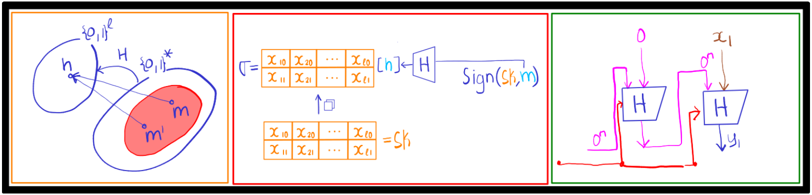

CS783: Theoretical Foundations of Cryptography

Lecture 12 (10/Sep/24)

Instructor: Chethan Kamath

Plan for Today's Lecture

Recall from Last Lecture

- Introduced digital signatures: public-key analogue of MAC

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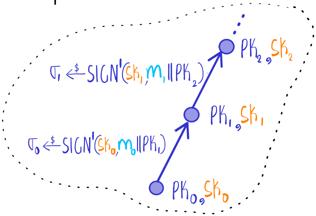

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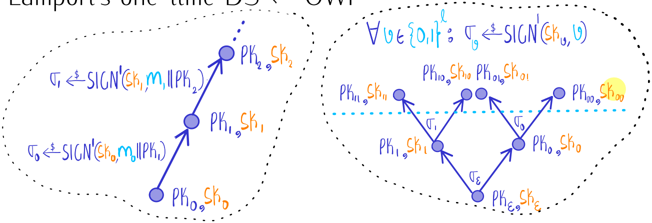

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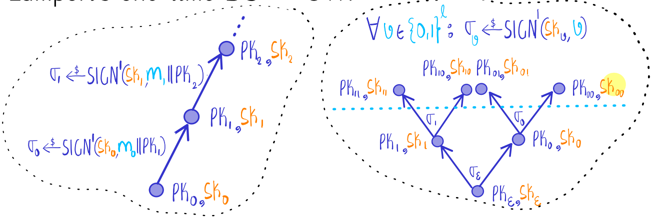

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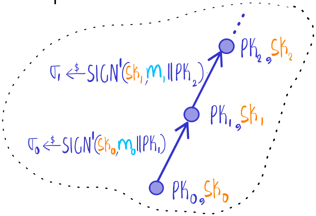

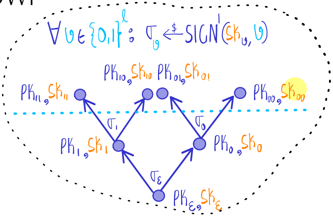

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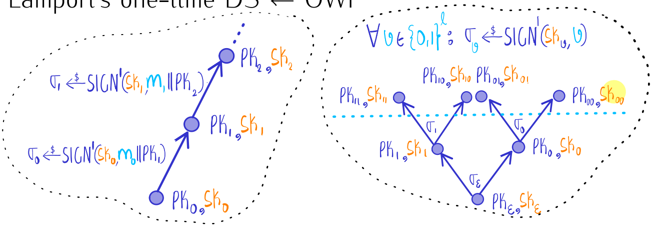

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  - Lectures 13: from *trapdoor* OWP via hash-then-invert
  - Lecture 15(?): via Fiat-Shamir transform (e.g., Schnorr)
- Takeaways:
  - Constructive: "bootstrapping" one-time to many-time signatures
  - Proof techniques: "plug and pray" $pk^* =$

y_{00}	y_{10}	y_{20}	y_{30}
y_{01}	y^*	y_{21}	y_{31}

$b^* \leftarrow \{0,1\}$

$i^* \leftarrow [\ell]$

Plan for Today's Lecture...

Theorem 1 (Theorem 1, Lecture 11)

If f is a OWF then Lamport's scheme is a one-time DS

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Theorem 2 (PRF→ MAC: Theorem 2, Lecture 7) $\mathrm{Tag}(k,m) := F_k(m)$

If $\{\mathsf{F}_{\mathsf{k}} : \{0,1\}^n \to \{0,1\}^n\}_{\mathsf{k}\in\{0,1\}^n}$ *is a PRF then Construction 3 (Lecture 7) is EU-CMA-secure for fixed-length messages* $\{0,1\}^n$.

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Exercise 1 (Exercise 3, Lecture 11 (Domain Extension)) $\to \boxed{H} \to$

Given a compressing function $H : \{0,1\}^{2\ell} \to \{0,1\}^{\ell}$*, construct a one-time DS for arbitrary-length messages. What are the properties you need from* H *to ensure that the one-time DS is secure?*

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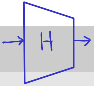

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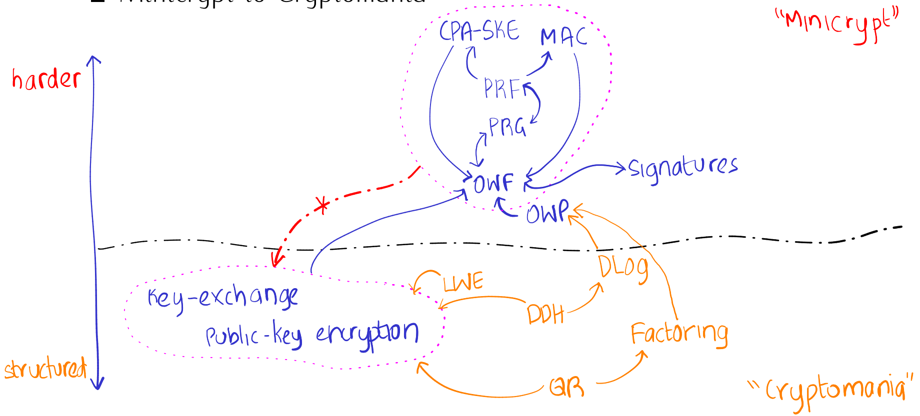
"Hash function"

- Minicrypt to Cryptomania



- Sub-task 5.a: domain-extension of digital signature/MAC
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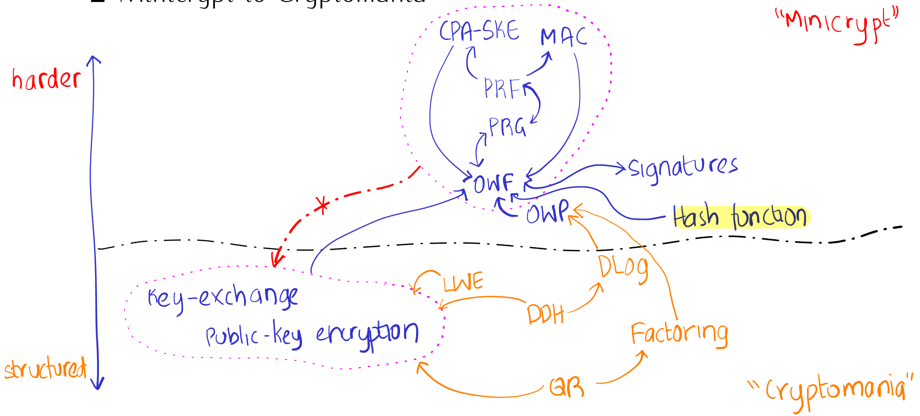

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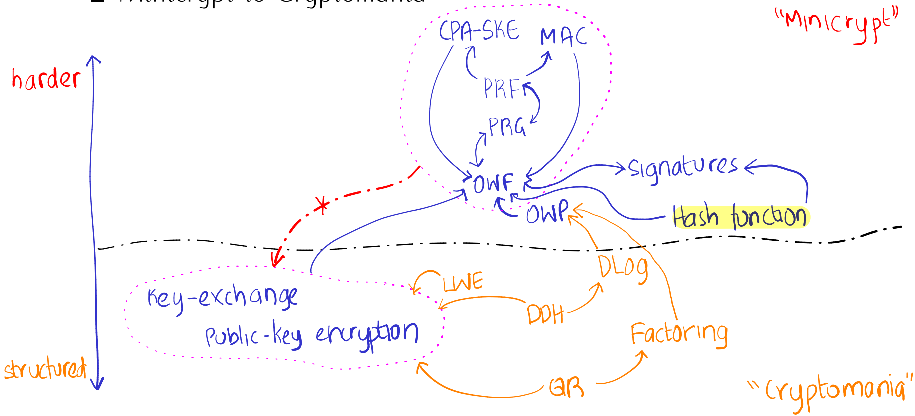

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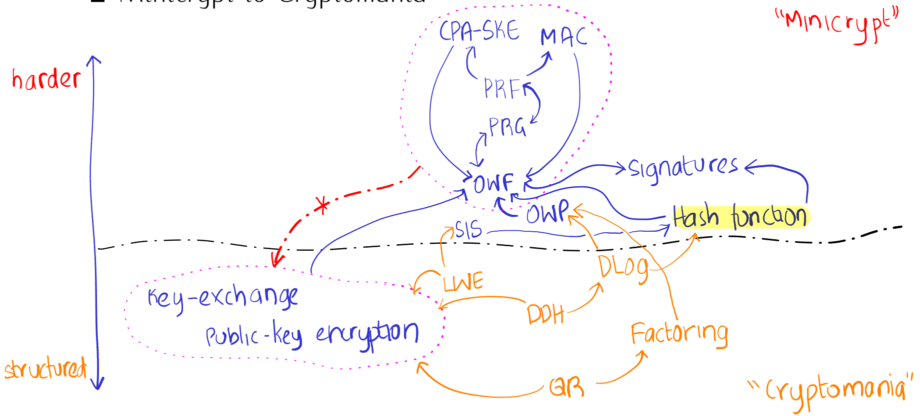

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Plan for Today's Lecture...

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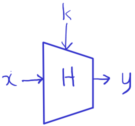
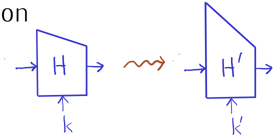


2 Compression Functions and Domain-Extension



3 How to Construct Compression Functions?

SIS DLOG

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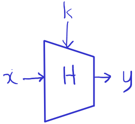

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SIS DLOG

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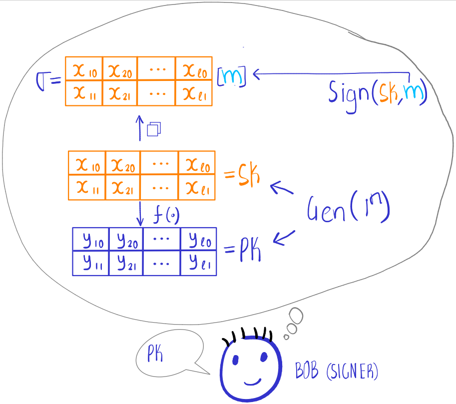

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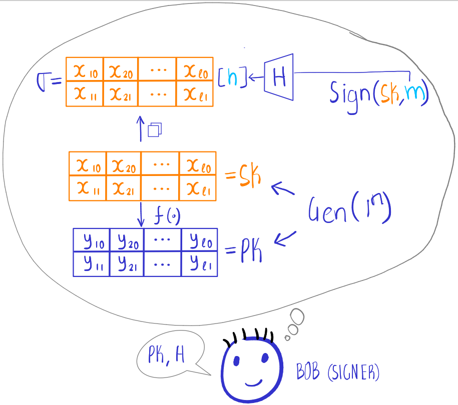
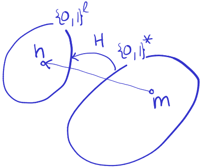
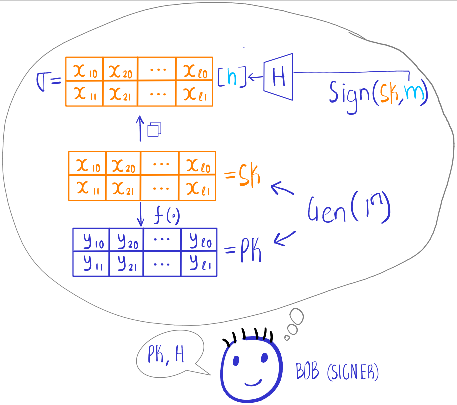
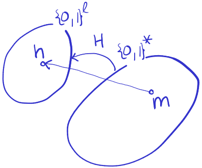




- Hash-then-sign: compute "hash" $h = \mathrm{H}(m)$ and then sign h





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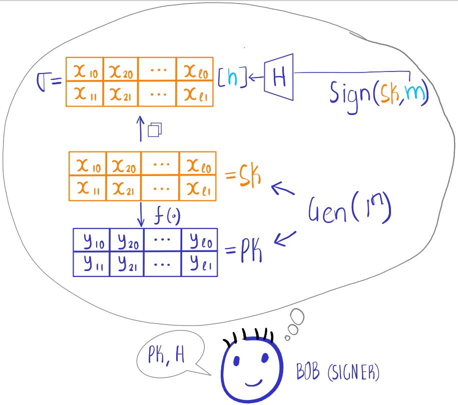

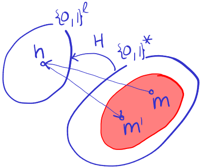

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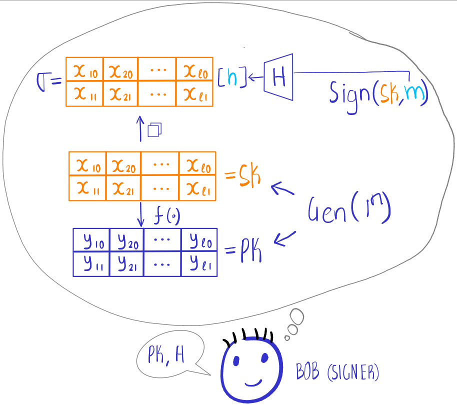

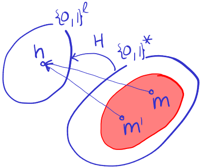

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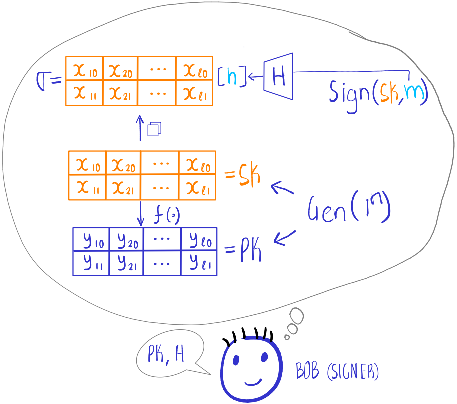

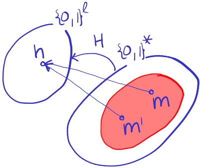

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Collision-Resistant Hash Function (CRHF)...

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A function (family) $\left\{\mathsf{H} : \{0,1\}^* \to \{0,1\}^n\right\}$ *is a CRHF if for every PPT collision-finder* F*, the following is negligible.*

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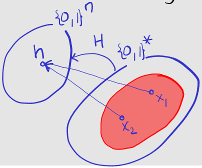

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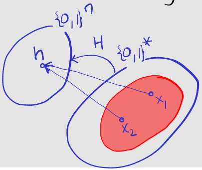

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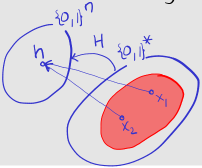

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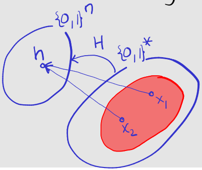

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Defintion 2 (CRHF, with key generation algorithm Gen)

A keyed function (family) $\left\{ \mathsf{H} : \mathcal{K} \times \{0,1\}^* \rightarrow \{0,1\}^n \right\}$ *is a CRHF if for every PPT collision-finder* F*, the following is negligible.*

$$\Pr_{\substack{\mathsf{k} \leftarrow \mathbf{Gen}(1^n) \\ (x_1, x_2) \leftarrow \mathsf{F}(\mathsf{k})}} [\mathsf{H}(\mathsf{k}, x_1) = \mathsf{H}(\mathsf{k}, x_2)]$$

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Exercise 2

Prove formally the cases where H is a CRHF; describe counter-example otherwise.

Let's (Slowly) Find Collisions in H!

$\{K \times \{0,1\}^* \to \{0,1\}^n\}$

What about a deterministic $O(2^n)$-time collision-finder?

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- What is the amount of space required? Naïvely, $O(n2^n)$

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  - Exploit birthday paradox
  - Compute hash of $N := O(2^{n/2})$ *random* inputs $x_1, \ldots, x_N$
    - With *noticeable probability*, there exist colliding pairs of inputs
  - Consequence: key-size/output length must be $2\times$ security level

Let's (Slowly) Find Collisions in H!

$\{K \times \{0,1\}^* \to \{0,1\}^n\}$

- What about a deterministic $O(2^n)$-time collision-finder?
  - Exploit pigeonhole principle
  - Compute (e.g.) hash of inputs $00^n, \ldots, 01^n, 10^n$
    - There *must exist* colliding pair of inputs
  - What is the amount of space required? Naïvely, $O(n2^n)$
- Randomised $O(2^{n/2})$-time+$O(n2^{n/2})$-space collision-finder
  - Exploit birthday paradox
  - Compute hash of $N := O(2^{n/2})$ *random* inputs $x_1, \ldots, x_N$
    - With *noticeable probability*, there exist colliding pairs of inputs
  - Consequence: key-size/output length must be $2\times$ security level

Exercise 3

1. *Is deterministic $O(2^{n/2})$-time+$O(n2^{n/2})$-space collision-finder possible?*
2. *Is rand. $O(2^{n/2})$-time+$O(n)$-space collision-finder possible?*

Extending Lamport's One-Time DS for Longer Messages...

Theorem 3

If f *is a OWF and* H *is CRHF then the "hash-then-sign" scheme is a one-time DS for arbitrarily-long messages.*

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(m, m^*) collision for H

sample(sk, PK)

k

Collision finder F

Tam

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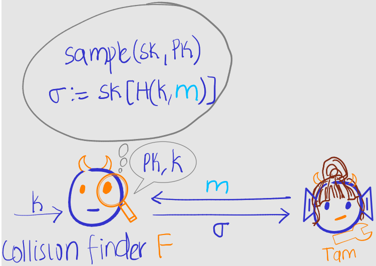

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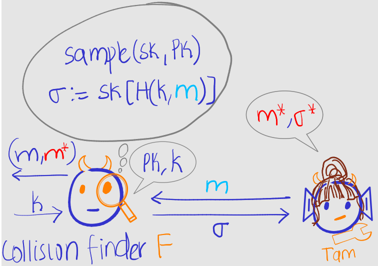

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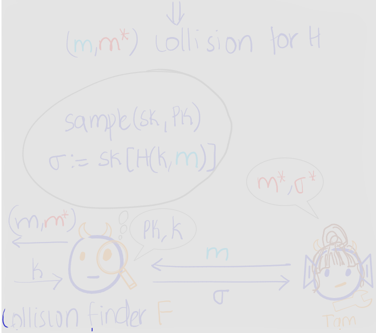

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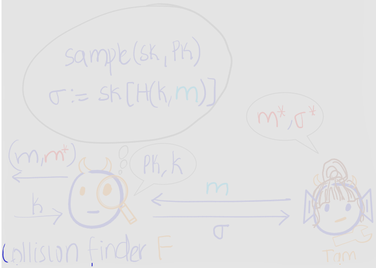

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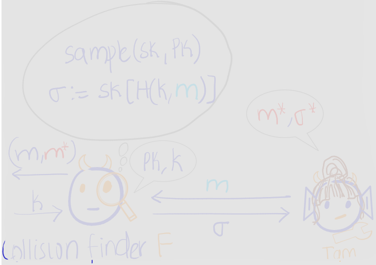

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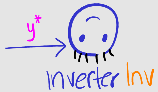

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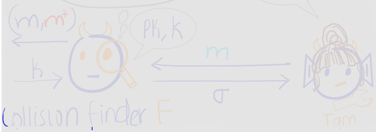
sample(sk, PK)

$\sigma := sk[H(k,m)]$



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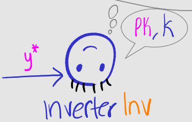

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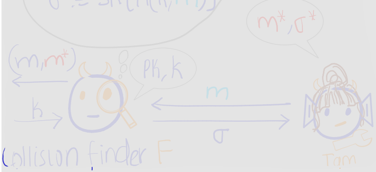

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(h^*, σ^*) forgery for Lamport's ⇒ invert f

"Plug" y^* at (i^*, b^*) to
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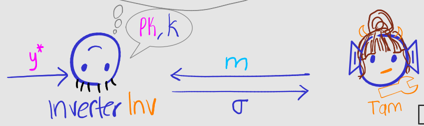
"Pray": $H(k,m)[i^*] \neq b^*$



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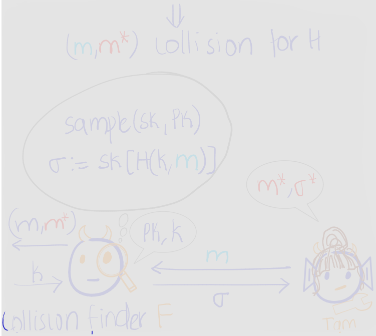

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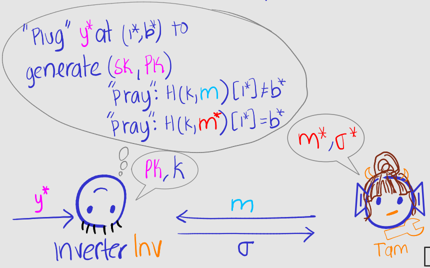

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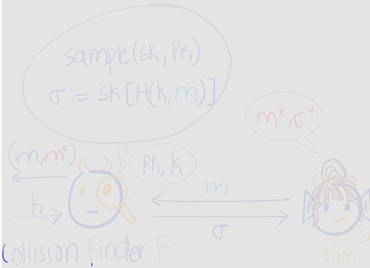

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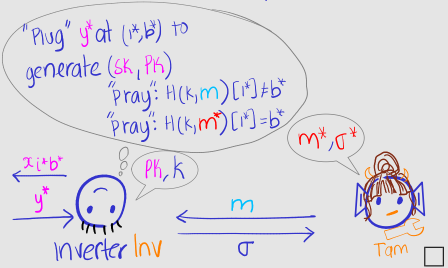

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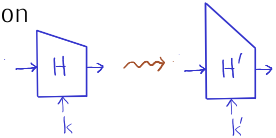
Plan for Today's Lecture

1 Hash Functions

2 Compression Functions and Domain-Extension

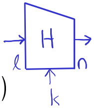


3 How to Construct Compression Functions?

SIS DLOG

Compression Functions and Domain-Extension

- (Collision-resistant) *compression function*: CRHF for fixed input length $\ell(n) > n$
  - Easier to construct in practice: e.g., MD5, SHA2 (unkeyed) compression function of certain block-size



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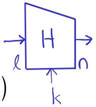

Defintion 3 ($\ell(n)$-compression function)

A keyed function (family) $\{\mathsf{H} : \mathcal{K} \times \{0,1\}^{\ell(n)} \to \{0,1\}^n\}$ *is an* $\ell(n)$*-compression function if for every PPT collision-finder* F*, the following is negligible.*

$$\Pr_{\substack{\mathsf{k} \leftarrow \mathsf{Gen}(1^n) \\ (x_1, x_2) \leftarrow \mathsf{F}(\mathsf{k})}} [\mathsf{H}(\mathsf{k}, x_1) = \mathsf{H}(\mathsf{k}, x_2)]$$

same length

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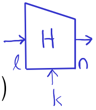

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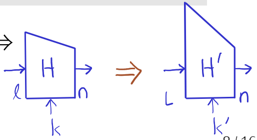
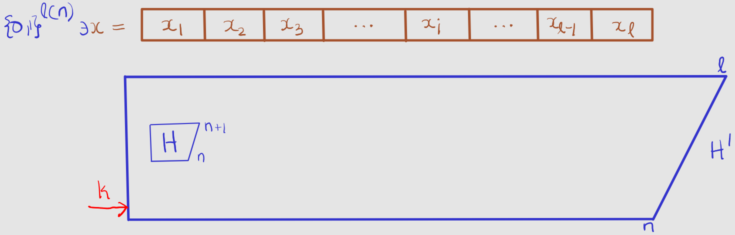
- Domain extension: $\ell(n)$-compression function $\Rightarrow$ $L(n)$-compression function for $L(n) > \ell(n)$



Merkle–Damgård Transform (Chaining)...

Construction 1 (($n+1$)-compression function H $\Rightarrow$ $\ell(n)$-compression function H′, for any polynomial $\ell(n)$)



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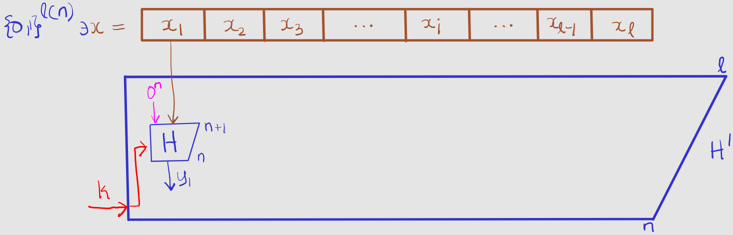

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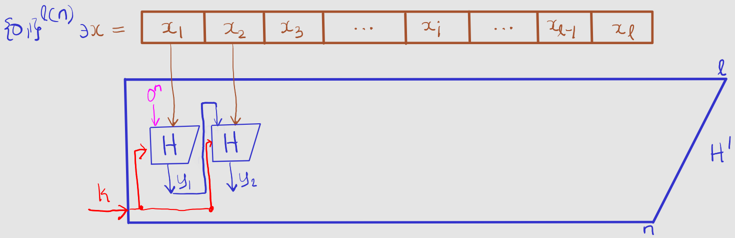

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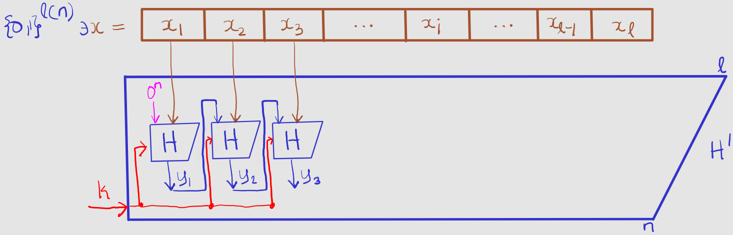

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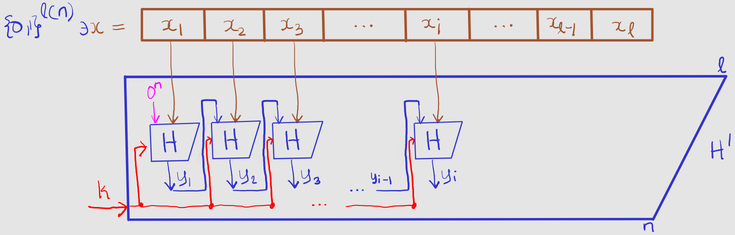

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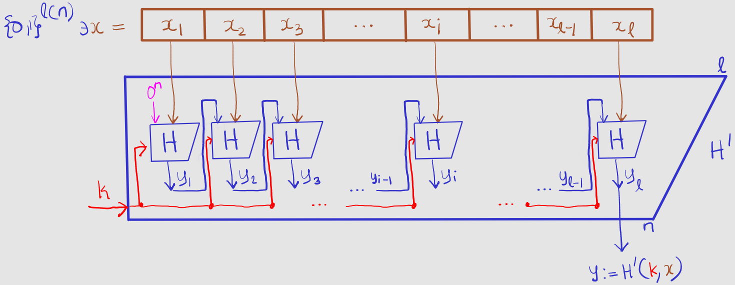

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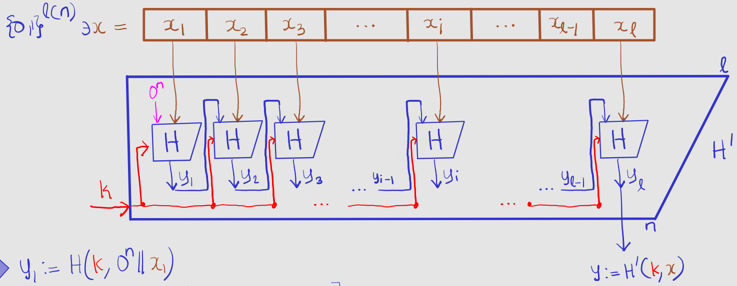

- $y_1 := H(k, 0^n \| x_1)$
- $y_i := H(k, y_{i-1} \| x_i)$, for $i \in [2, \ell]$
- $H'(k, x) := y_\ell$

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Theorem 4

If H *is a compression function then so is* H'

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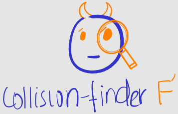
Proof sketch: ∃ collision finder F for H ⇐ ∃ coll. finder F′ for H′.



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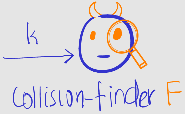

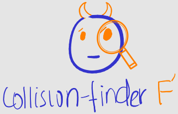

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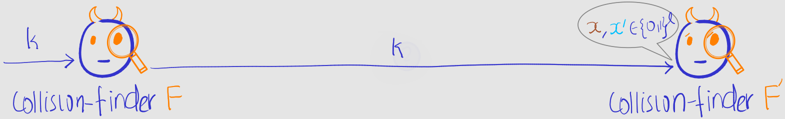

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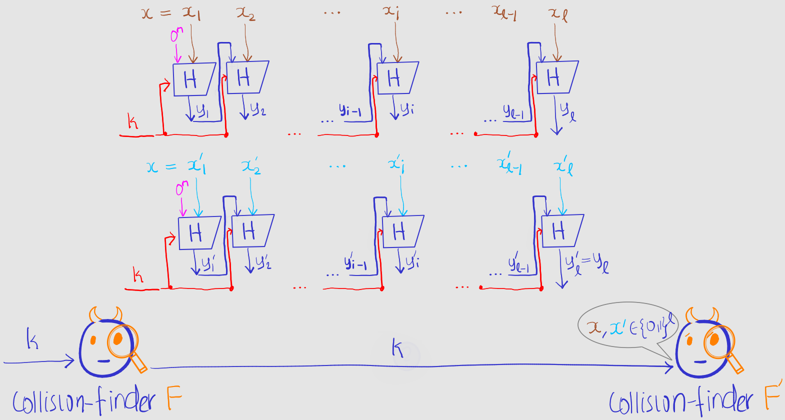

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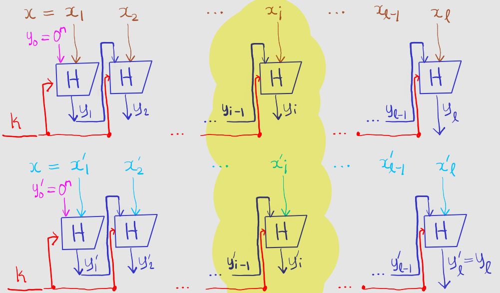


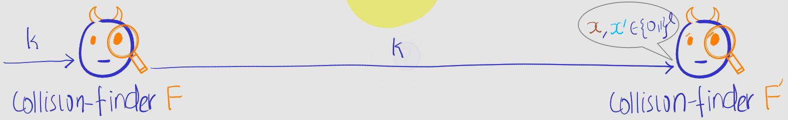
Claim: since $x \neq x'$

$\exists i \in [1,\ell]$ s.t.

i) $y_{i-1} \| x_i \neq y'_{i-1} \| x'_i$

and

ii) $H(k, y_{i-1} \| x_i) = H(k, y'_{i-1} \| x'_i)$



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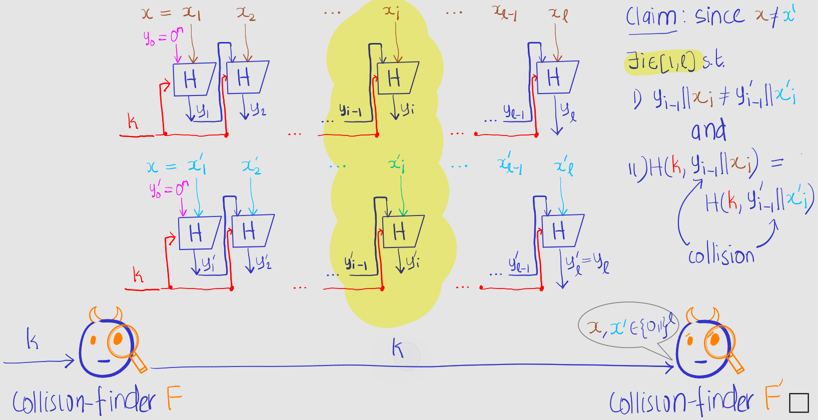
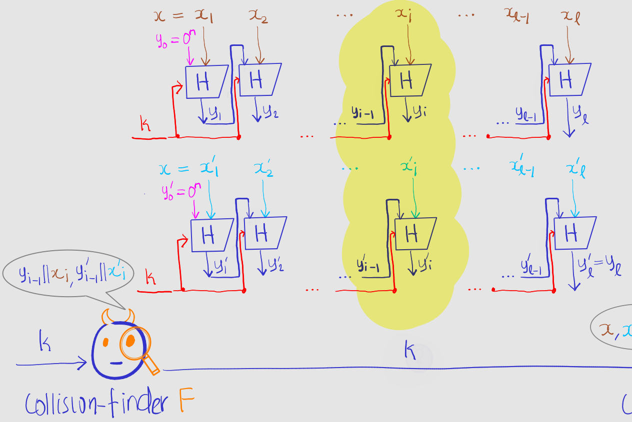


Merkle–Damgård Transform (Chaining)...

Theorem 4

If H *is a compression function then so is* H′

Proof sketch: $\exists$ collision finder F for H $\Leftarrow$ $\exists$ coll. finder F′ for H′.



Claim: since $x \neq x'$

$\exists i \in [1,\ell]$ s.t.

I) $y_{i-1}\|x_i \neq y'_{i-1}\|x'_i$

and

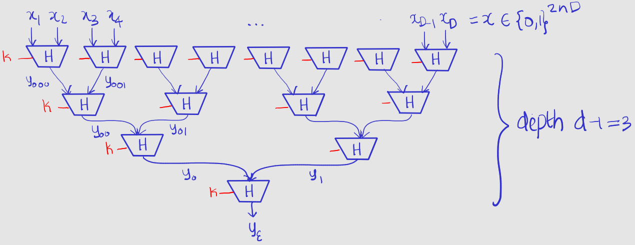
II) $H(k, y_{i-1}\|x_i) = H(k, y'_{i-1}\|x'_i)$

collision

$\square$

Construction 2 (2n-compression function H $\Rightarrow$ $2^d 2n$-compression function H′, for any $d \in \mathbb{N}$)

$\overset{\Vert}{D}$



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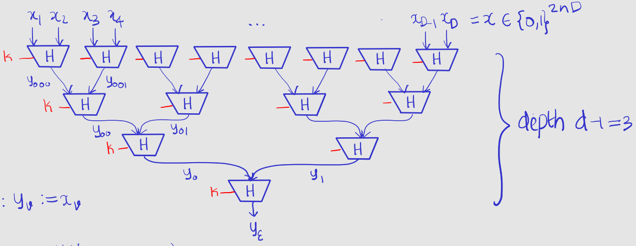

- $\forall v \in \{0,1\}^d : y_v := x_v$
- $\forall v \in \{0,1\}^{<d} : y_v := H(k, y_{v\|0}, y_{v\|1})$

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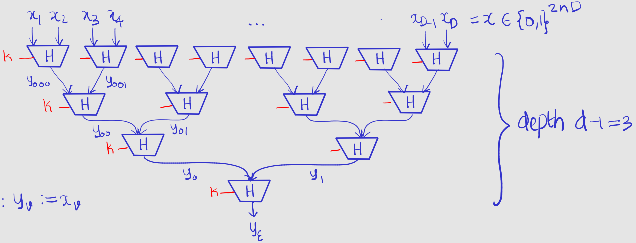

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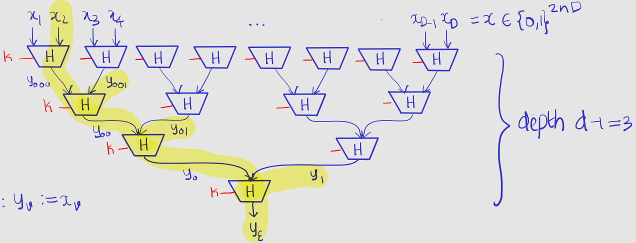
Exercise 4

Show that if H *is a compression function then so is* H′

Construction 2 ($2n$-compression function H $\Rightarrow$ $2^d 2n$-compression function H′, for any $d \in \mathbb{N}$)

$2^d = D$



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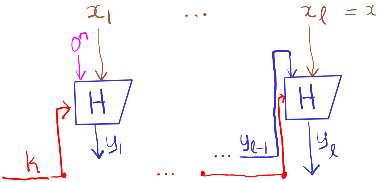
Exercise 4

Show that if H *is a compression function then so is* H′

- Has several interesting properties:
  1. Parallelisable: computable in depth $O(d)$
  2. Locally verifiable: parts of input can be verified

What If We Use Construction 1 for $\{0,1\}^*$?

Is it possible to find collisions of *different* length?



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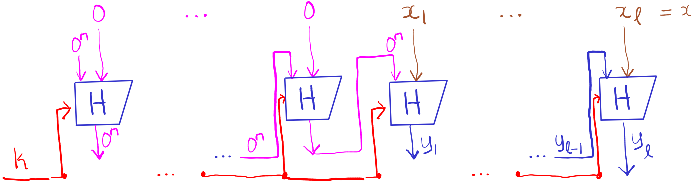

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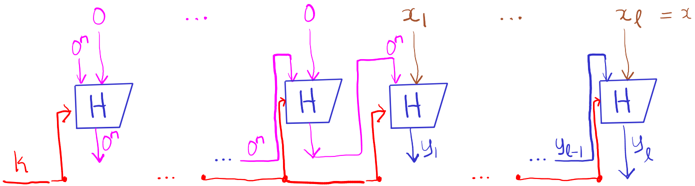

Exercise 5

1. *Find similar "length-extension" attack for Construction* 2
2. *Tweak Constructions* 1 *and* 2 *to obtain CRHF (i.e., for domain* $\{0,1\}^*$*)*
   - *Hint: add appropriate padding in the end*

Plan for Today's Lecture

1. Hash Functions
2. Compression Functions and Domain-Extension
3. How to Construct Compression Functions?

How to Construct Compression Functions in Practice?

- *Unkeyed* compression function for fixed input (block) length/output length

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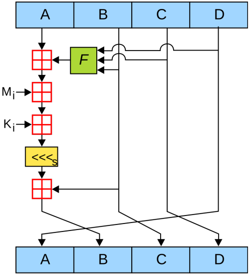
  - Message Digest (MD) family
    - ⚠ MD5 (512/128): collisions have been found!

```
d131dd02c5e6eec4 693d9a0698aff95c 2fcab58712467eab 4004583eb8fb7f89
55ad340609f4b302 83e488832571415a 085125e8f7cdc99f d91dbdf280373c5b
d8823e3156348f5b ae6dacd436c919c6 dd53e2b487da03fd 02396306d248cda0
e99f33420f577ee8 ce54b67080a80d1e c69821bcb6a88393 96f9652b6ff72a70
```

```
d131dd02c5e6eec4 693d9a0698aff95c 2fcab50712467eab 4004583eb8fb7f89
55ad340609f4b302 83e4888325f1415a 085125e8f7cdc99f d91dbd7280373c5b
d8823e3156348f5b ae6dacd436c919c6 dd53e23487da03fd 02396306d248cda0
e99f33420f577ee8 ce54b67080280d1e c69821bcb6a88393 96f965ab6ff72a70
```



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55ad340609f4b302 83e4888325 71415a 085125e8f7cdc99f d91dbdf280373c5b
d8823e3156348f5b ae6dacd436c919c6 dd53e2b487da03fd 02396306d248cda0
e99f33420f577ee8 ce54b67080a80d1e c69821bcb6a88393 96f9652b6ff72a70
```

```
d131dd02c5e6eec4 693d9a0698aff95c 2fcab50712467eab 4004583eb8fb7f89
55ad340609f4b302 83e4888325f1415a 085125e8f7cdc99f d91dbd7280373c5b
d8823e3156348f5b ae6dacd436c919c6 dd53e23487da03fd 02396306d248cda0
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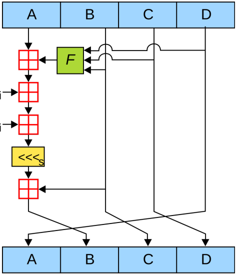

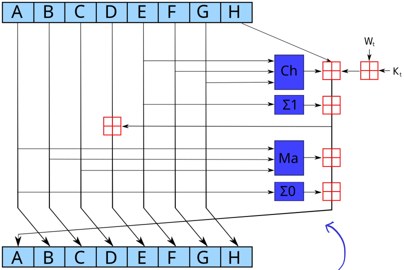

  - Secure-Hashing Algorithm (SHA) family
    - SHA2 (512/256,1024/512...): Davis-Meyer compression function
    - SHA3 (1152/224,576,512): "Sponge"-based compression function

Image credit: Wikipedia (Surachit,kockmeyer)

How to Construct Compression Functions in Theory?

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? How to solve SIS given a collision $(\bar{x}, \bar{x}')$?

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- Some constructions:
    - Practical/unkeyed: SHA2, MD5
    - Theoretical/keyed: DLog- and SIS-based

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  - (RSA) Full-domain hash
- TDP $\rightarrow$ PKE
  - New constructions of PKE: RSA

References

1. As discussed in Lecture 7, hash functions were first studied in [WC81], but they considered pairwise-independence/universal hashing
2. Collision resistance, and other cryptographic properties of hash functions were studied later [Dam88, Dam90, NY89, Mer90] a thorough historical perspective can be found in [RS04]

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