (SLWE))

The (n, m, p, E)*-SLWE assumption holds if for all quantum polynomial-time (QPT) inverters* Inv *the following is negligible*

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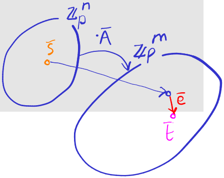

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poly(n) E_α, discrete Gaussian

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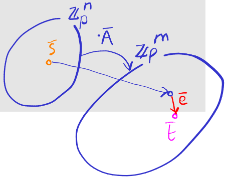

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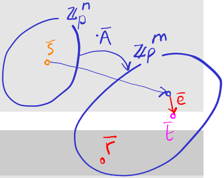

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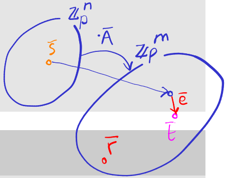

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real world

random world

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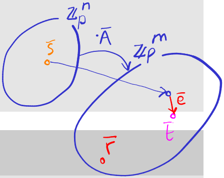

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real world   random world

Exercise 1

Are DLWE and SLWE random self-reducible?

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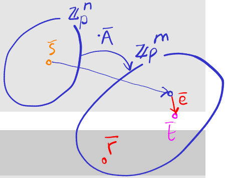

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real world

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Decision and Search LWE are Equivalent!

- Note: this is not true for, e.g, CDH and DDH!

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Claim 1 (Search to decision reduction for LWE)

For any $n \in \mathbb{N}$, m, $p \in \mathsf{poly}(n)$ and E, and sufficiently large m', (n, m', p, E)-SLWE problem reduces to (n, m, p, E)-DLWE problem.

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Proof sketch. $\exists \mathsf{Inv}$ for SLWE $\Leftarrow \exists \mathsf{D}$ for DLWE.

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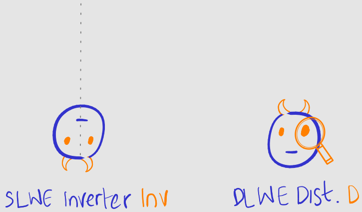

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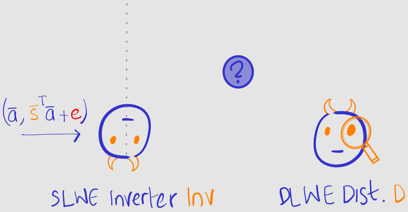

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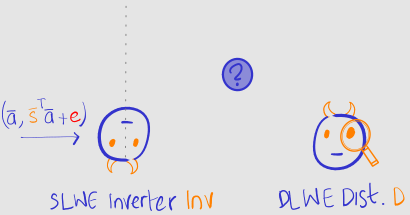

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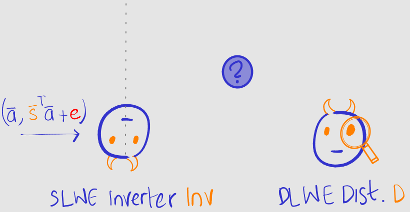

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$$(\bar{a}, \bar{s}^\top \bar{a} + e) \mapsto (\bar{a} + (a_1', 0, \ldots, 0), \bar{s}^\top \bar{a} + e + a_1' s_1)$$
$$= (\bar{a}', \bar{s}^\top \bar{a}' + e)$$

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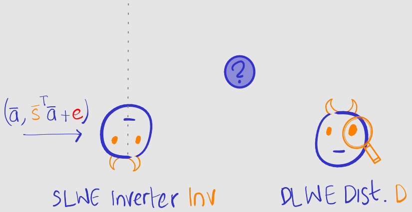

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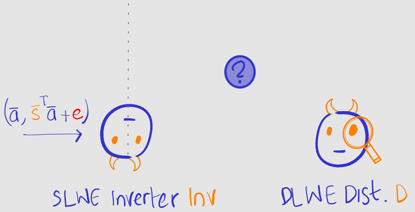

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For any $n \in \mathbb{N}$, m, $p \in \mathsf{poly}(n)$ and E, and sufficiently large m', (n, m', p, E)-SLWE problem reduces to (n, m, p, E)-DLWE problem.

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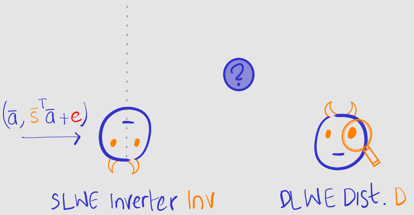

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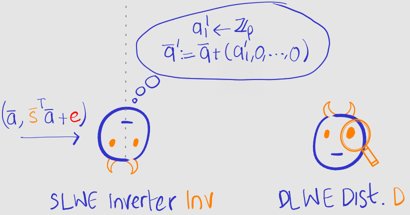

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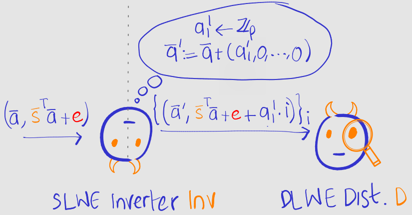

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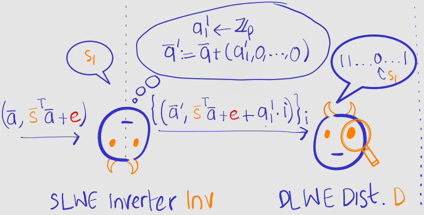

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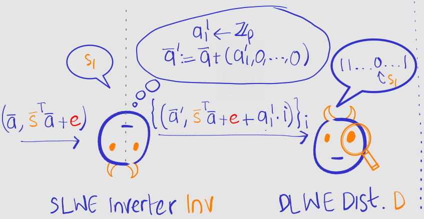

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$\square$

Plan for this Lecture

1 Motivation: Quantum Adversaries

2 Learning with Errors (LWE)

3 Cryptography from LWE

4 LWE and Lattices

1-Bit Key-Exchange Protocol ← DLWE...

- The protocol:

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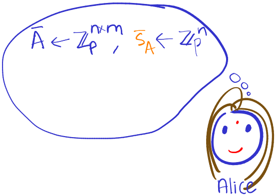

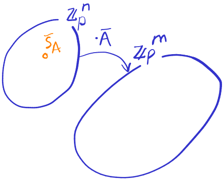

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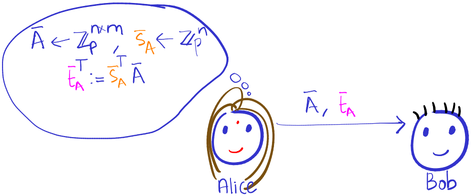

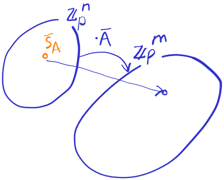

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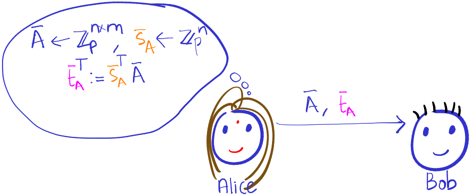

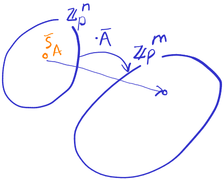

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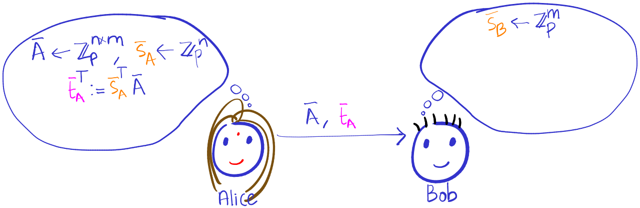

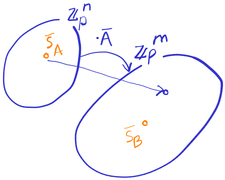

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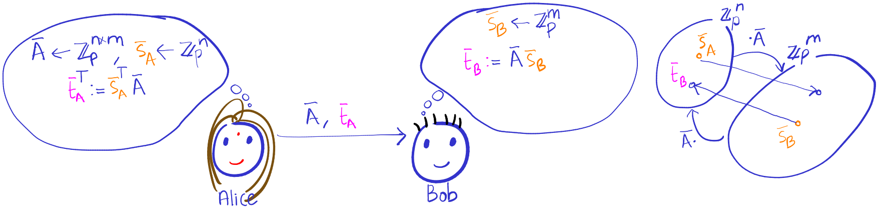

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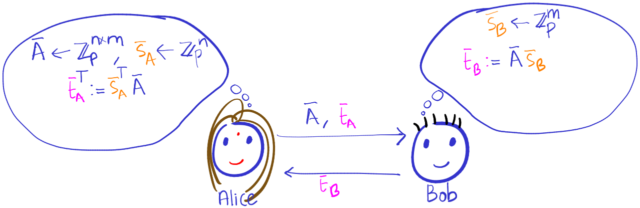

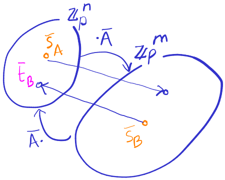

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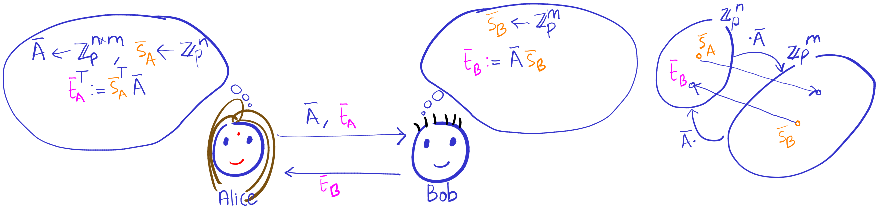

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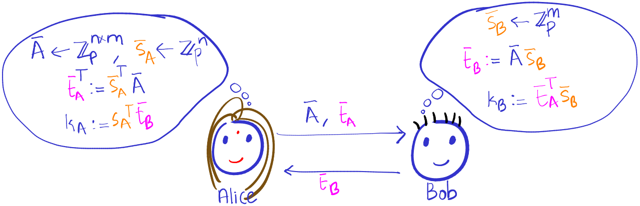

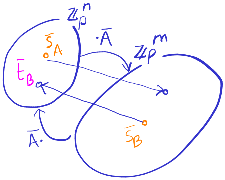

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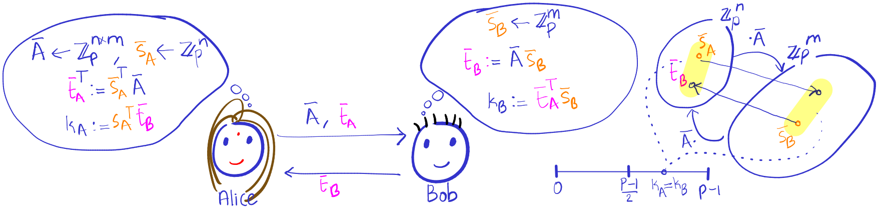

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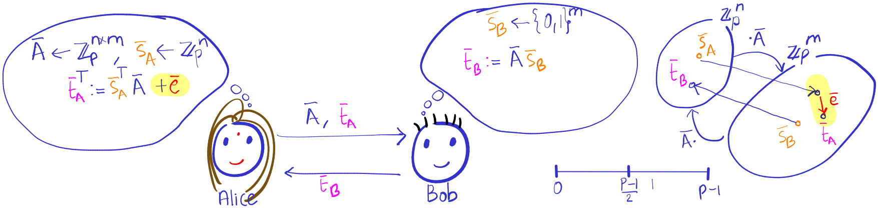

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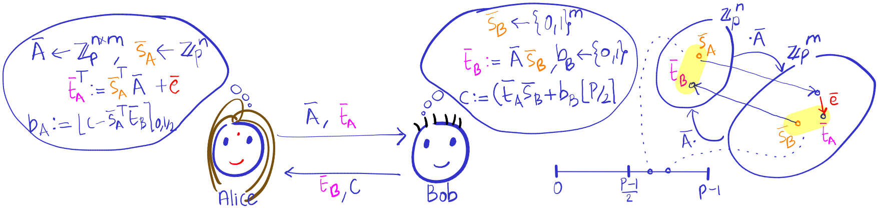

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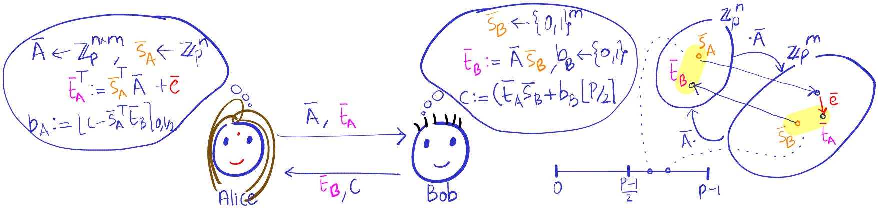
1-Bit Key-Exchange Protocol $\leftarrow$ DLWE...



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     round

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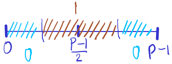

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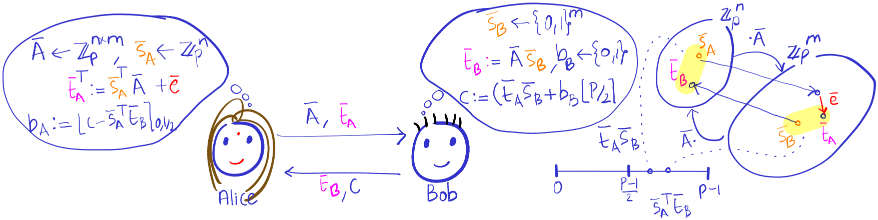

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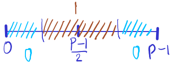

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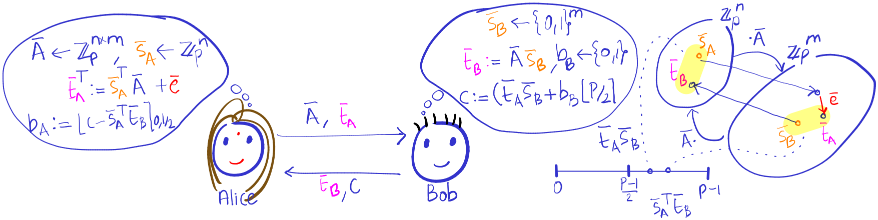

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- Scheme has negligible *key-exchange error* if $\alpha \leq 1/\tilde{O}(\sqrt{n})$

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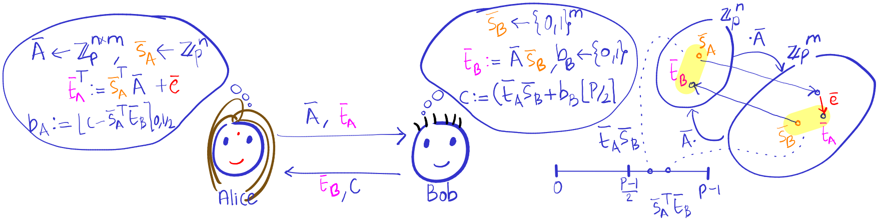

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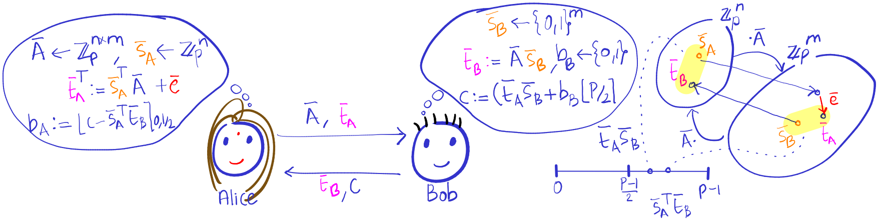

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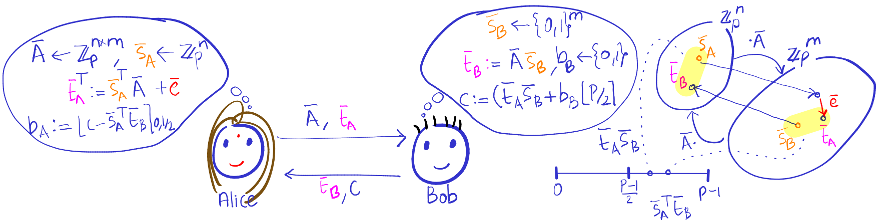

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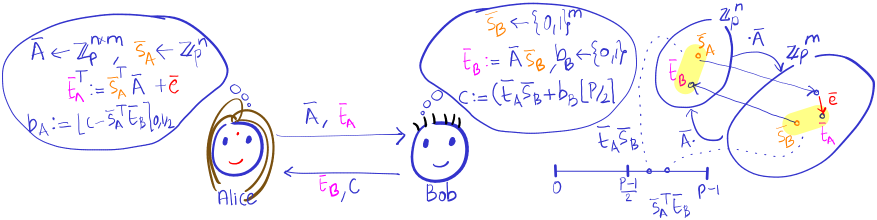

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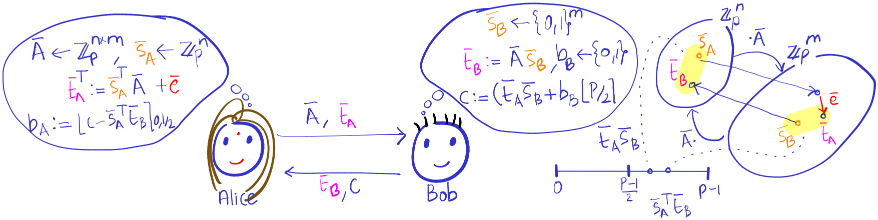
Note that $c - \bar{s}_A^T \bar{t}_B = \bar{t}_A^T \cdot \bar{s}_B - \bar{s}_A^T \bar{t}_B + b_A \lfloor p/2 \rceil$

$$= (\cancel{\bar{s}_A^T \bar{A}} + e^T)\, \bar{s}_B - \cancel{\bar{s}_A^T \bar{A} \bar{s}_B} + b_A \lfloor p/2 \rceil$$

$$= e^T \bar{s}_B + b_A \lfloor p/2 \rceil \implies b_A = b_B \text{ if } |e^T \bar{s}_B| < p/4$$

- Scheme has negligible *key-exchange error* if $\alpha \leq 1/\tilde{O}(\sqrt{n})$



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Regev's Encryption: 1-Bit PKE ← DLWE

Construction 1

- *Key generation* $\mathsf{Gen}(1^n)$*:*
  1. *Sample matrix* $\bar{A} \leftarrow \mathbb{Z}_p^{n \times m}$ *for* $m, p = \mathrm{poly}(n)$
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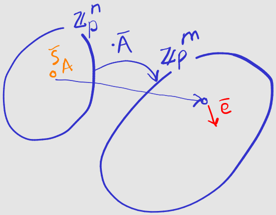

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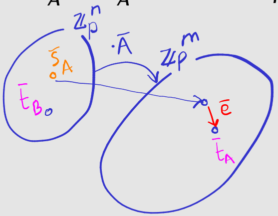

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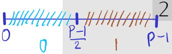

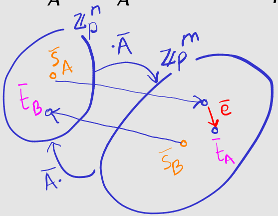

Regev's Encryption: 1-Bit PKE $\leftarrow$ DLWE

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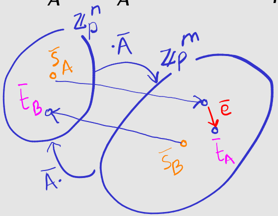

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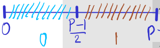

$$\begin{pmatrix} \bar{A}\bar{s}_B \\ \bar{t}_A^\top \bar{s}_B \end{pmatrix}$$

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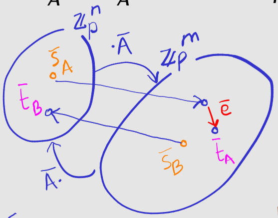

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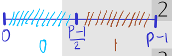
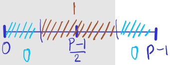


$\begin{pmatrix} \bar{A}\bar{s}_B \\ \bar{t}_A^\top \bar{s}_B \end{pmatrix}$

$-\bar{s}_A^\top \bar{A}\bar{s}_B + \bar{t}_A^\top \bar{s}_B$

- *Decryption* $\mathsf{Dec}(\mathrm{sk}, \bar{c})$: *output* $\left\lfloor (-\bar{s}_A^\top, 1)\bar{c} \bmod p \right\rceil_{0,1/2}$



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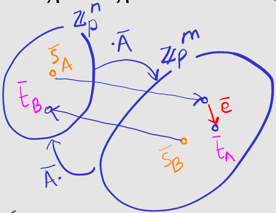

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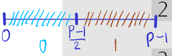

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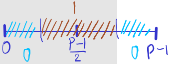

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- *Correctness of decryption: similar argument to key exchange*

Regev's Encryption is Quantum Secret...

Theorem 3 (LWE → Quantum CPA-PKE)

Regev PKE is quantum CPA-secret under DLWE assumption.

Proof sketch. Hybrid argument with two steps.

Step 1: Real world H_0

$\downarrow$

Random world H_1

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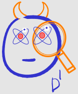

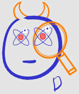

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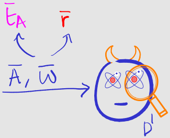

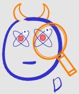

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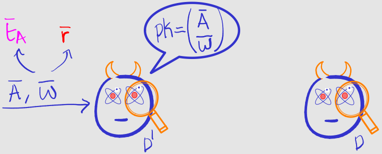

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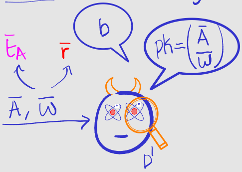

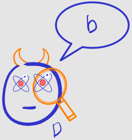

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Step2

Claim 2: PKE in H_1 is statistically secure.

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Claim 2: PKE in H_1 is statistically secure.

Proof: $c = \underbrace{\begin{pmatrix} \bar{A} \\ \bar{r} \end{pmatrix}}_{n+1 \times m} \bar{s}_B^{\,m} + \begin{pmatrix} 0^n \\ \tilde{b} \end{pmatrix}$

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$$c = \underbrace{\begin{pmatrix} \bar{A} \\ \bar{r} \end{pmatrix}}_{n+1 \times m} \overbrace{\bar{s}_B}^{m} + \begin{pmatrix} \bar{0}^n \\ \tilde{b} \end{pmatrix} \Rightarrow \begin{pmatrix} \bar{A} \\ \bar{r} \end{pmatrix} \bar{s}_B \text{ loses information about } \bar{s}_B$$

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Matrix "Leftover hash lemma": For $\begin{pmatrix} \bar{A} \\ \bar{r} \end{pmatrix} \leftarrow \mathbb{Z}_p^{(n+1) \times m}$, $\left(\begin{pmatrix} \bar{A} \\ \bar{r} \end{pmatrix}, \begin{pmatrix} \bar{A} \\ \bar{r} \end{pmatrix} \bar{s}_B \right) \approx_s \left(\begin{pmatrix} \bar{A} \\ \bar{r} \end{pmatrix}, \bar{r}' \leftarrow \mathbb{Z}_q^{n+1} \right)$ ∎

Regev's Encryption is Quantum Secret...

Exercise 2

Show that the following variants of Regev's scheme are also quantum secret assuming DLWE:

1. *Gaussian secret-keys: same as in Construction 1 except sample the secret key as* $\bar{\boldsymbol{s}}_{\boldsymbol{A}} \leftarrow \mathrm{E}_{\alpha}^{\boldsymbol{n}}$
2. *Gaussian random coins: same as in Construction 1 except sample the random coin as* $\bar{\boldsymbol{s}}_{\boldsymbol{B}} \leftarrow \mathrm{E}_{\alpha}^{\boldsymbol{m}}$

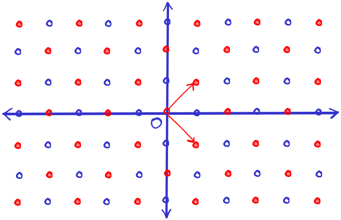
Plan for This Lecture...

1 Motivation: Quantum Adversaries

2 Learning with Errors (LWE)

3 Cryptography from LWE

4 LWE and Lattices



What has LWE to Do with Lattices?...

Defintion 1 (Lattice)

A n-dimensional lattice $\mathbb{L}$ is a discrete, additive subgroup of $\mathbb{R}^n$.

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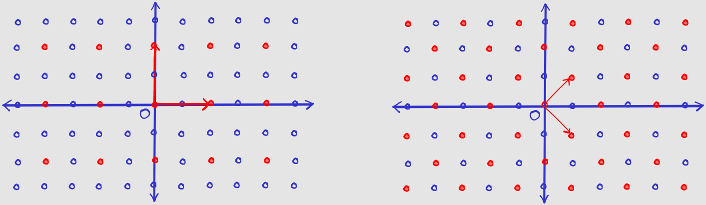
Example 4 (2D scaled integer and 2D checkerboard lattice)



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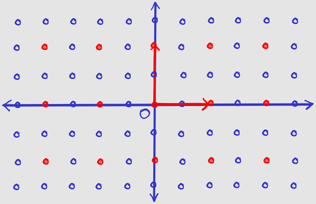
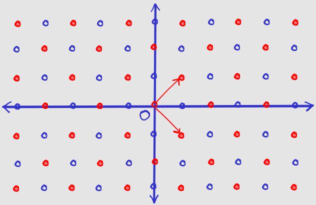
subset that is also group

Example 4 (2D scaled integer and 2D checkerboard lattice)





- Represented using a basis $\bar{B} = (\bar{b}_1, \cdots, \bar{b}_n) \in \mathbb{R}^{n\times n}$ as its integer linear combination:

$$\mathbb{L}(\bar{B}) := \left\{ \bar{v} := \sum_{i\in[n]} a_i \bar{b}_i \text{ for } (a_1, \ldots, a_n) \in \mathbb{Z}^n \right\}$$

What has LWE to Do with Lattices?...

- Some *worst-case* hard problems on lattices:
  1. Shortest vector problem (SVP)
     - Input: lattice $\mathbb{L}$ via basis $\bar{B}$
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- Compare with factoring
  - Only weakly one-way and most instances are easy
  - Worst-case to average case reduction not known

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- Related computational problem: learning *parity* with noise
  - "Modulus 2 version" of LWE
  - Open: PKE from LPN

Next Lecture

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- Next lecture: integrity + authentication in *public-key* setting
- New cryptographic primitive: *digital signatures*
  - Two construction, both quantum secure
    - Lamport's one-time signature $\leftarrow$ OWF
    - Theoretic construction of stateless signature
  - New proof technique: plug and pray!

References

1. [KL14, §14.3] for details of this chapter
2. For a formal introduction to quantum computing, use [NC10]; a quick introduction can be found in [AB09, Chapter 10] (including Grover's and Shor's algorithms)
3. For a formal introduction to lattice-based cryptography, refer to Peikert's survey [Pei16] or lecture notes of Vaikuntanathan's CS294 course.
4. The LWE-based encryption in Construction 1 is from [Reg05], but the presentation is from [Pei16, §5.2.1]
5. The worst-case to average-case reduction for LWE in the form stated in Theorem 5 is due to a series of works: [Reg05, Pei09, LM09]

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