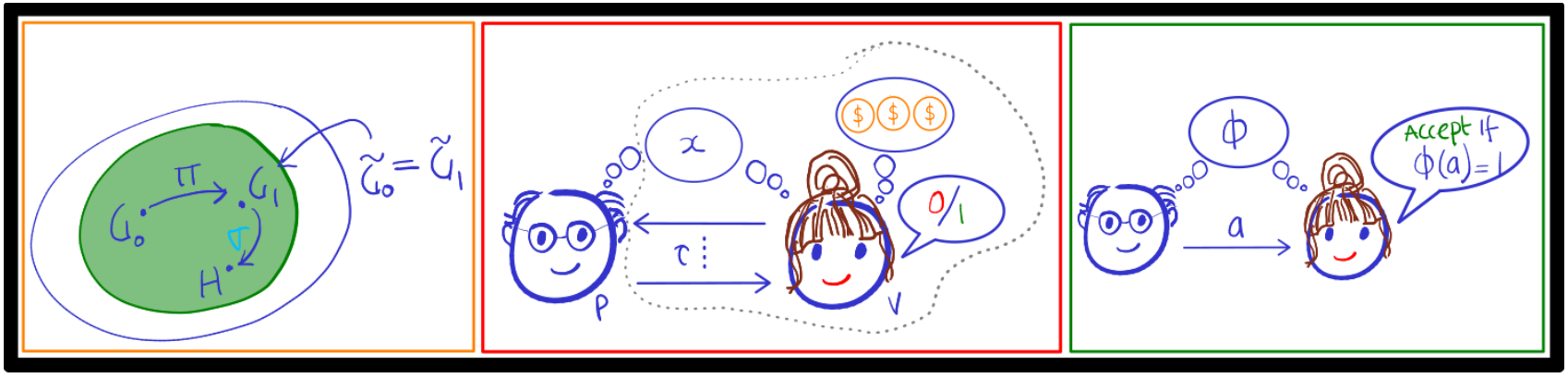


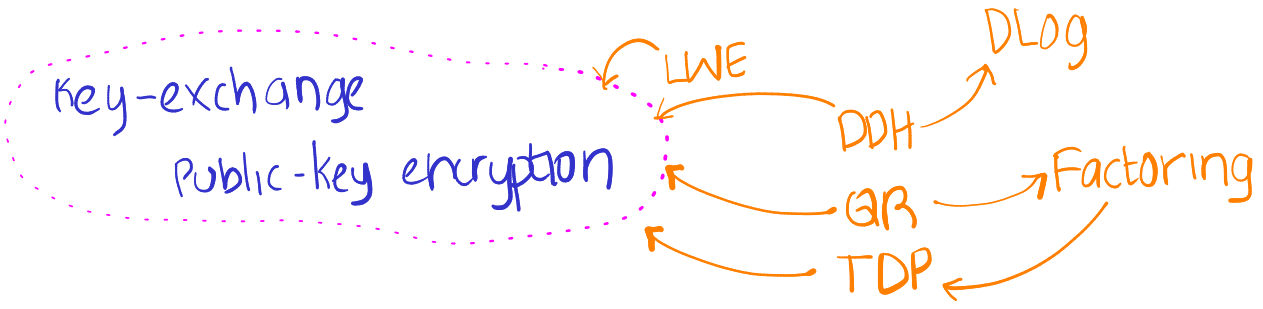
CS783: Theoretical Foundations of Cryptography

Lecture 14 (24/Sep/24)

Instructor: Chethan Kamath

To Recap Previous Module

- We learnt: secure communication in the *public-key* setting
- Cryptographic primitives encountered: key-exchange, public-key encryption, signature, hash function, TDP
- Hardness assumptions: Factoring, DLog, QR, LWE, RSA



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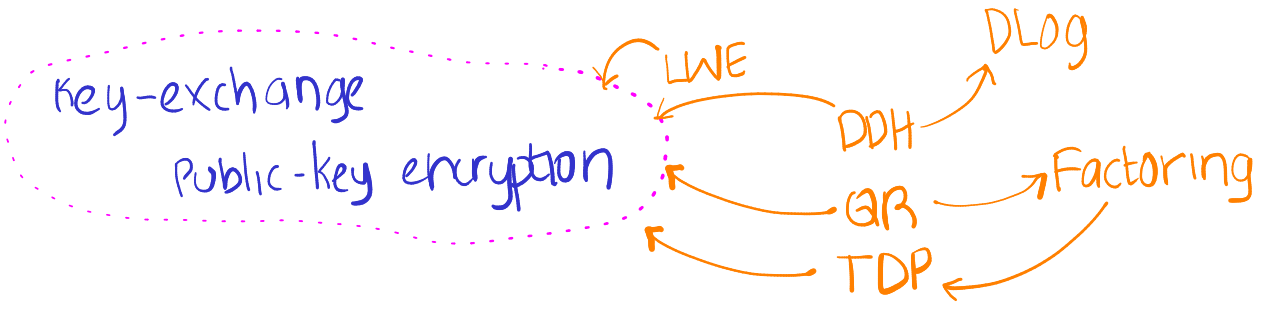
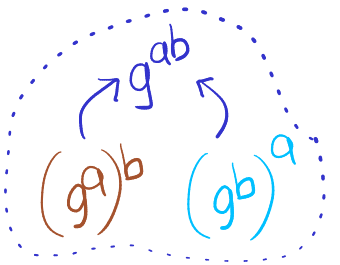
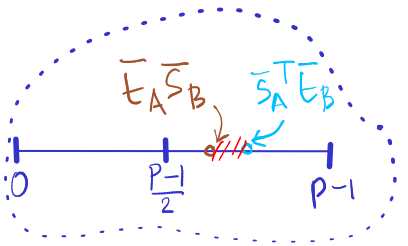




- Key conceptual takeaway: structure vs. hardness



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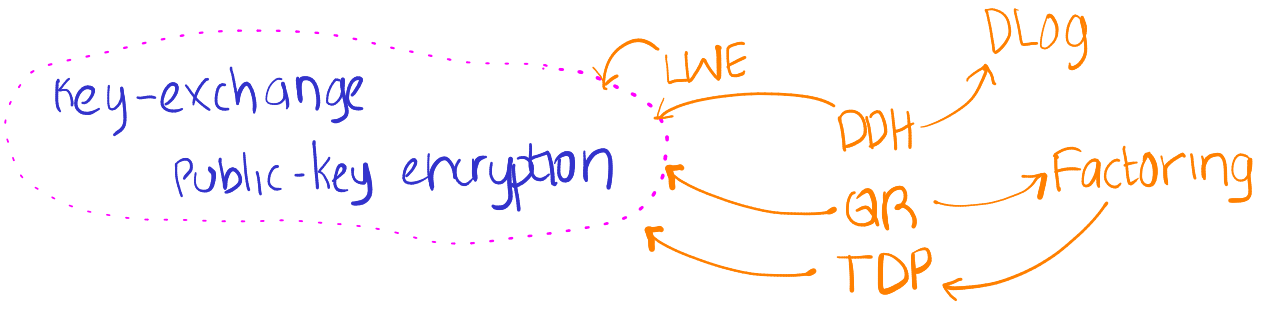

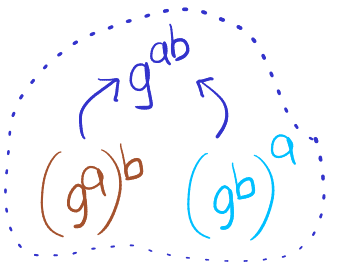

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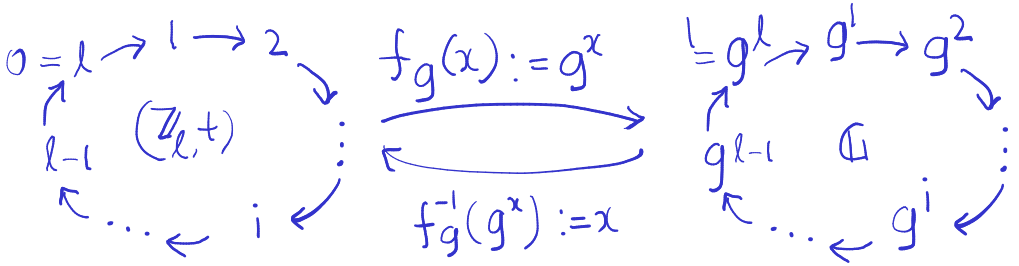
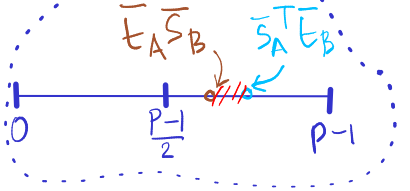




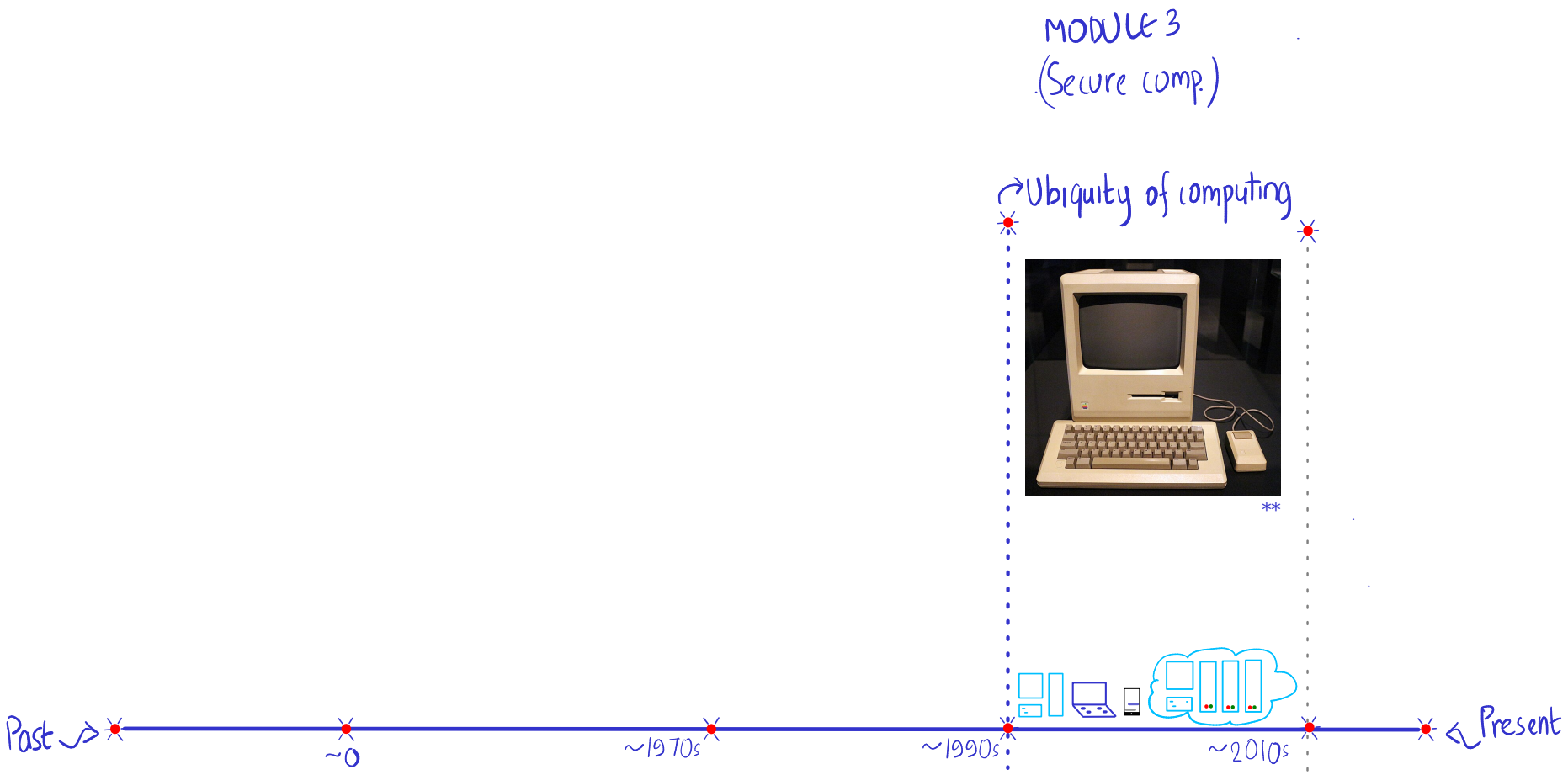
$PK^* =$

y_{00}	y_{10}	y_{20}	y_{30}
y_{01}	y^*	y_{21}	y_{31}

$i^* \leftarrow [\ell]$ $\quad b^* \leftarrow \{0,1\}$

- Key tools: groups, random self-reducibility, plug and pray

This Module: Secure Computation...



Credit for images: Wikipedia (**User:Saliko)

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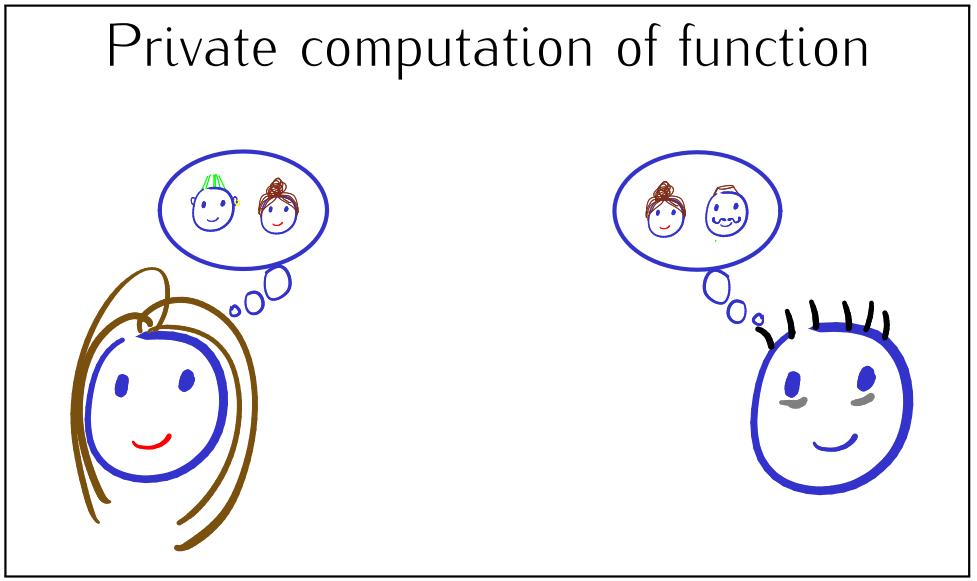

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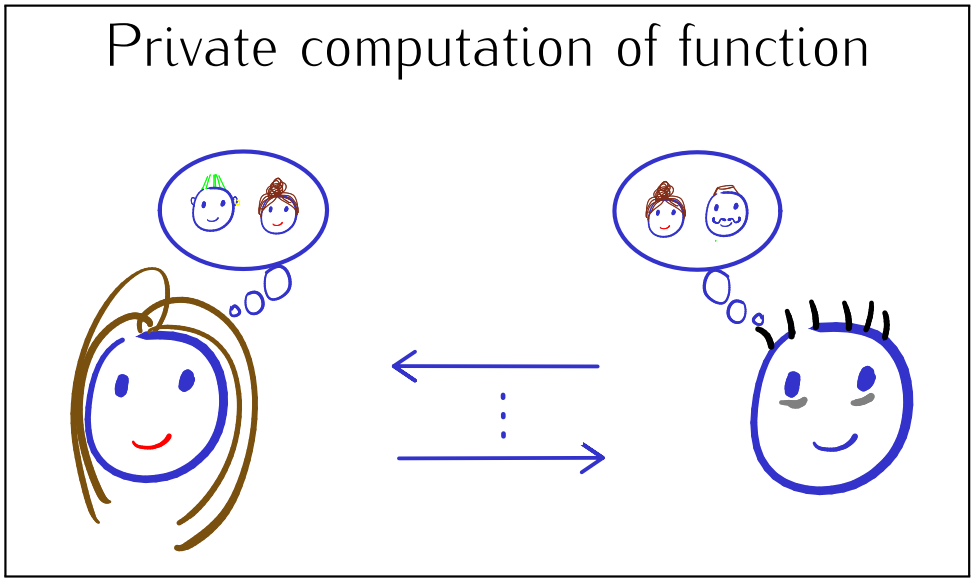
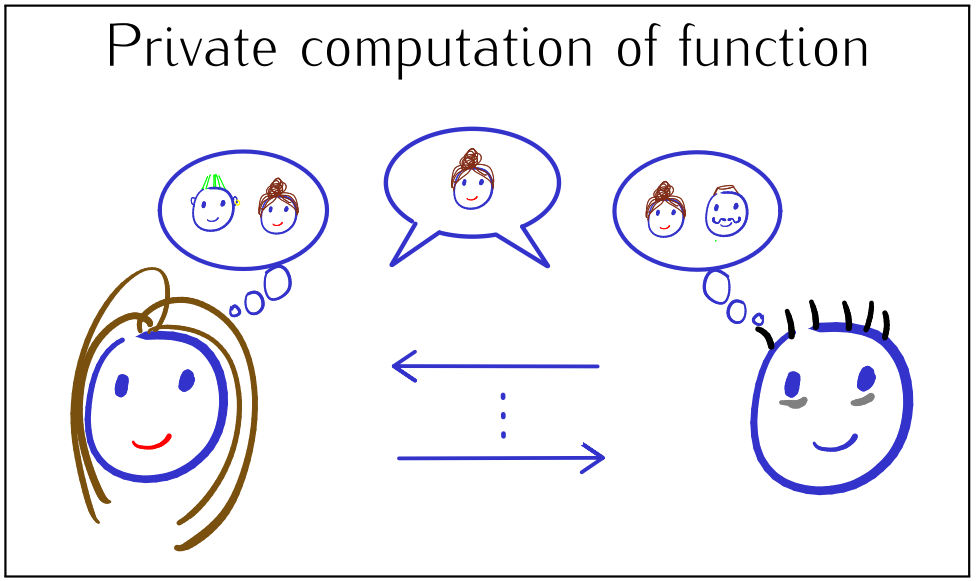


This Module: Secure Computation...

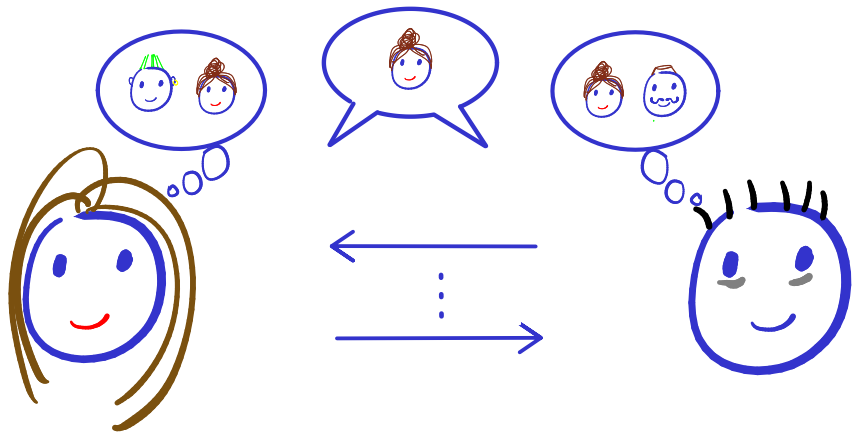
L17-18



This Module: Secure Computation...

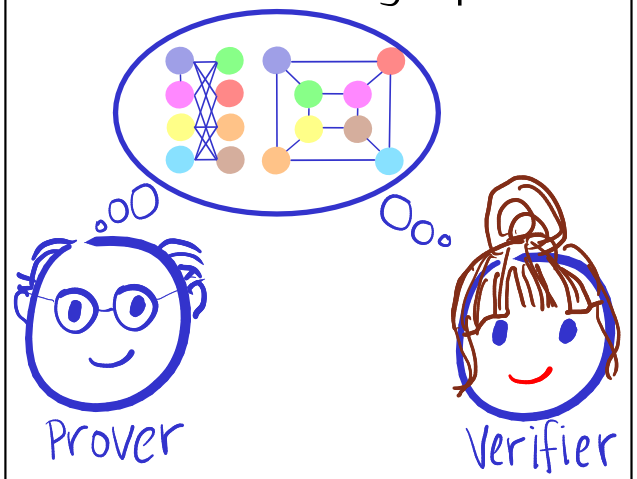
L17-18

Private computation of function



L14-16

Zero-knowledge proofs



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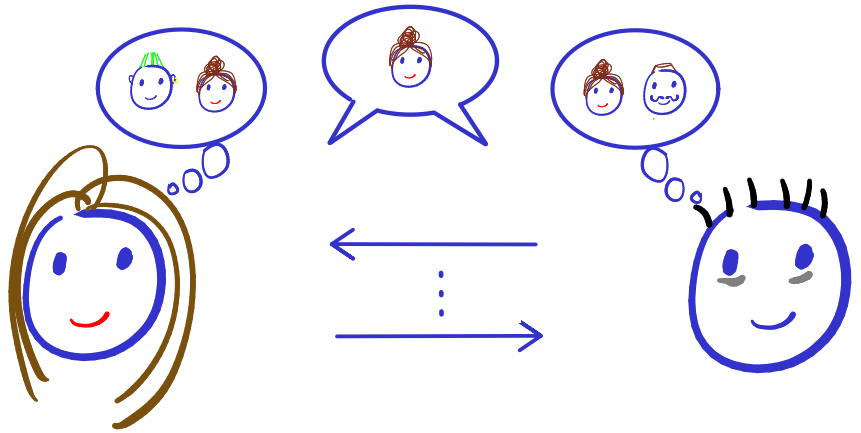

L14-16

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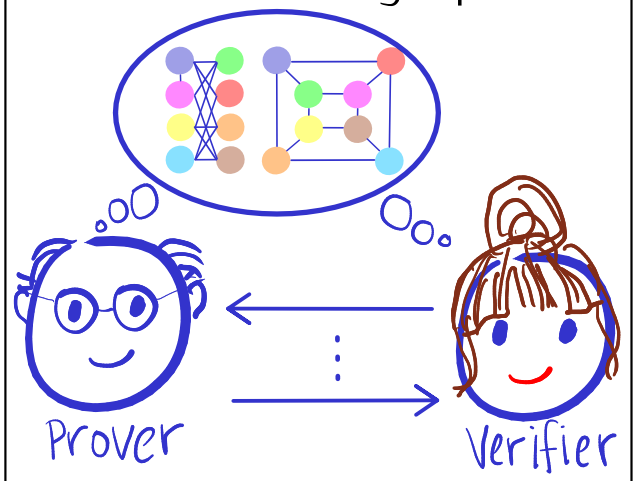

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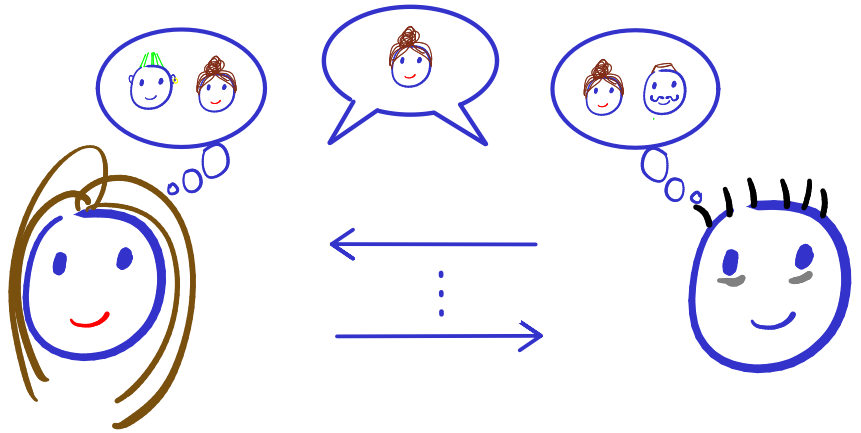

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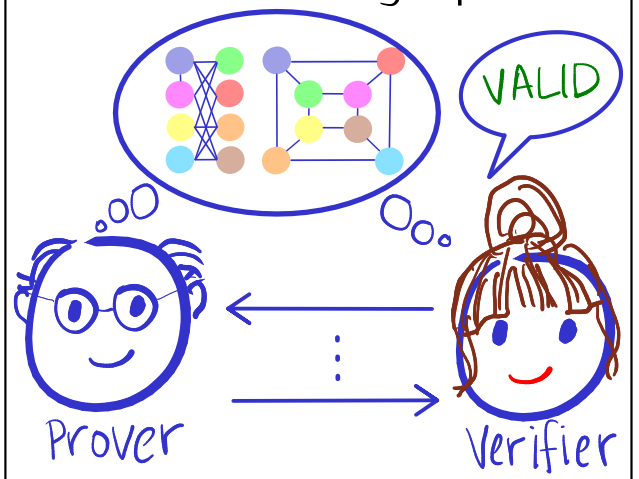

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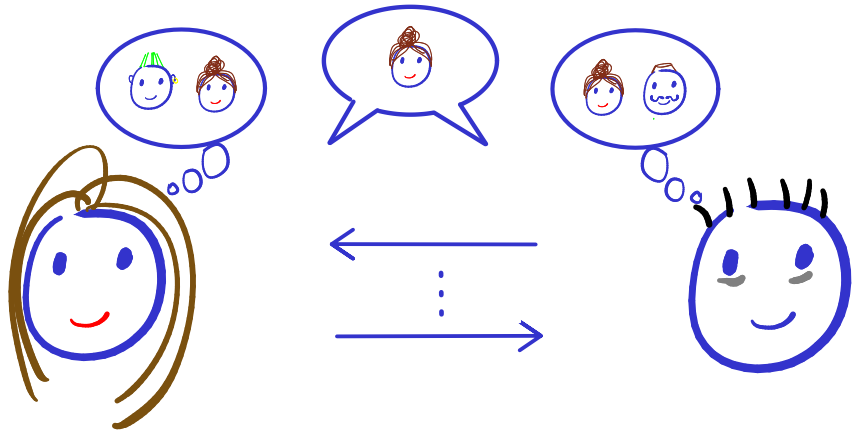

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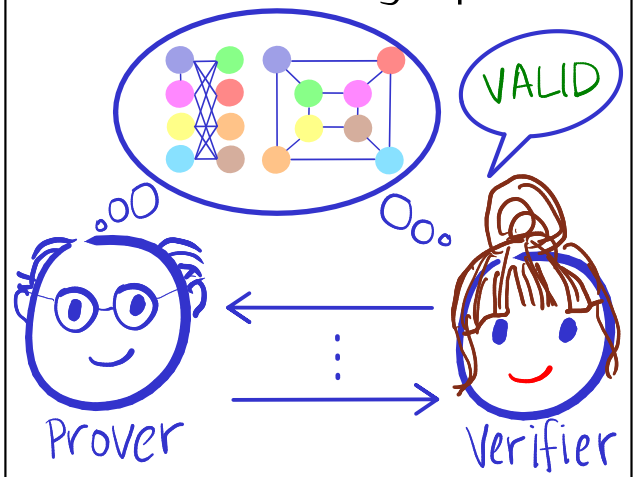
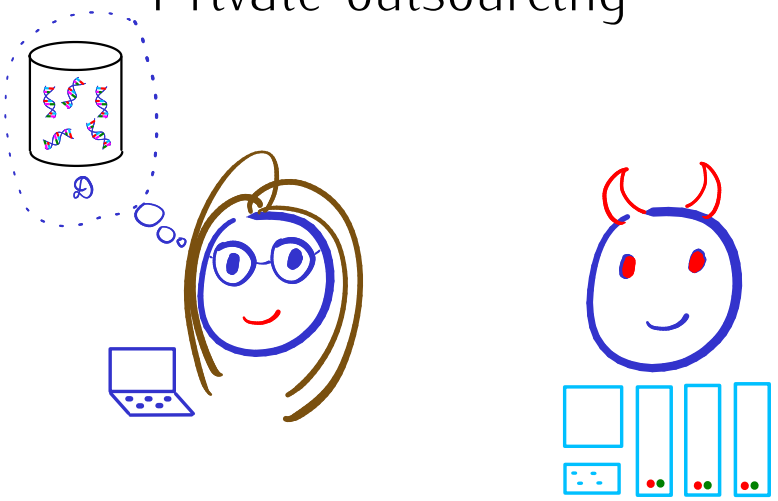


Private outsourcing



L19

Verifiable outsourcing

L20

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L17-18

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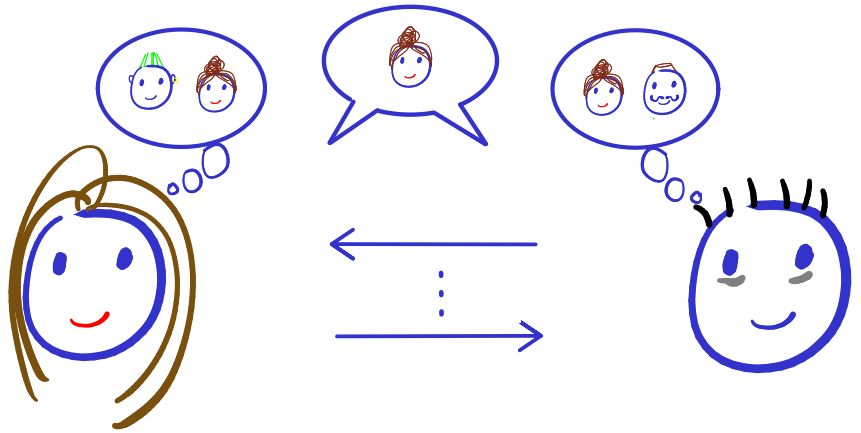

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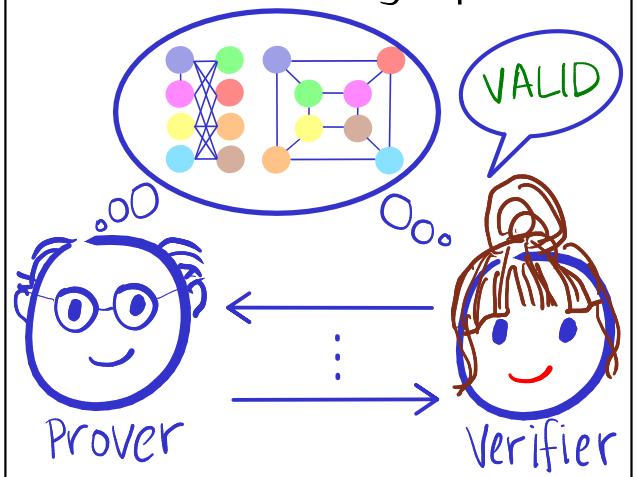

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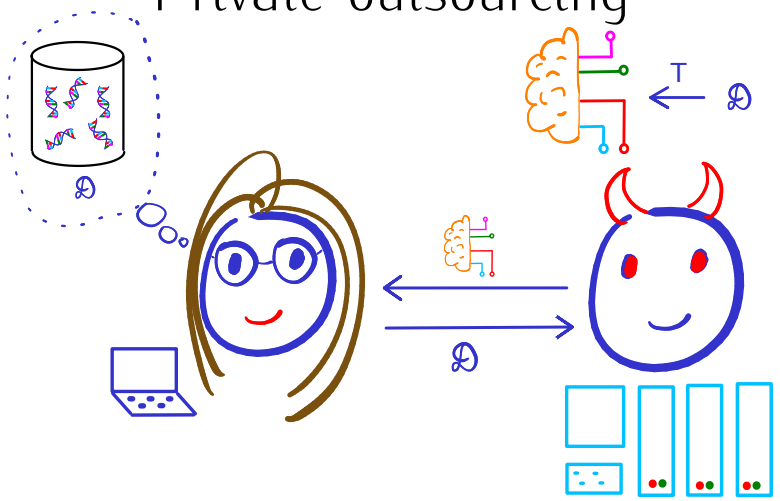

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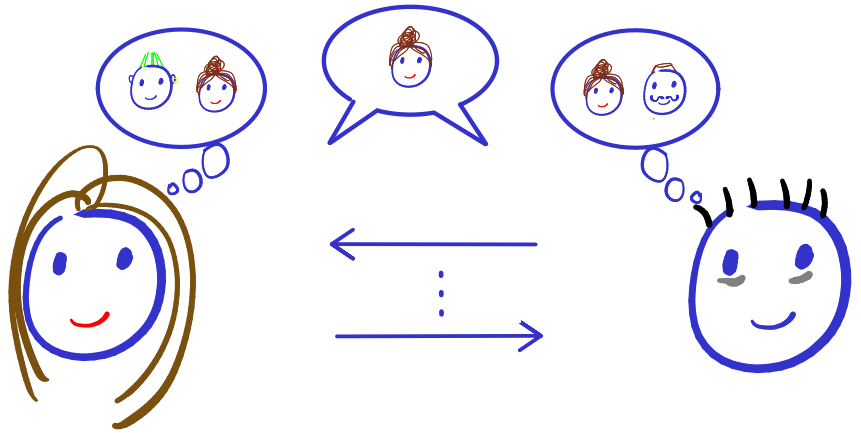

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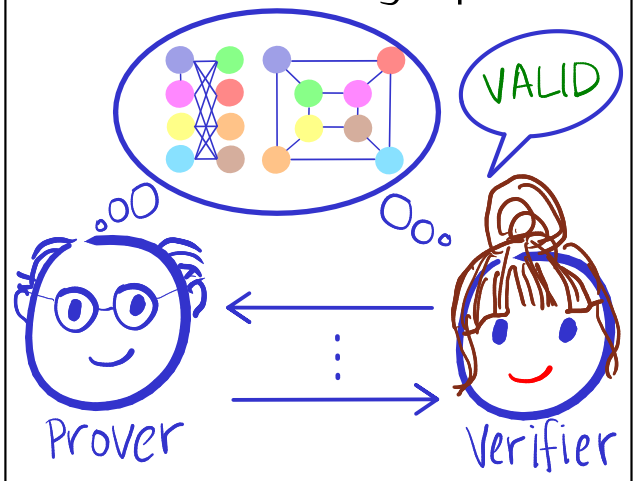

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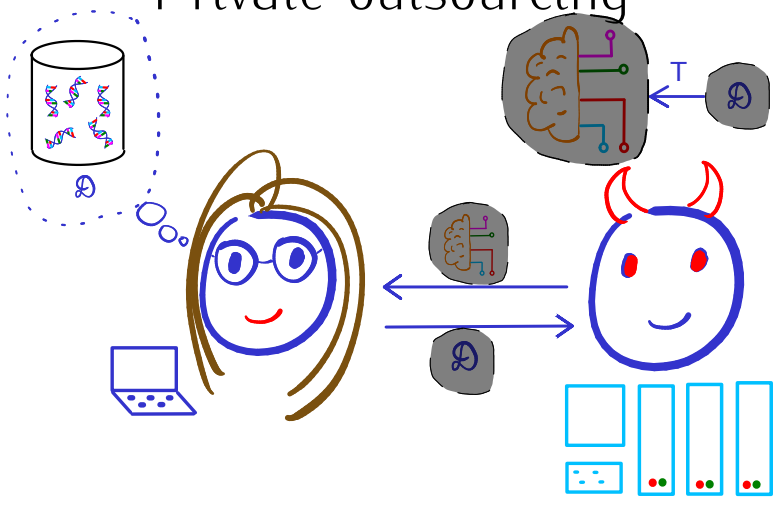

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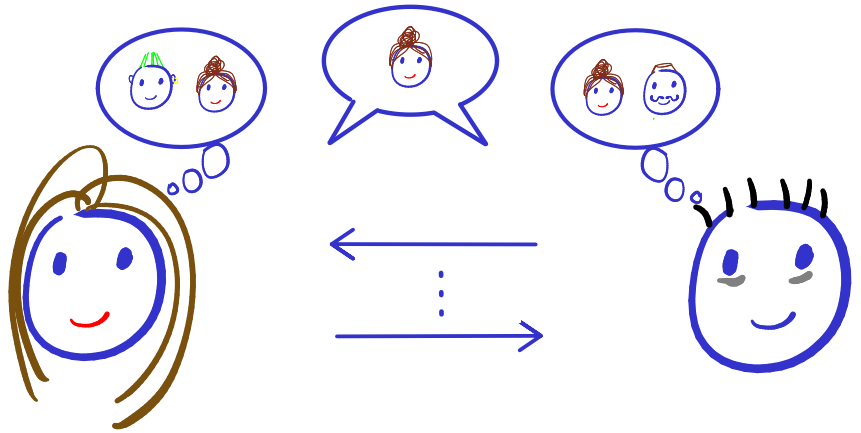

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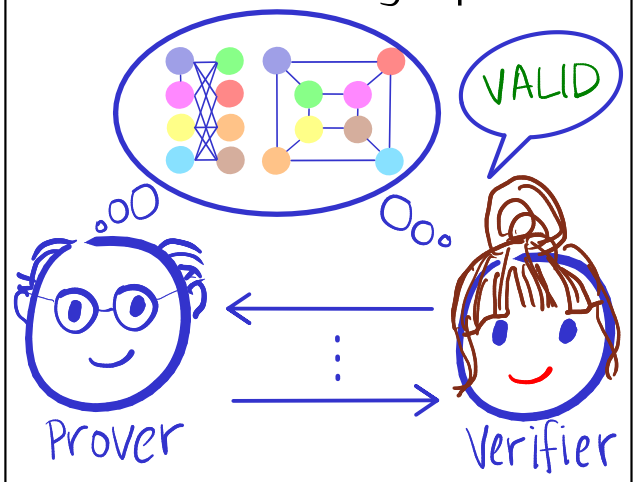

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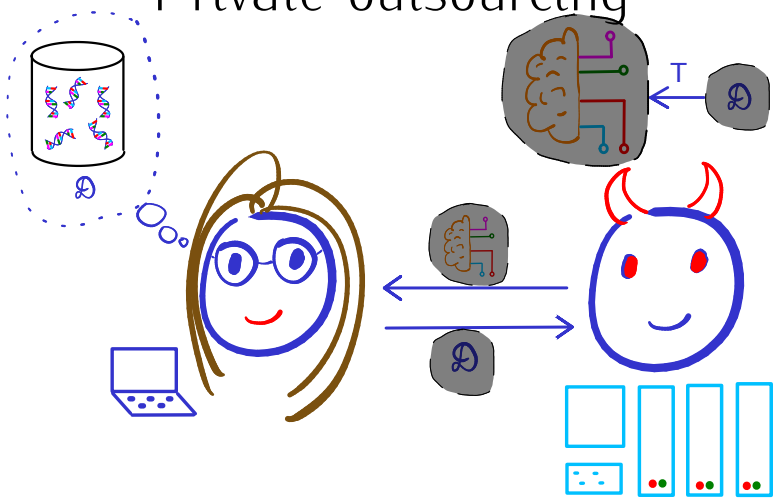

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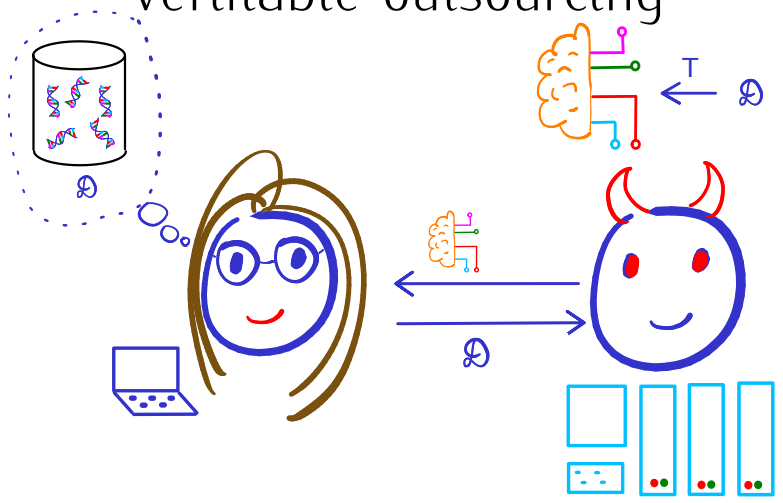

L20

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L17-18

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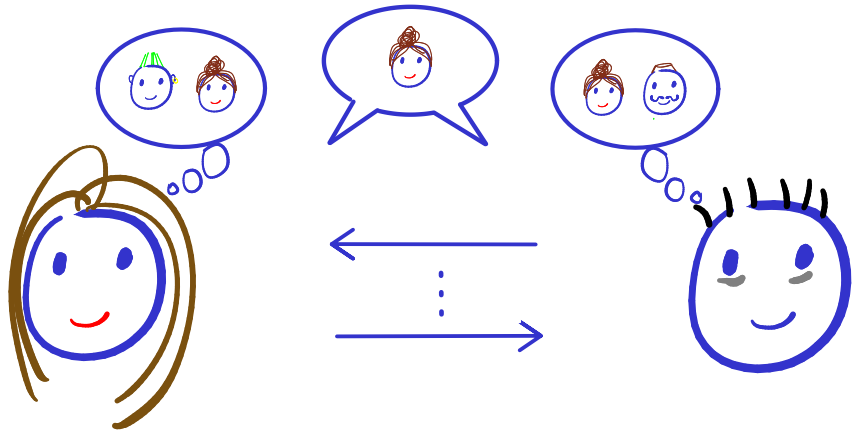

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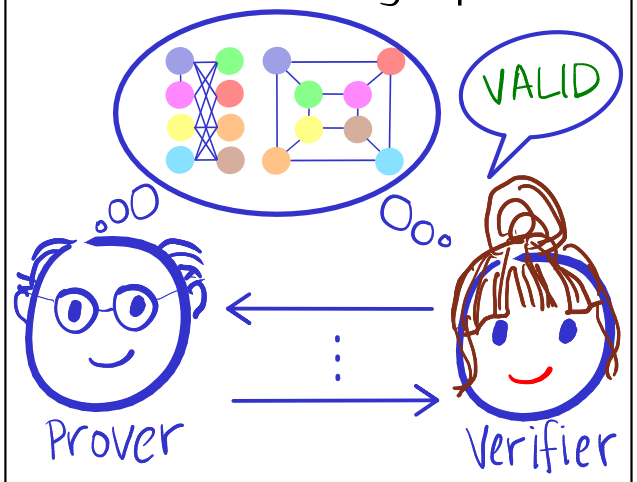
Zero-knowledge proofs



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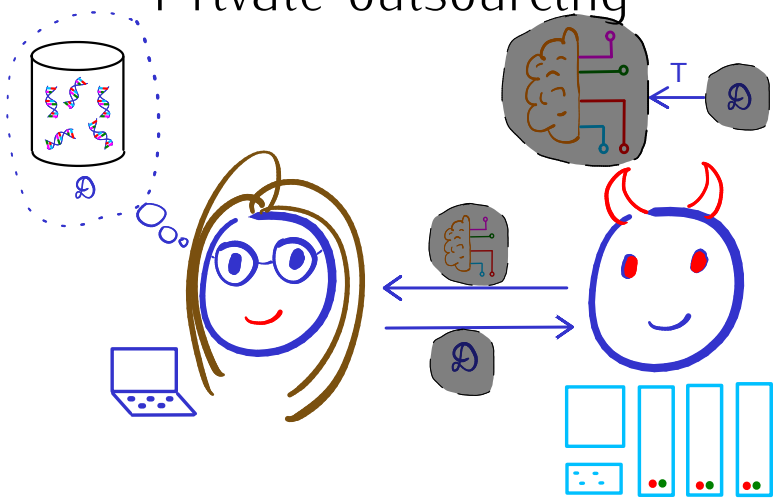

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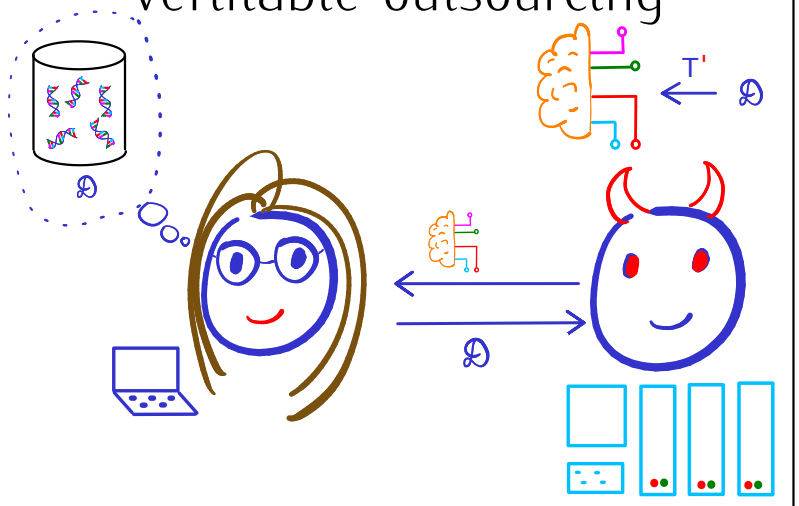

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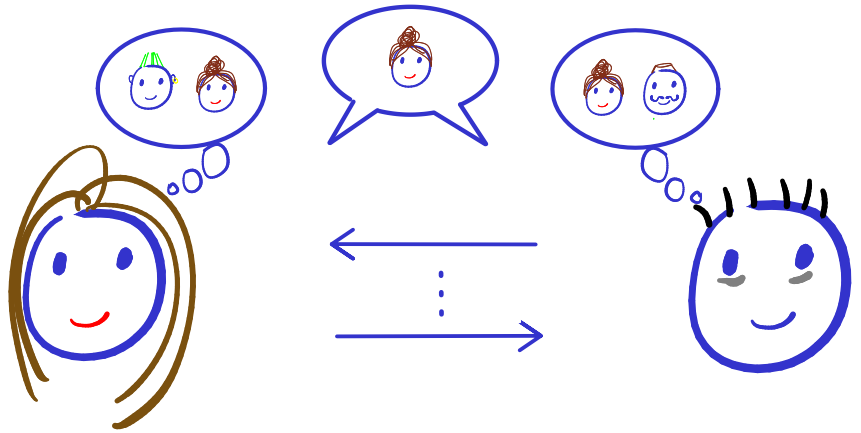

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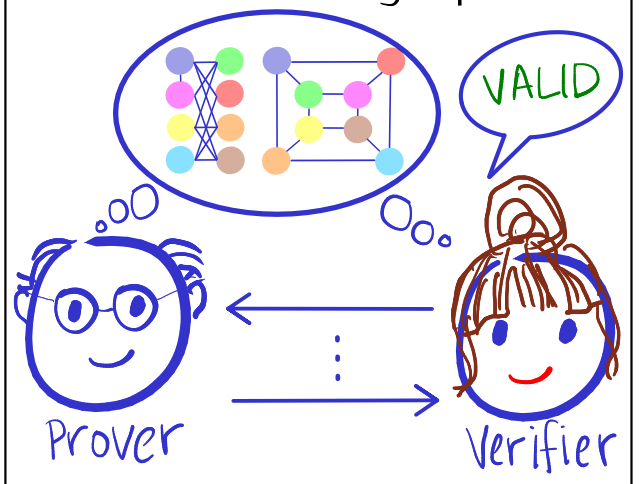

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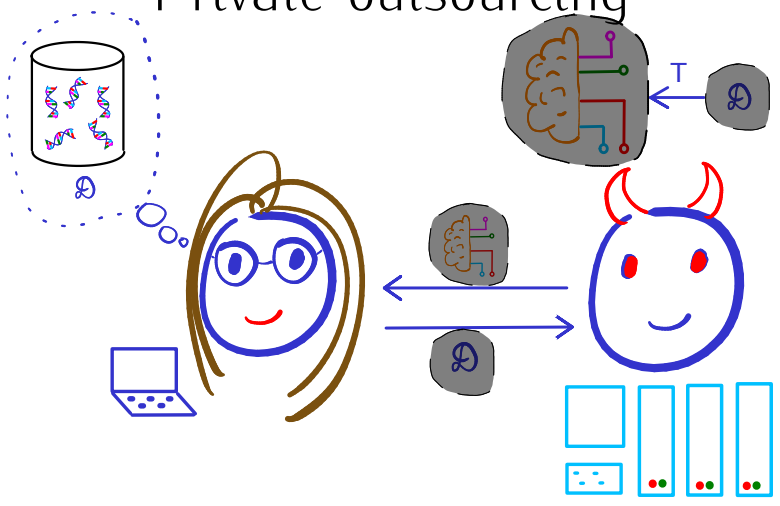

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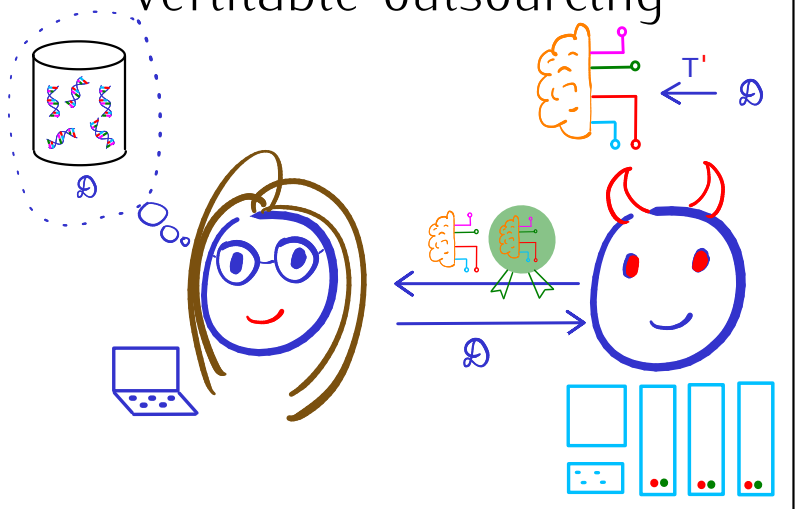

L20

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L17-18

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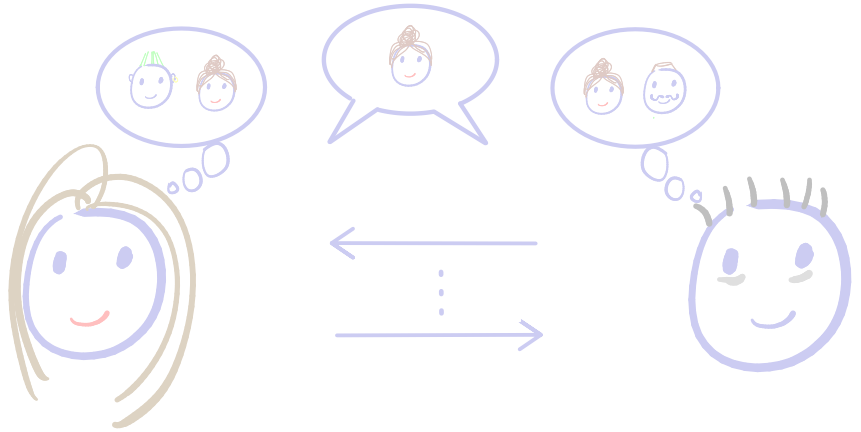
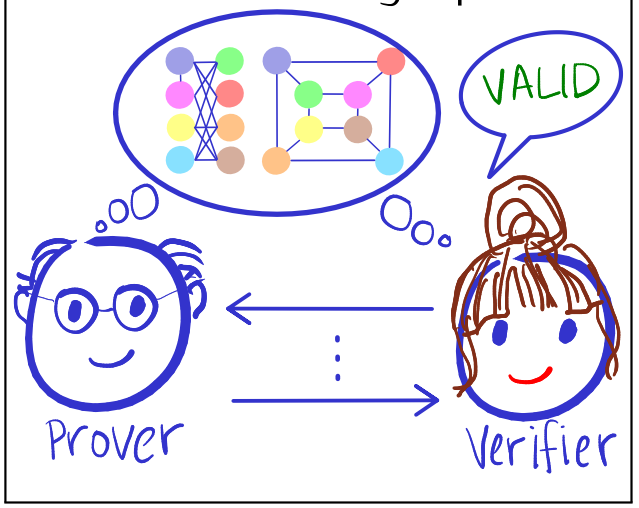


L14-16

Zero-knowledge proofs



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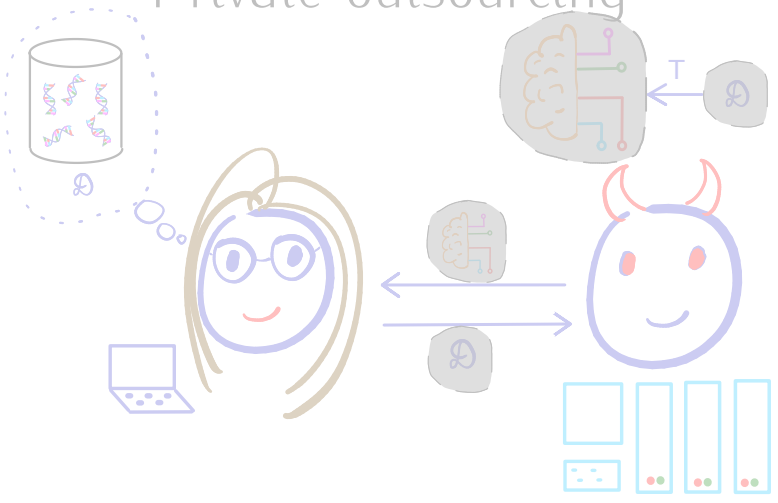

L19

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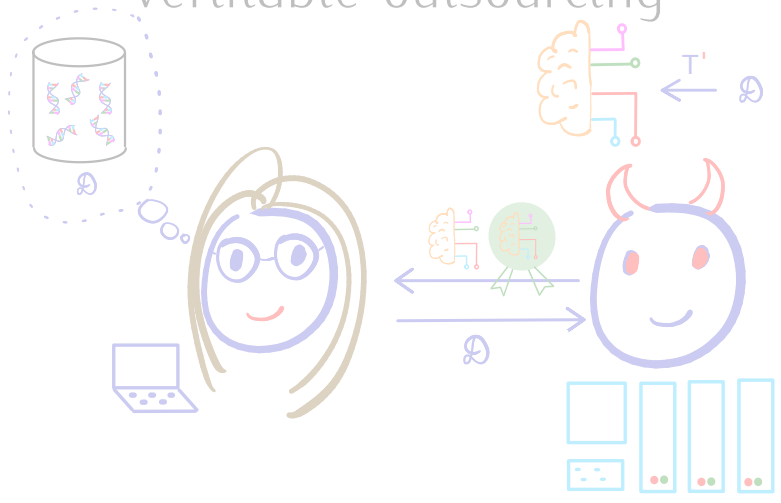
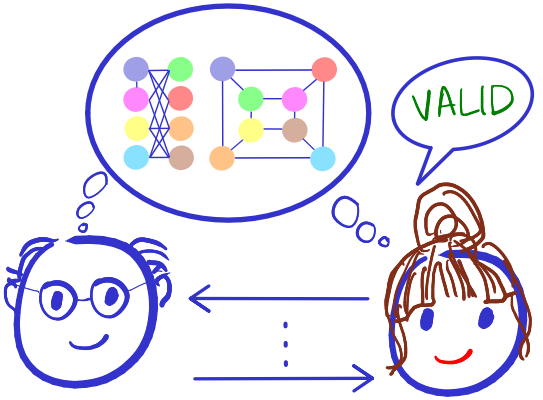


L20

Plan for Today's Lecture

- What constitutes a proof?
  - Traditional "NP" proofs vs *interactive* proofs



Plan for Today's Lecture

- What constitutes a proof?
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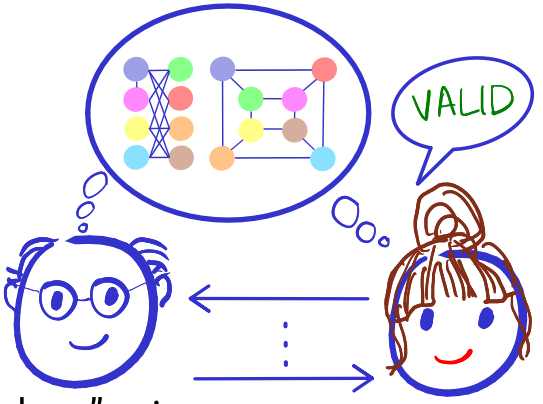
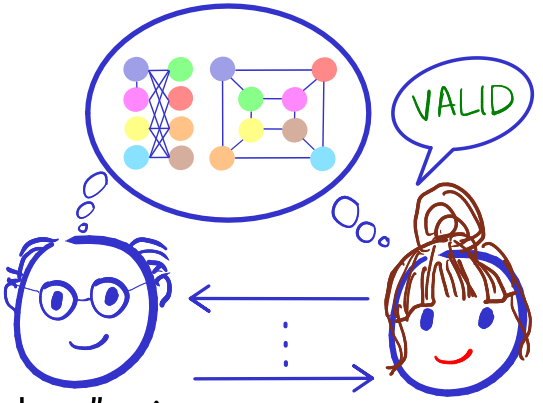
- Zero-knowledge proofs: capture "zero knowledge" via simulation paradigm



Plan for Today's Lecture



- What constitutes a proof?
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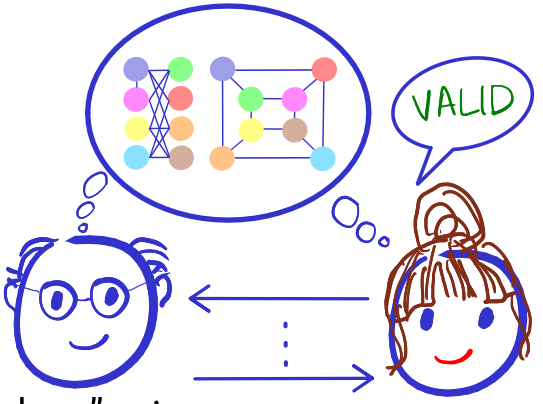
- Examples. ZKP for:
  - Graph isomorphism (GI)
  - Quadratic residuosity (QR)
  - Graph non-isomorphism (GNI)
  - Quadratic non-residuosity (QNR)

Plan for Today's Lecture



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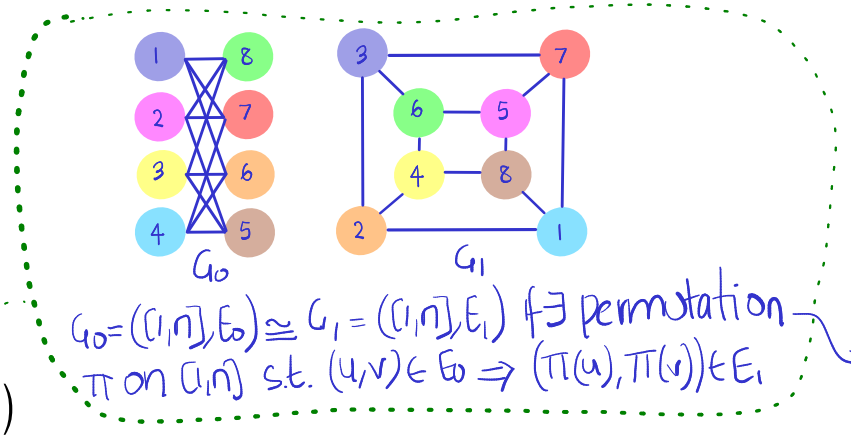

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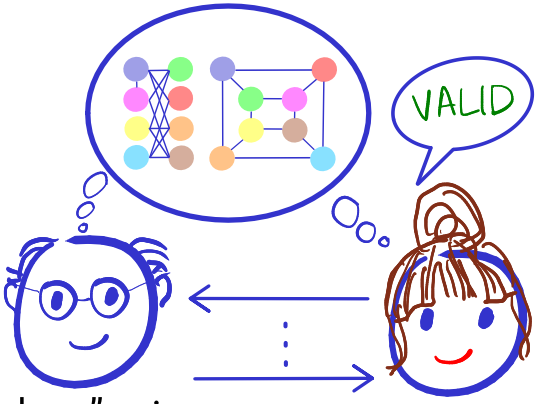
Plan for Today's Lecture



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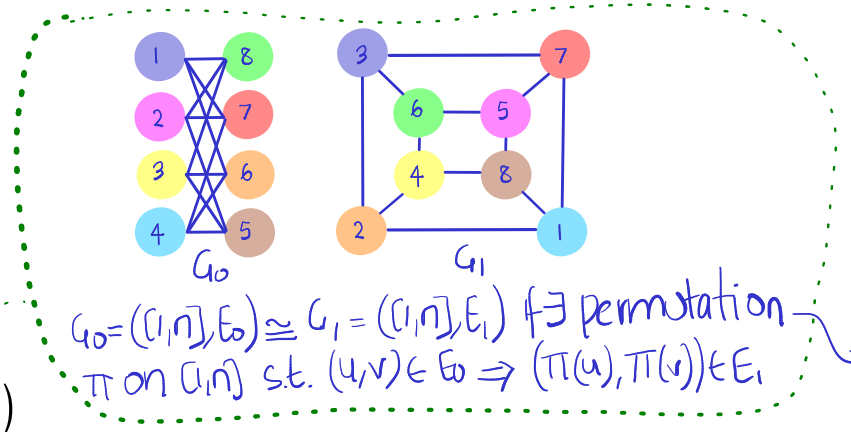

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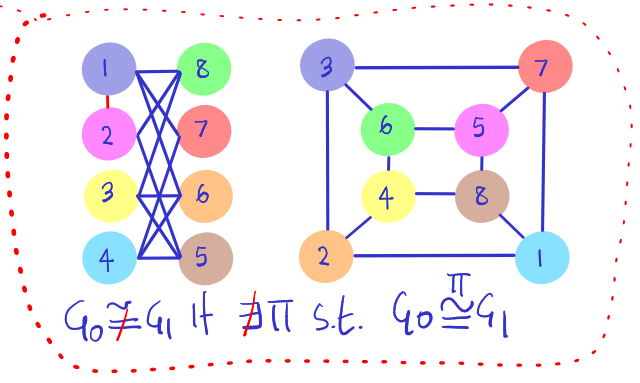

Plan for Today's Lecture

1. Interactive Proof (IP)
2. Zero Knowledge (Interactive) Proof (ZKP)
3. Honest Verifier ZKP for Graph (Non-) Isomorphism

Plan for Today's Lecture

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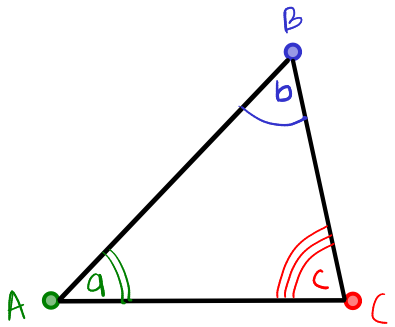
2. Zero Knowledge (Interactive) Proof (ZKP)
3. Honest Verifier ZKP for Graph (Non-) Isomorphism

Traditional "NP" Proof...

- Axioms $\xrightarrow{\text{derivation rules}}$ theorems=true statements
  - E.g.: Axioms of Euclidean geometry

    → Theorem: "Sum of angles of a triangle equals 180°"



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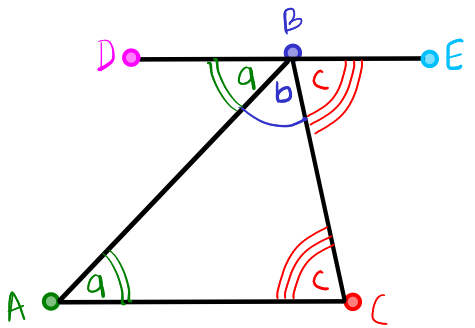

- Prover vs. verifier
  - Prover does the heavy lifting: derives the proof
    1. Construct a line through B parallel to $\overline{AC}$
    2. $\angle DBA = \angle a$ and $\angle EBC = \angle c$ (alternate interior angles)
    3. $2 \Rightarrow \angle a + \angle b + \angle c = \angle DBA + \angle b + \angle EBC = 180°$

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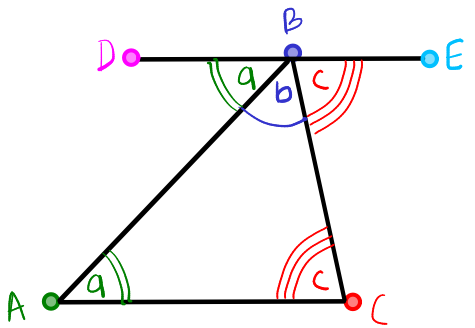

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  - Verifier checks the proof, step by step

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- Corresponds to class NP
  - A language $\mathcal{L} \in \mathsf{NP}$ if there exists a polynomial-time *deterministic* machine V such that

$$\forall x \in \mathcal{L}\ \exists \pi \in \{0,1\}^{\mathsf{poly}(|x|)} : \mathsf{V}(x, \pi) = 1$$

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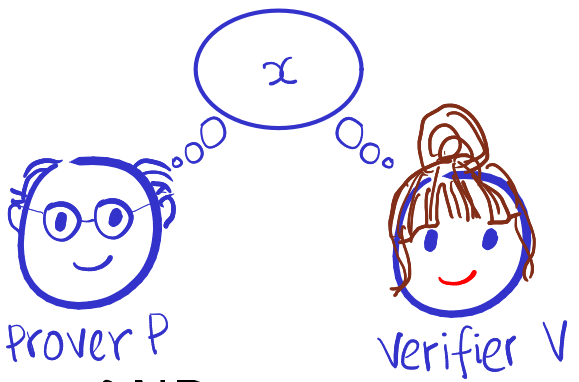

- "Proof system" view of NP
  - Prover P is *unbounded*: finds short proof π for x (if one exists)
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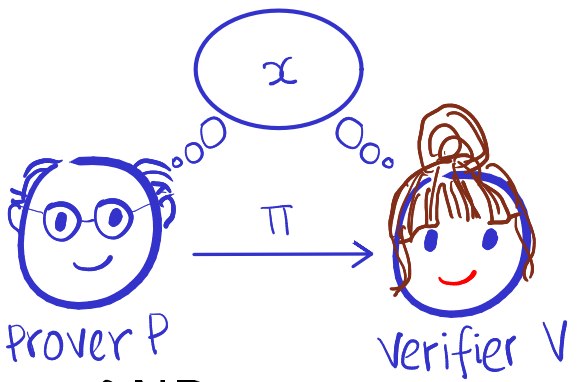

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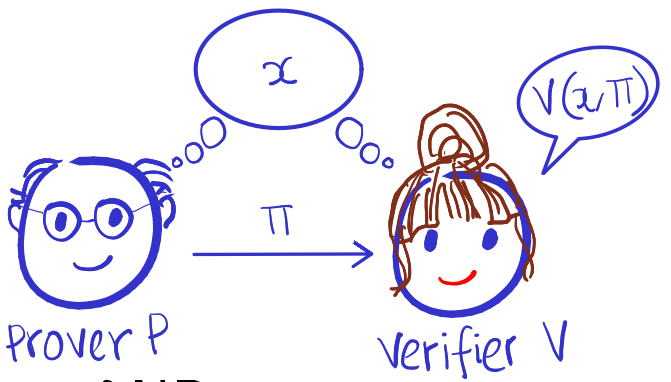

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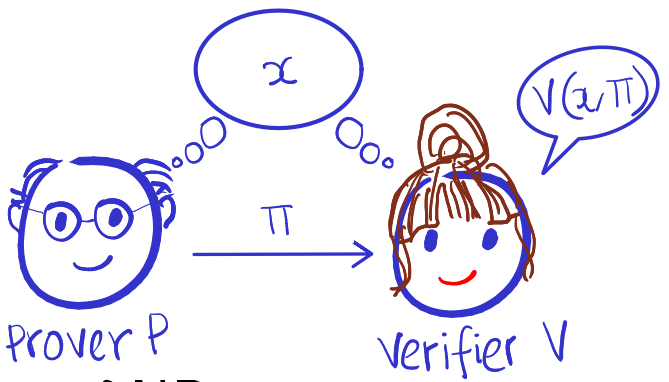

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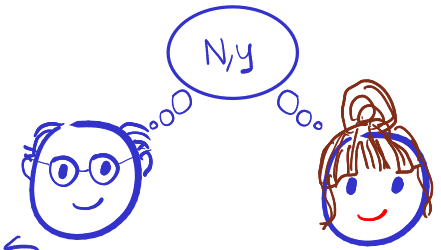
  - Completeness: $x \in \mathcal{L} \Rightarrow \mathsf{P}$ finds $\pi \Rightarrow \mathsf{V}(x, \pi) = 1$
  - Soundness: $x \notin \mathcal{L} \Rightarrow \nexists \pi \in \{0,1\}^{\mathsf{poly}(|x|)}$ s.t. $\mathsf{V}(x, \pi) = 1$

Which Languages have "NP" Proofs?

Quadratic residuosity (QR)



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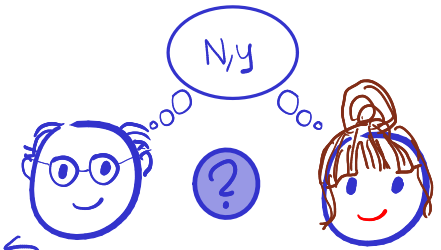

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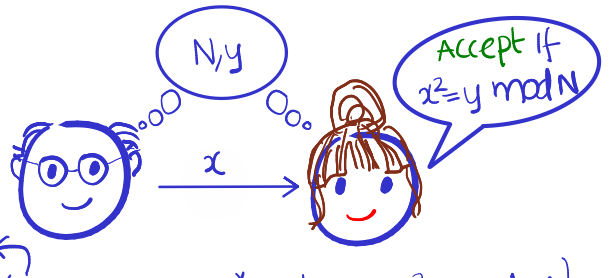

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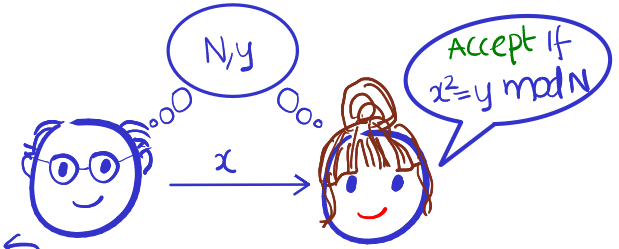

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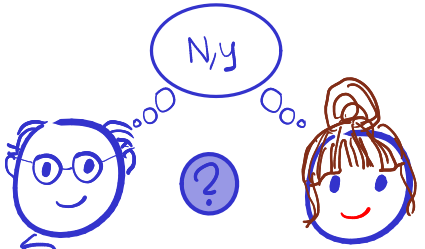

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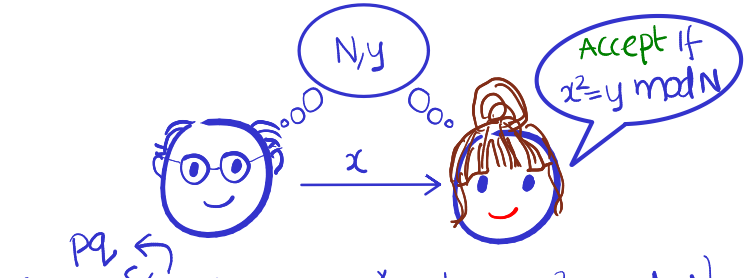

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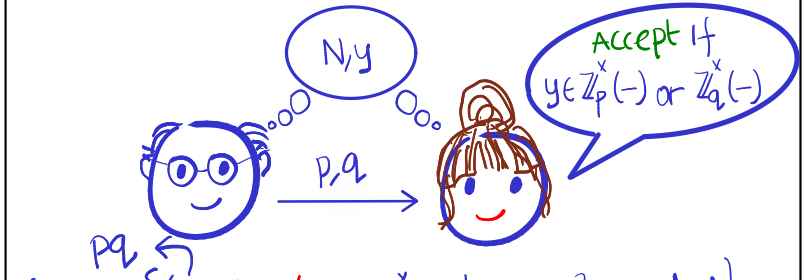

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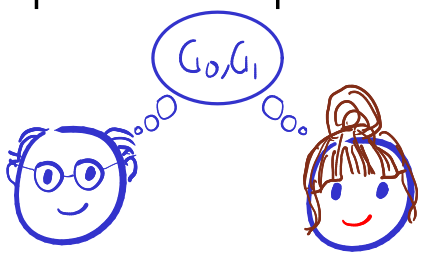

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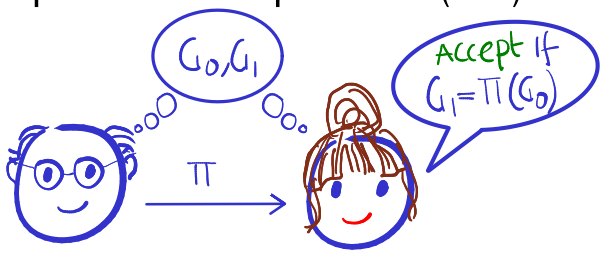

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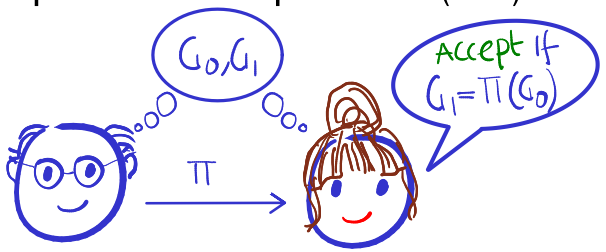

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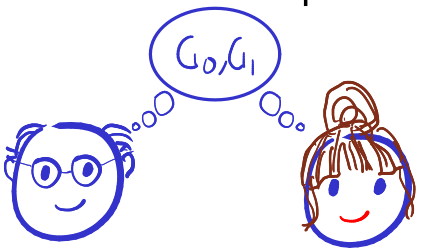

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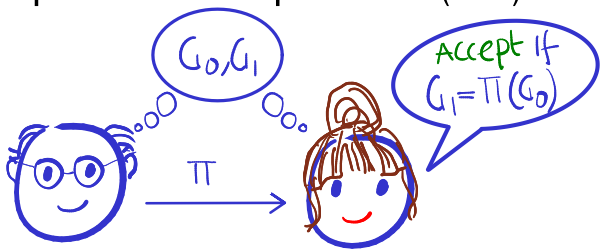

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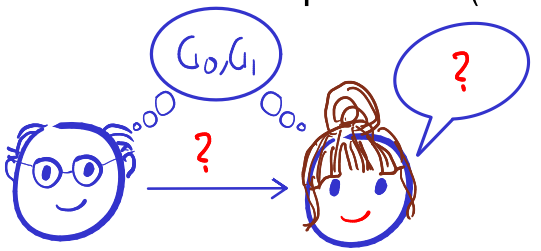

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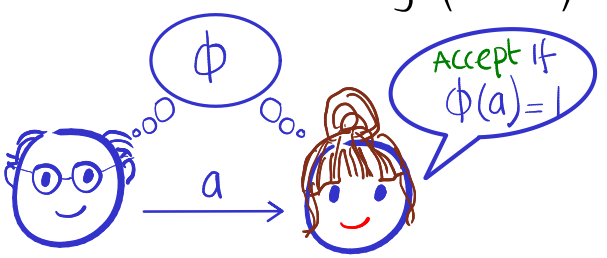
Boolean satisfiability (SAT)



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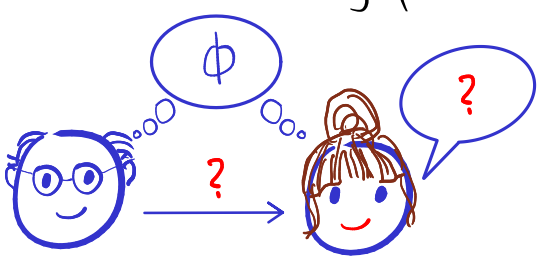
Bool. unsatisfiability (UNSAT)



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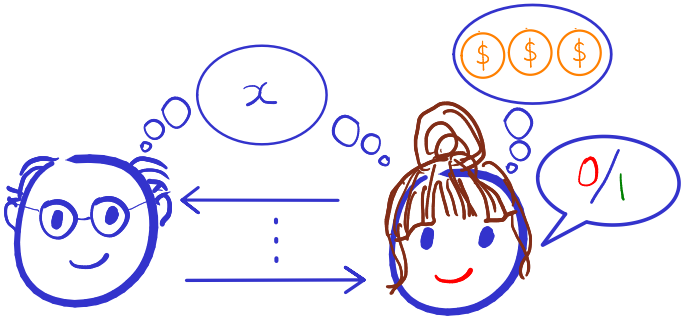

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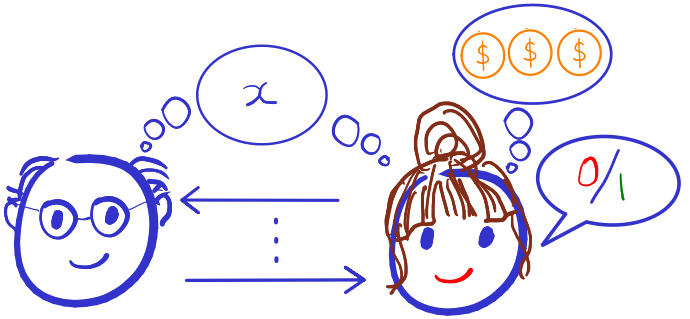

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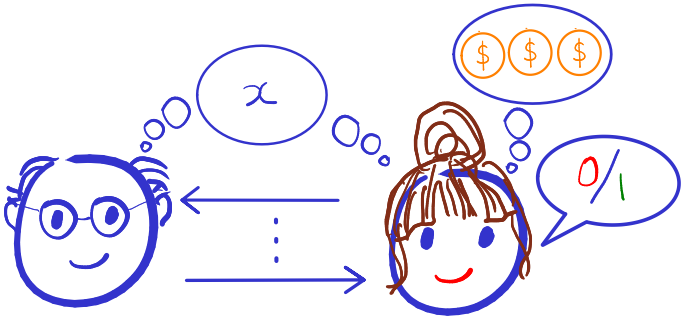

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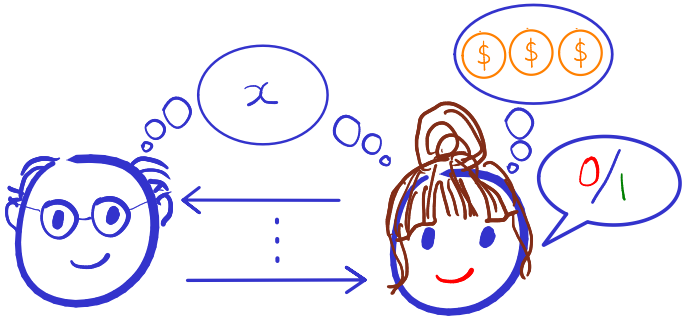

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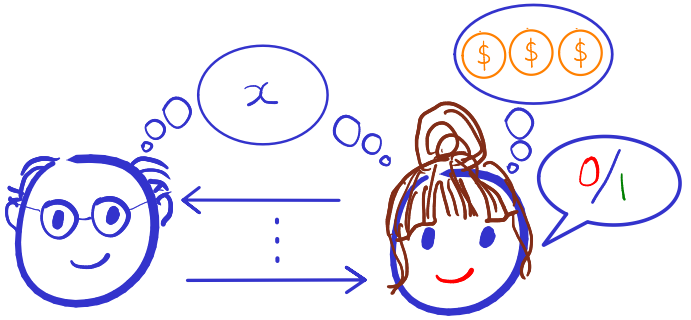

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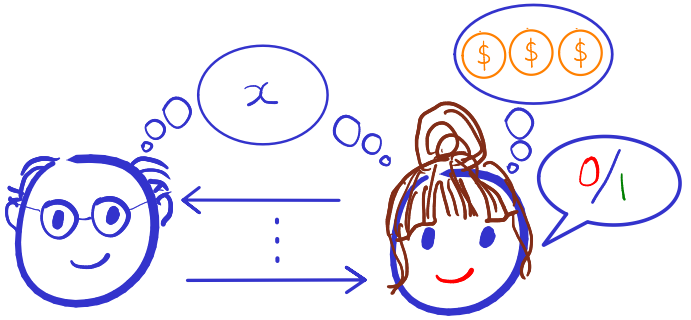

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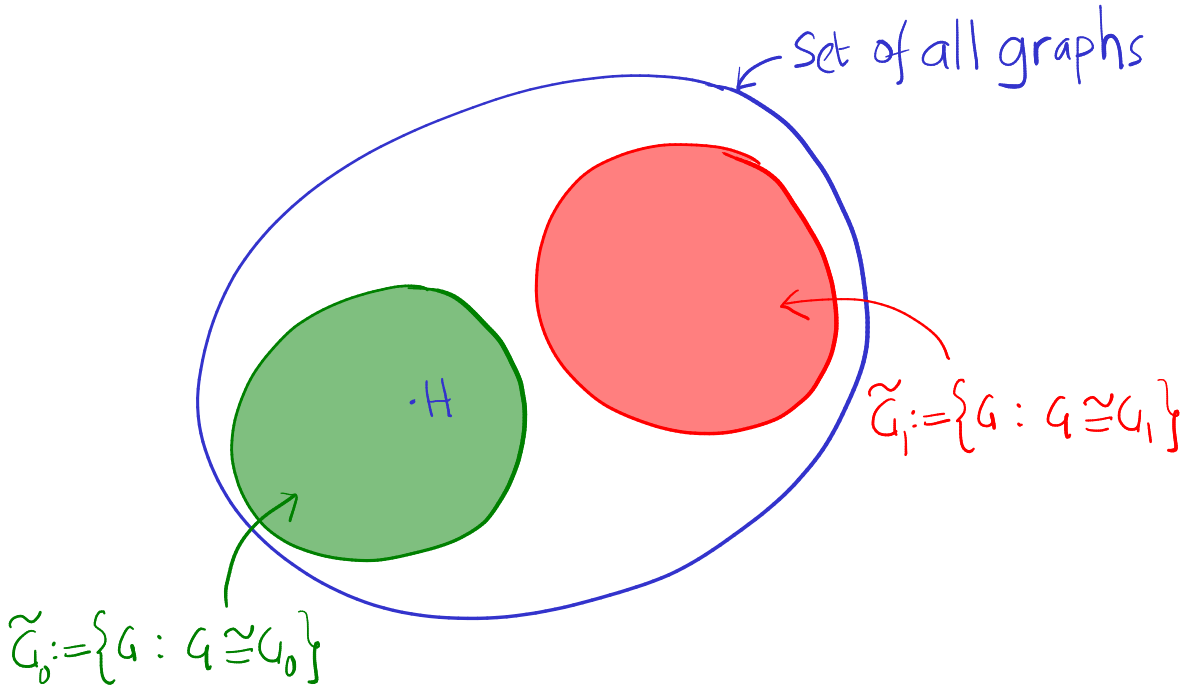
Exercise 1 (Robustness of Defintion 1)

Show that languages captured by Defintion 1 doesn't change when 1) $\epsilon_c \leq 1/2^{|x|}, \epsilon_s \leq 1/2^{|x|}$*; 2)* $\epsilon_c \leq 1/2 - 1/|x|, \epsilon_s \leq 1/2 - 1/|x|$

Power of Randomness+Interaction: IP for GNI...

Idea: $G_0 \not\cong G_1 \Rightarrow$ for any graph H, $G_0 \cong H$ *and* $G_1 \cong H$ *both cannot hold*



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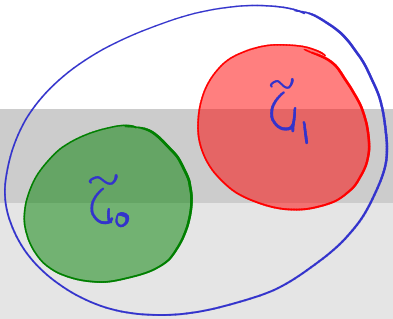

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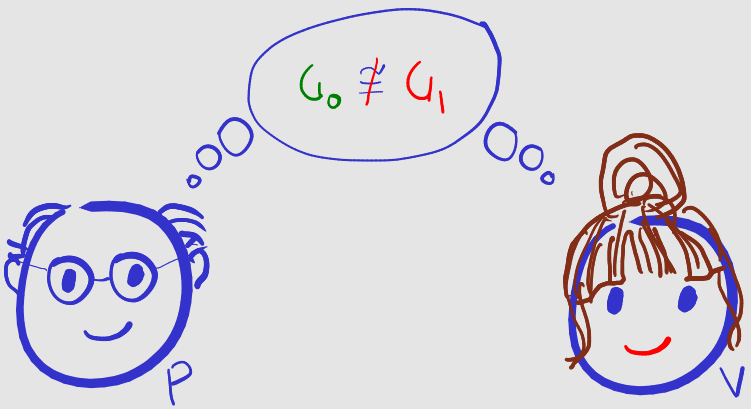

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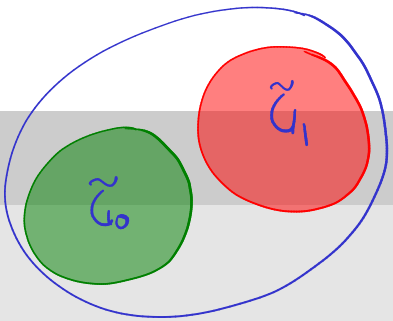

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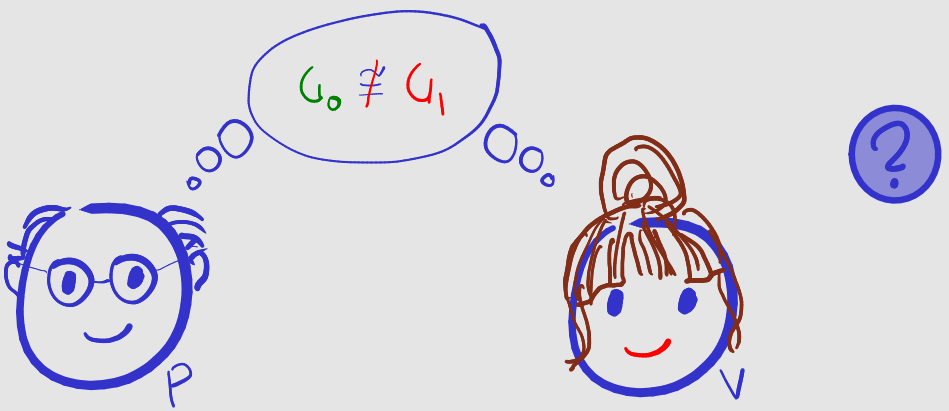

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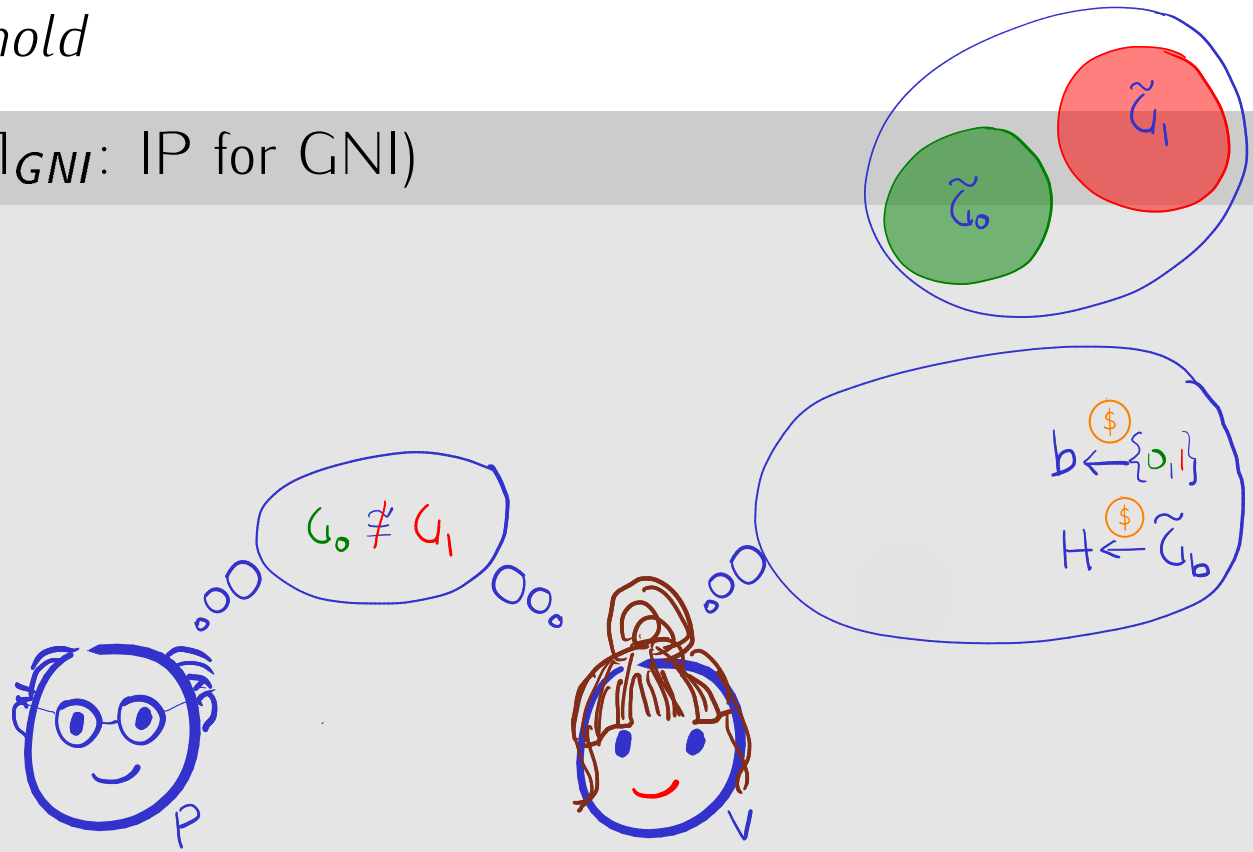

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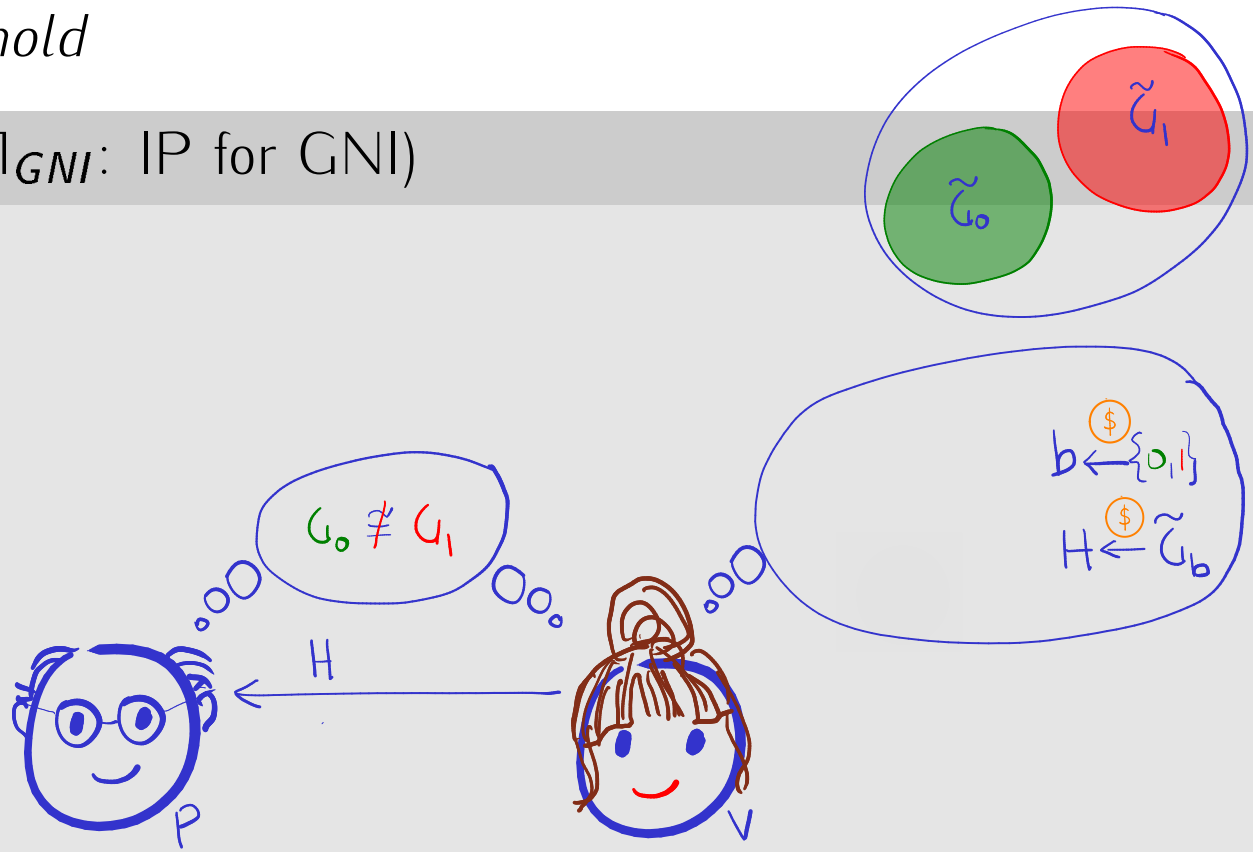

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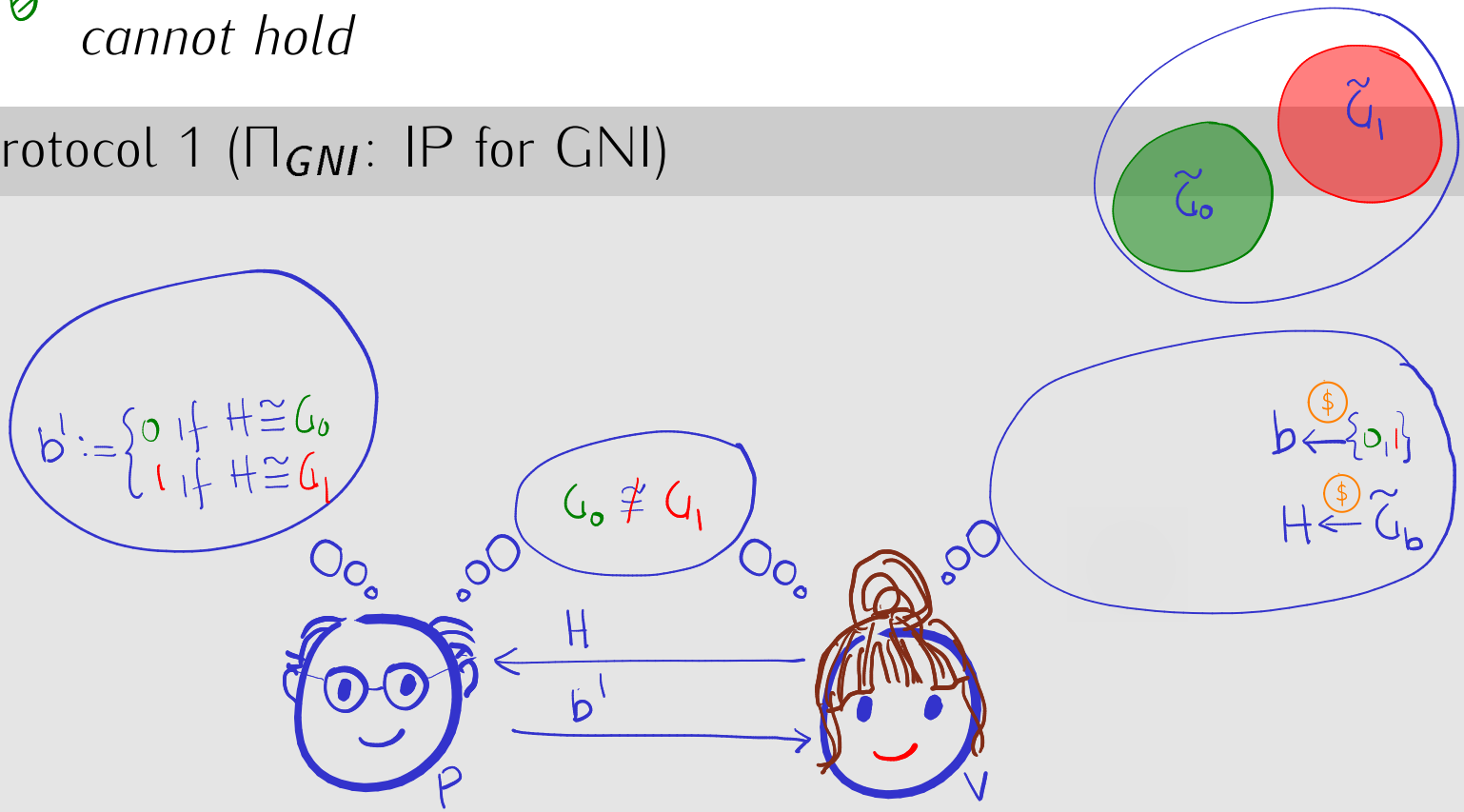

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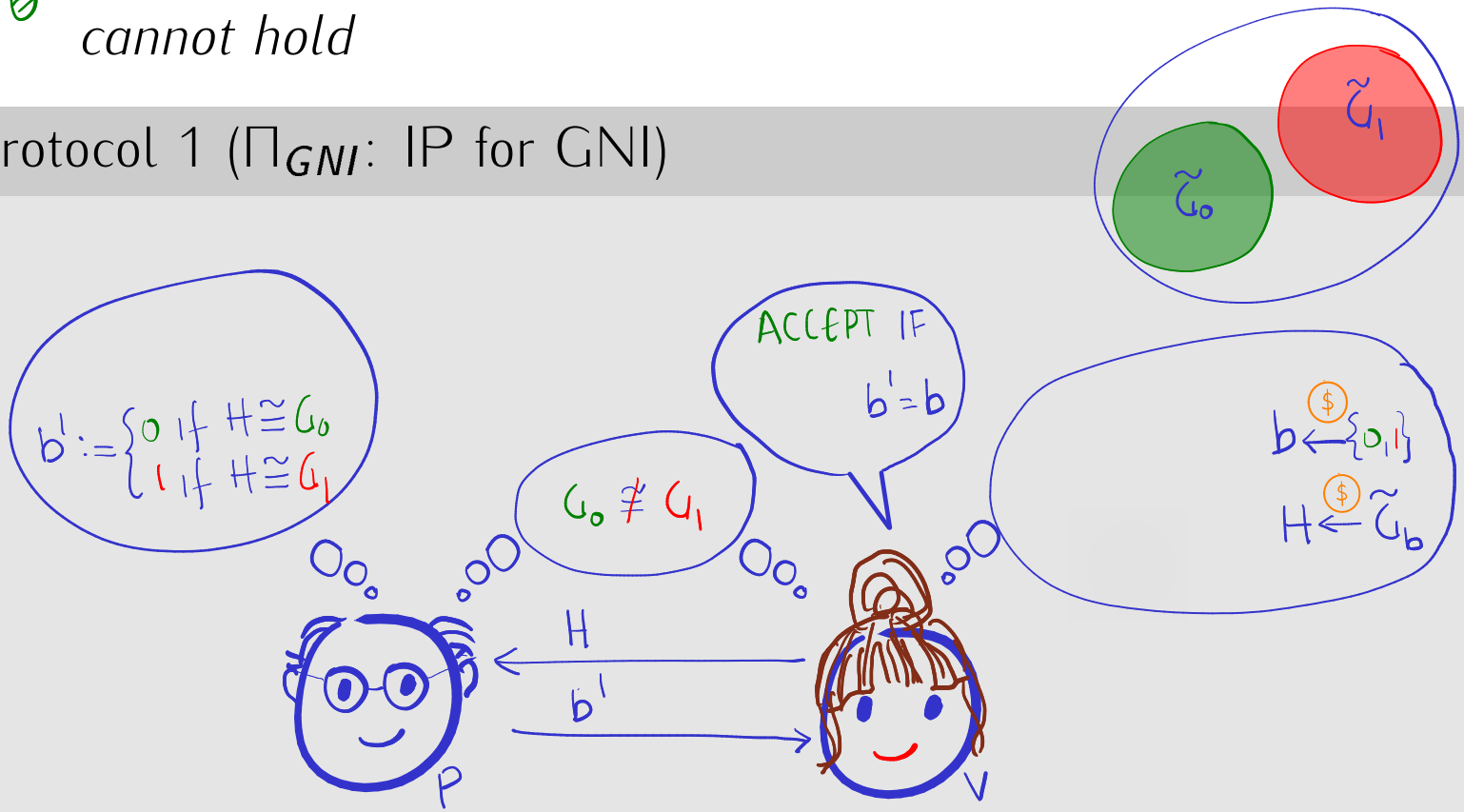

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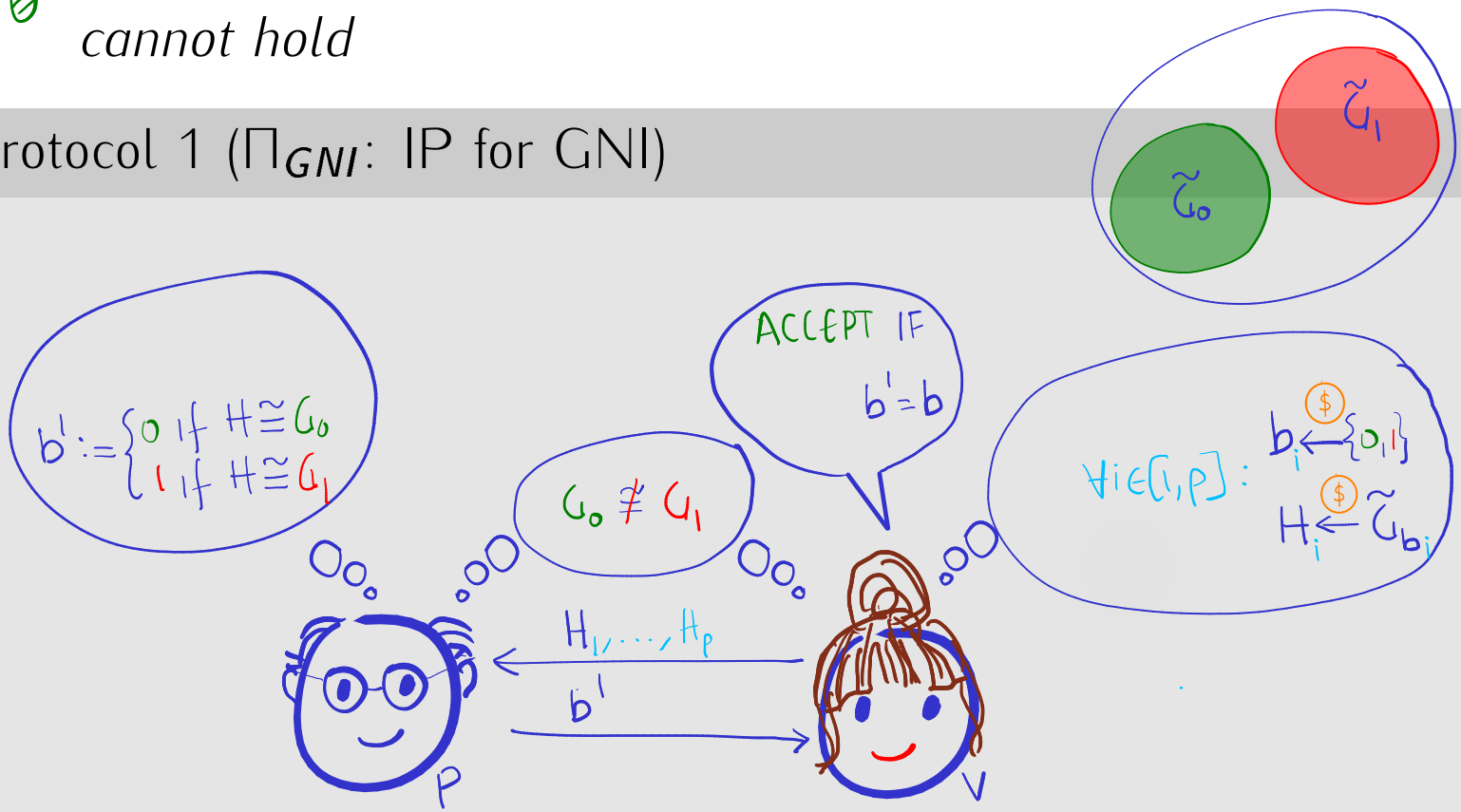

Parallel/sequentially repeat to boost soundness

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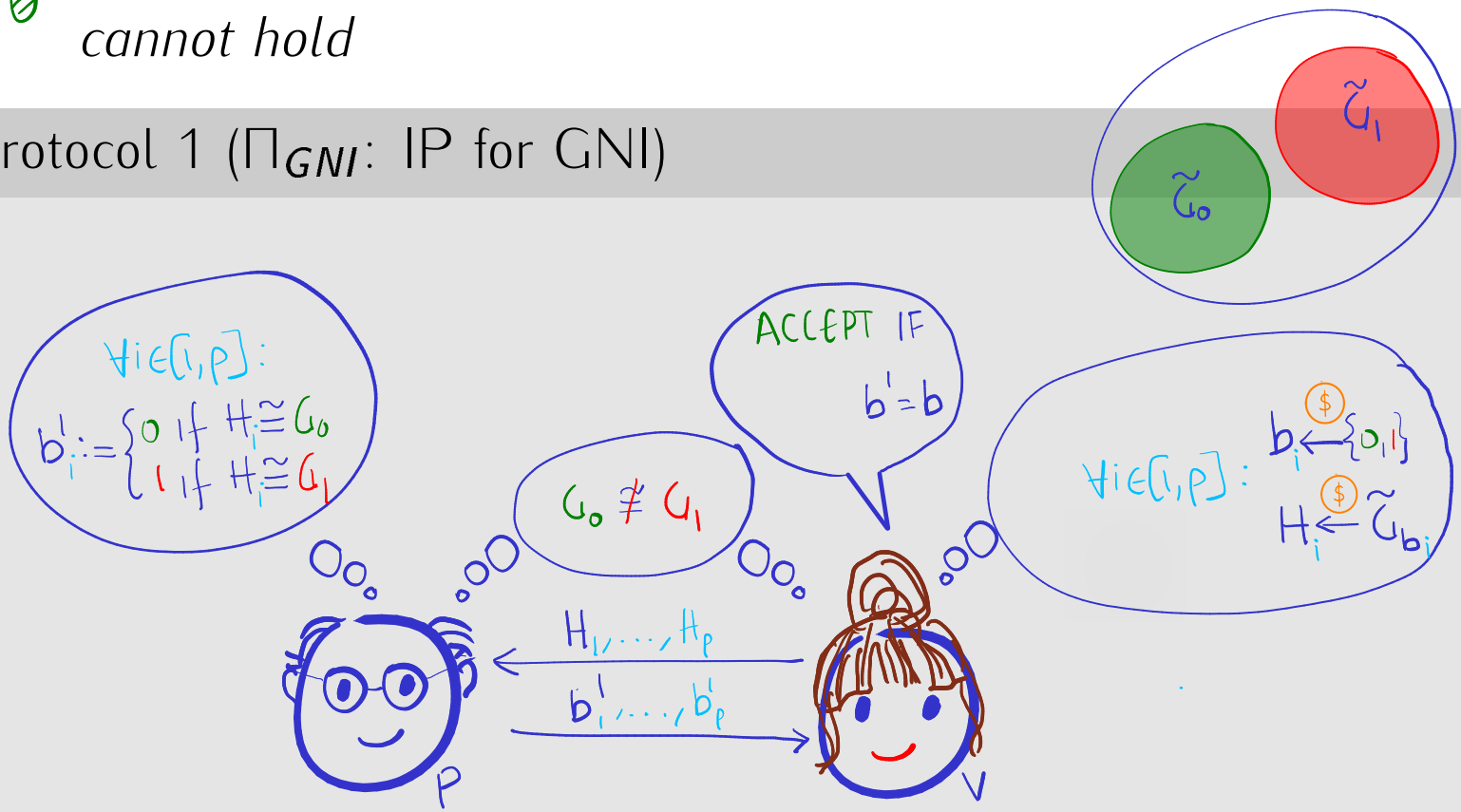

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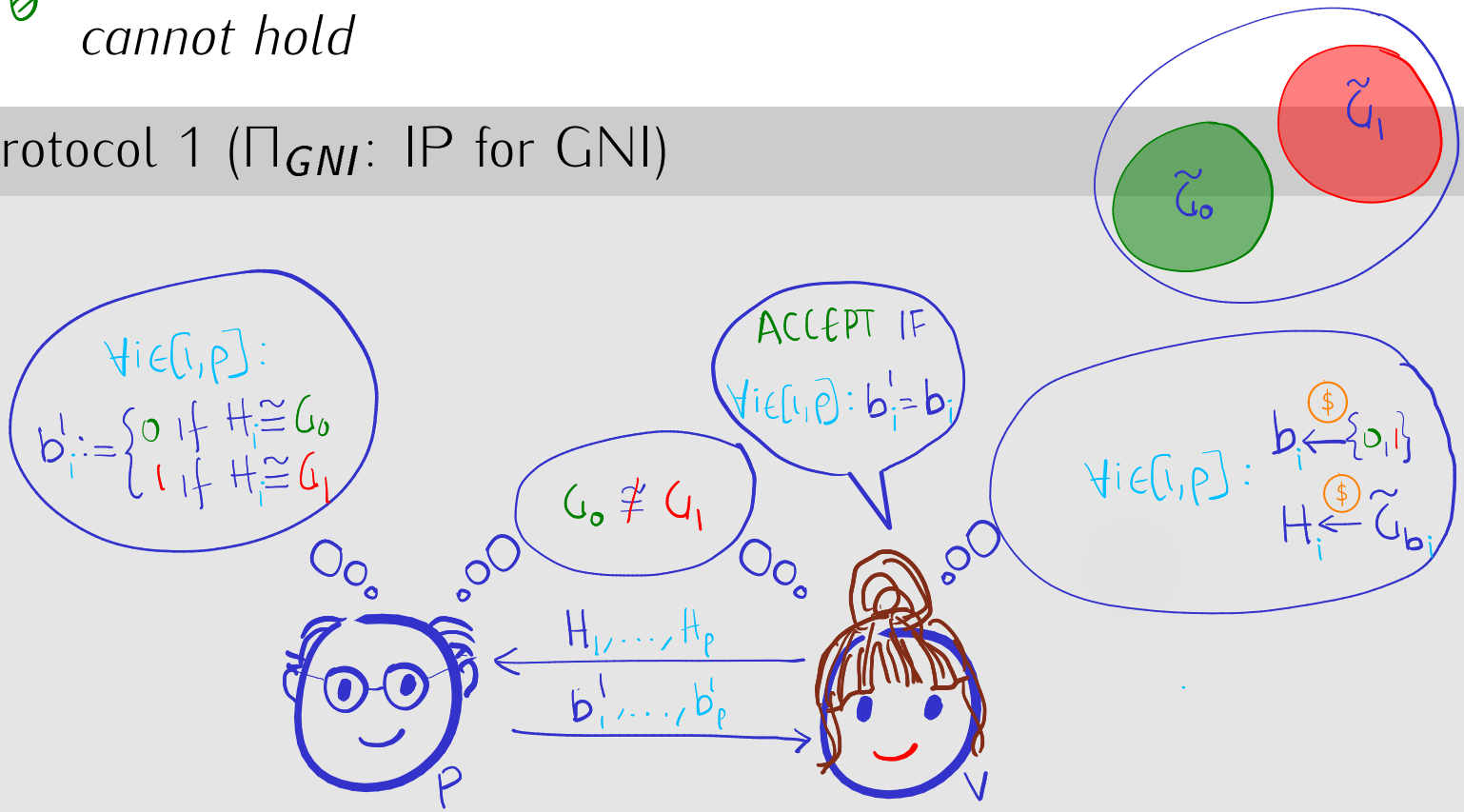

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Theorem 1

Π_{GNI} *is an IP for* $\mathcal{L}_{GNI}$

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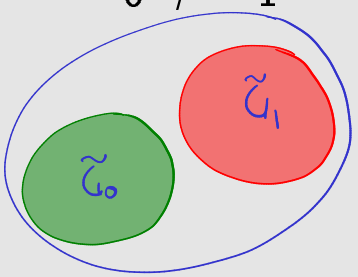
Proof.

- Completeness:
  - $G_0 \not\cong G_1 \Rightarrow \mathsf{P}$ can recover b_i from H_i with certainty



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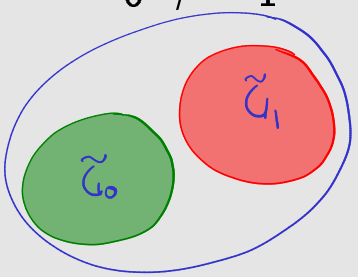

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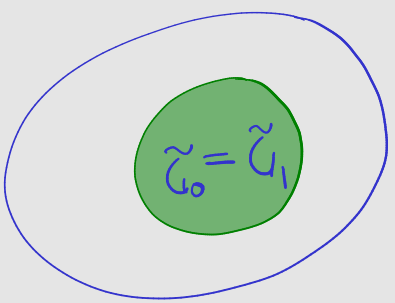
- Soundness:
  - $G_0 \cong G_1 \Rightarrow H_i$ loses information about bits b_i
  - Hence best P^* can do is guess b_is

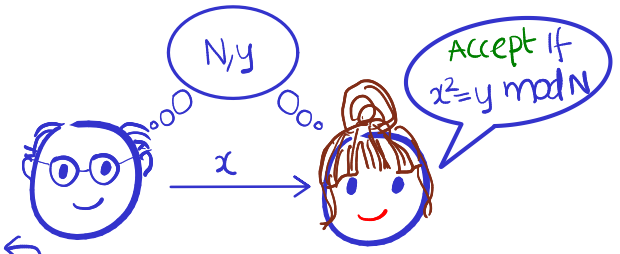


$$\Pr[1 \leftarrow \langle \mathsf{P}^*, \mathsf{V} \rangle (G_0, G_1)] = 1/2^{\rho} < 1/3$$

□

Which Languages have IPs?

Quadratic residuosity (QR)



$\mathcal{L}_{QR} = \{(N,y) : \exists x \in \mathbb{Z}_N^\times \text{ s.t. } y = x^2 \bmod N\}$

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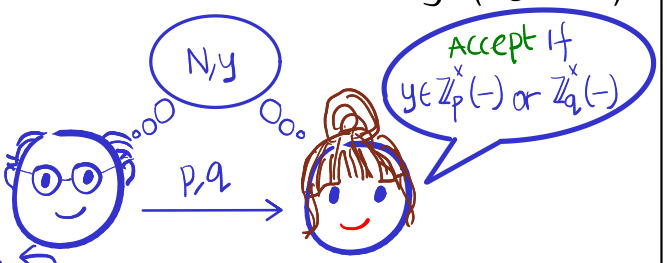
Quad. non-residuosity (QNR)



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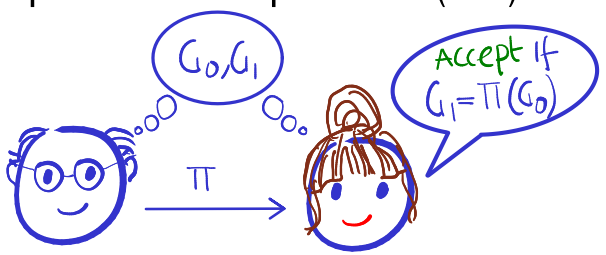

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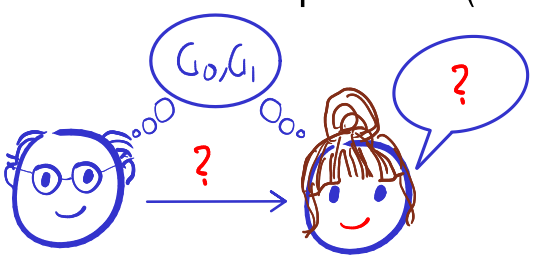

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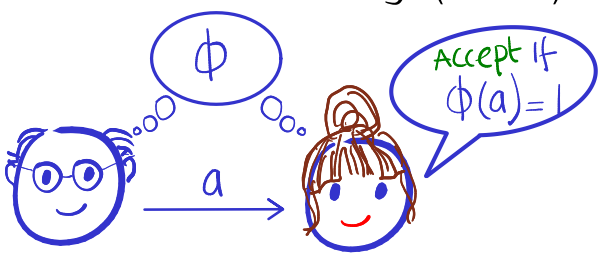

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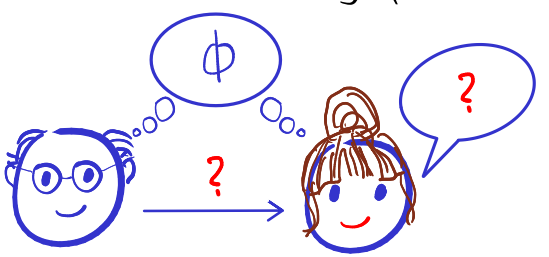

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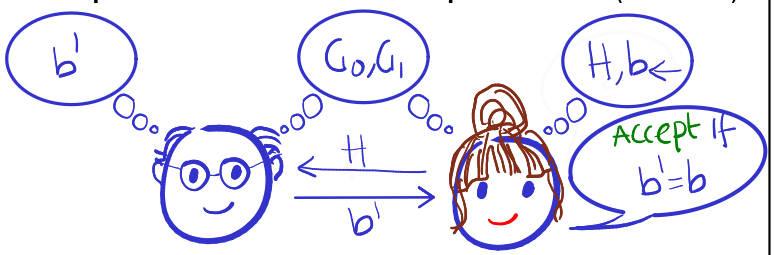

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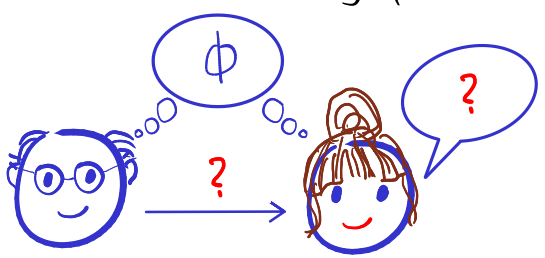

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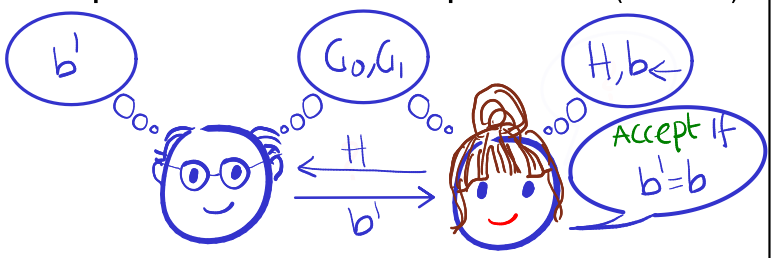

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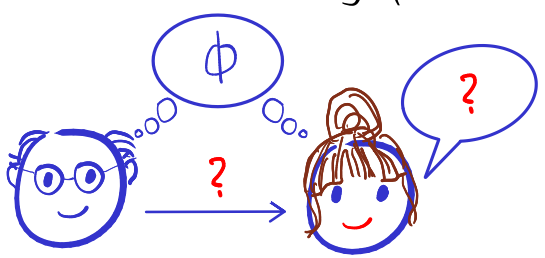

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"Sumcheck"

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Which Languages have IPs? PSPACE Languages

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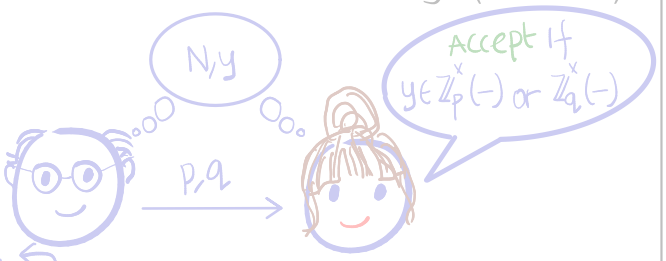

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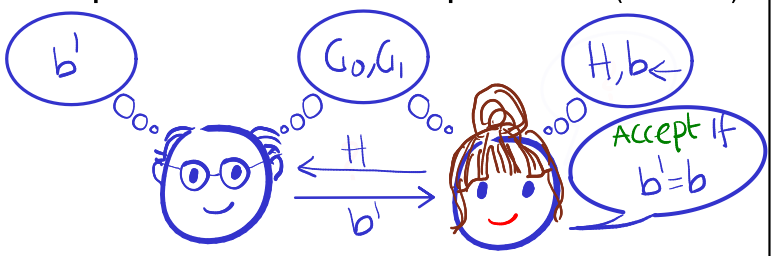

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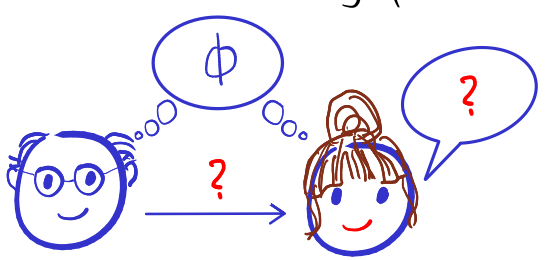

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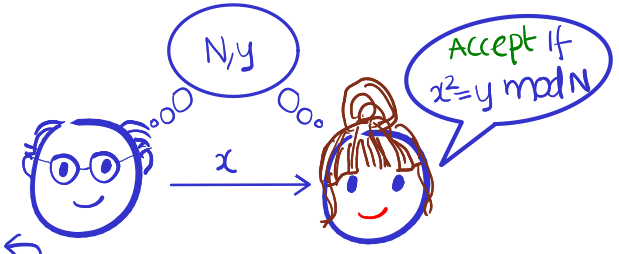
Plan for Today's Lecture

1. Interactive Proof (IP)
2. Zero Knowledge (Interactive) Proof (ZKP)
3. Honest Verifier ZKP for Graph (Non-) Isomorphism

Any Issues with the NP Proofs We Saw?

Quadratic residuosity (QR)



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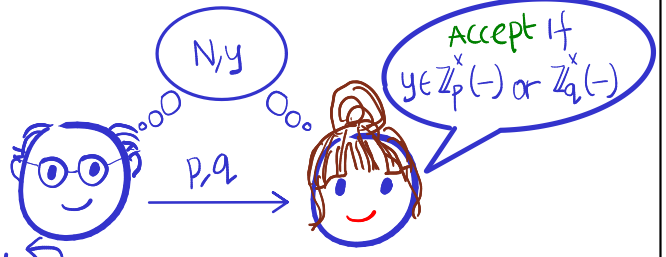

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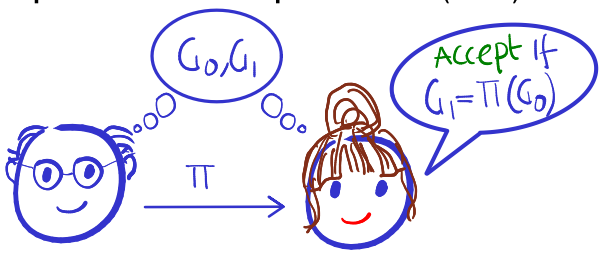

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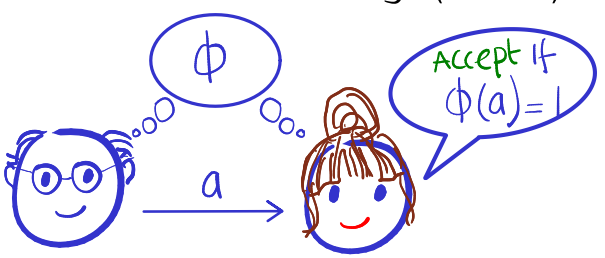

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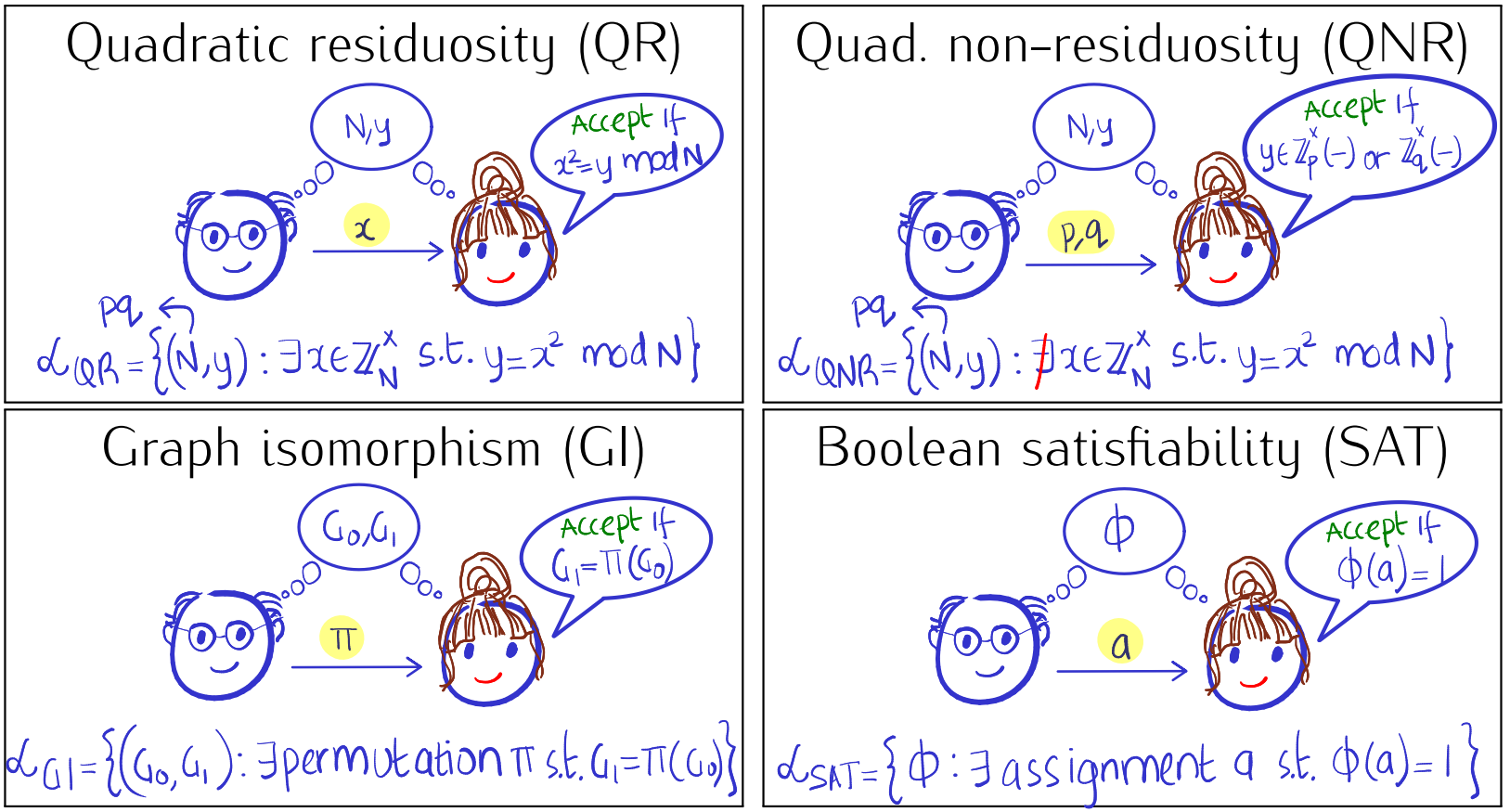
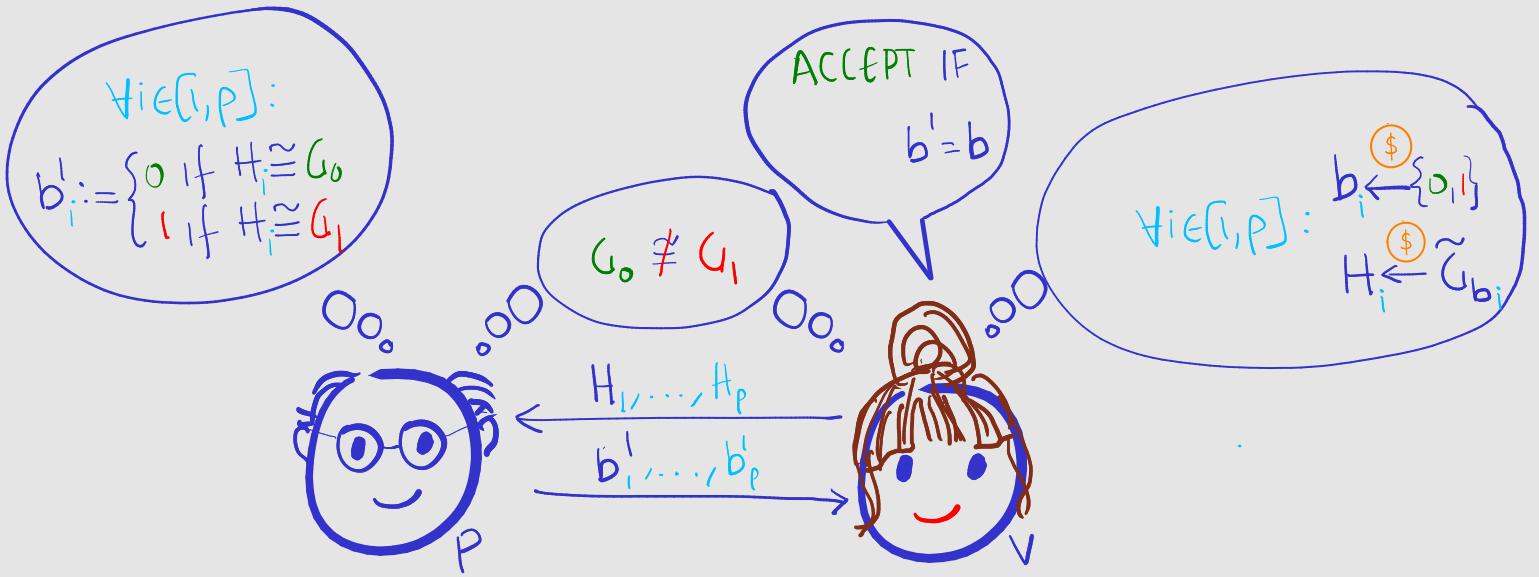


- Verifier gains "non-trivial knowledge" about witness w
  - Not desirable, e.g., when $x = pk$ and $w = sk$ (identification)

What About the IP We Saw?

Protocol 1 (Π_{GNI}: IP for GNI)



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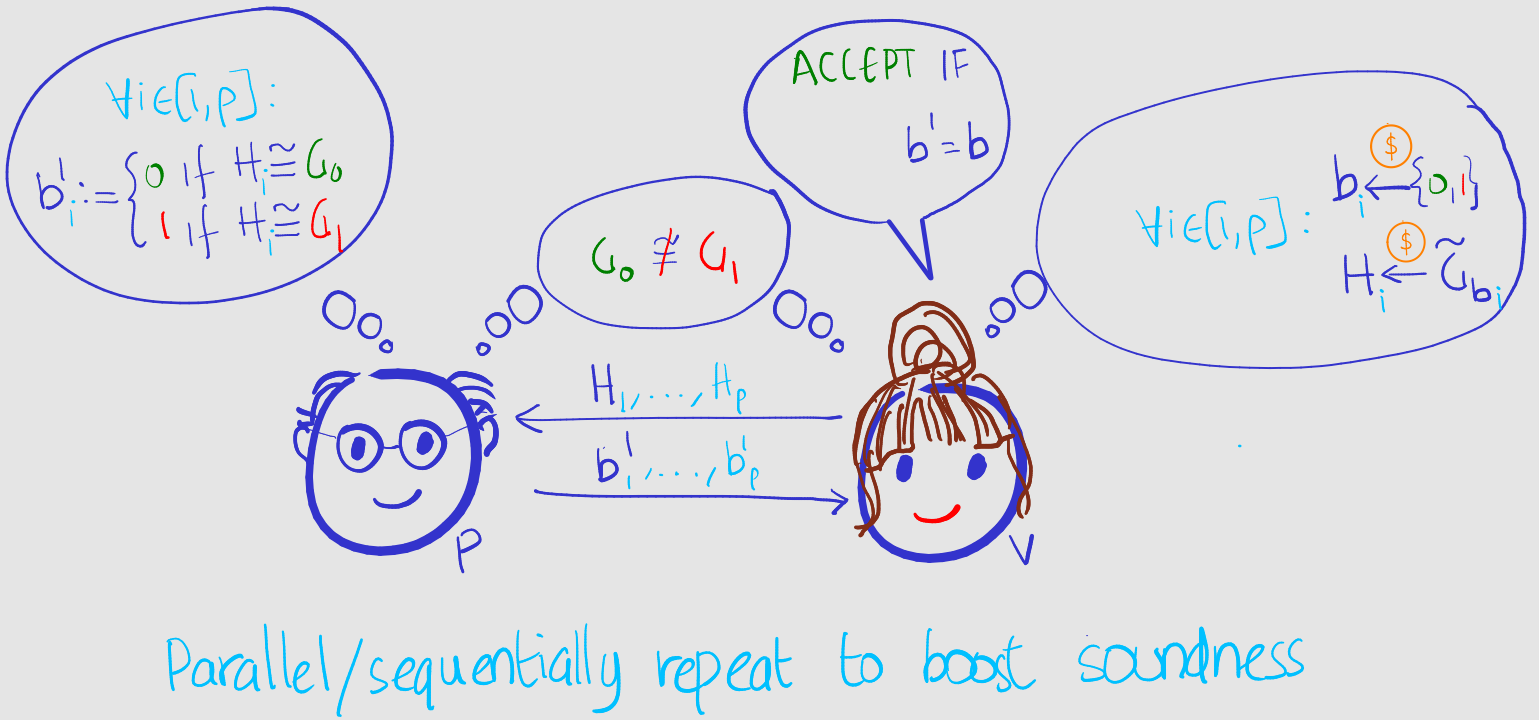

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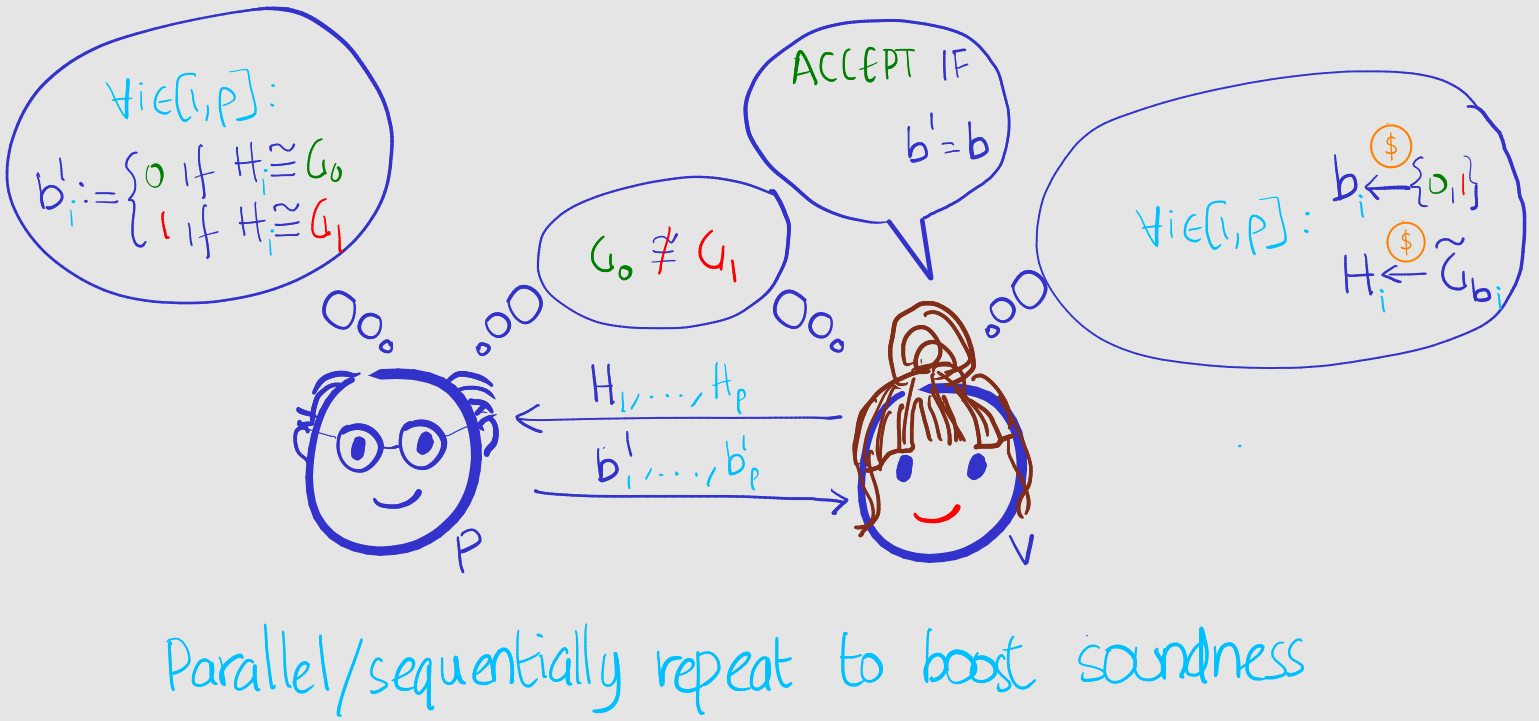


Parallel/sequentially repeat to boost soundness

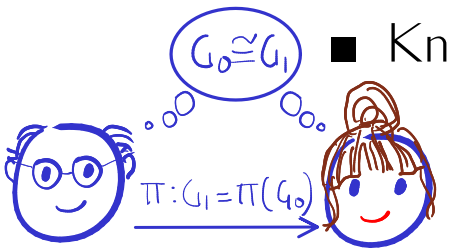
- Seems V *gains no knowledge* beyond validity of the statement
- We will see that Π_{GNI} is (honest-verifier) zero-knowledge!

How to Capture "V Gains No Knowledge"?

- Knowledge vs. information ← in the information-theoretic sense
  - Knowledge is computational

How to Capture "V Gains No Knowledge"?

- Knowledge vs. information ← in the information-theoretic sense
  - Knowledge is computational: e.g., consider NP proof for GI



  - Given (G_0, G_1), the isomorphism π contains no *information*
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How to Capture "V Gains No Knowledge"?

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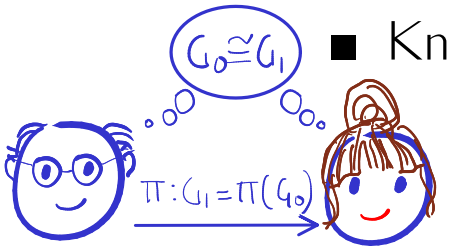

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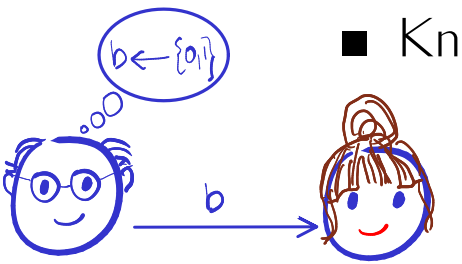

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    - Flipping a *private* fair coin b and (later) revealing its outcome leads to V gaining *information*
    - But V *does not gain knowledge*: she could herself have tossed the private coin and revealed it

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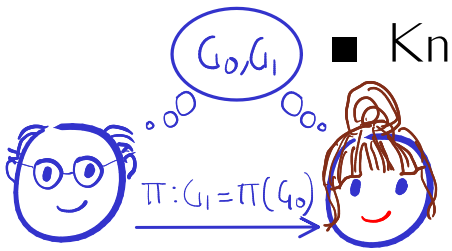

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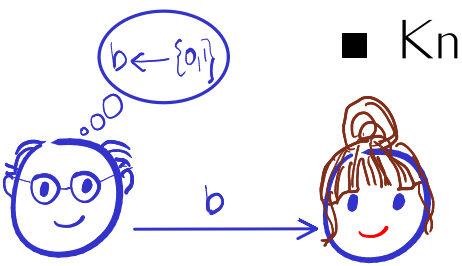

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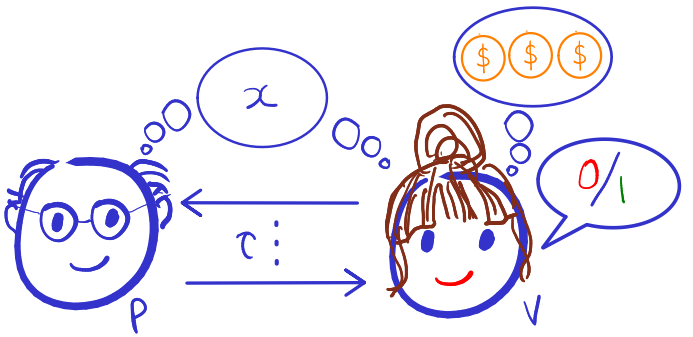
Intuitively, "V gains no knowledge" if anything V can *compute* (other than the validity of x) after the interaction, V could have computed *without it*

Defining Zero Knowledge via Simulators...

- Formalised via "simulation paradigm": $\mathsf{View}_{\mathsf{V}}(\langle \mathsf{P}, \mathsf{V} \rangle(x))$ can be *efficiently* simulated given only the instance



Defining Zero Knowledge via Simulators...

V's "view" = x + transcript τ + coins

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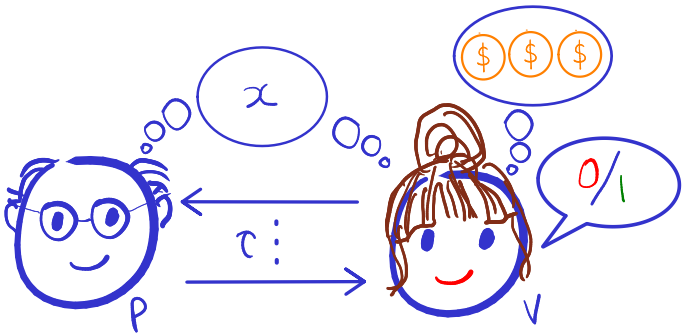

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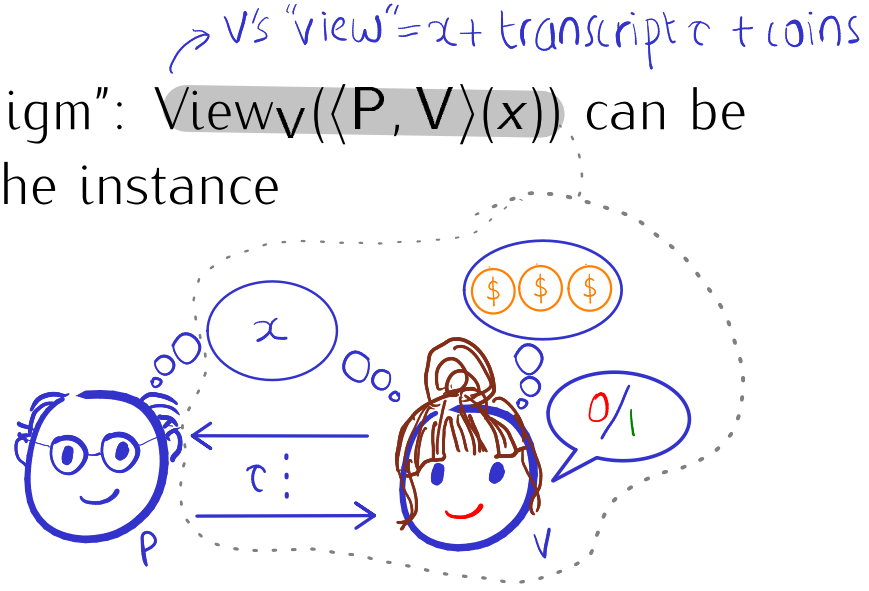

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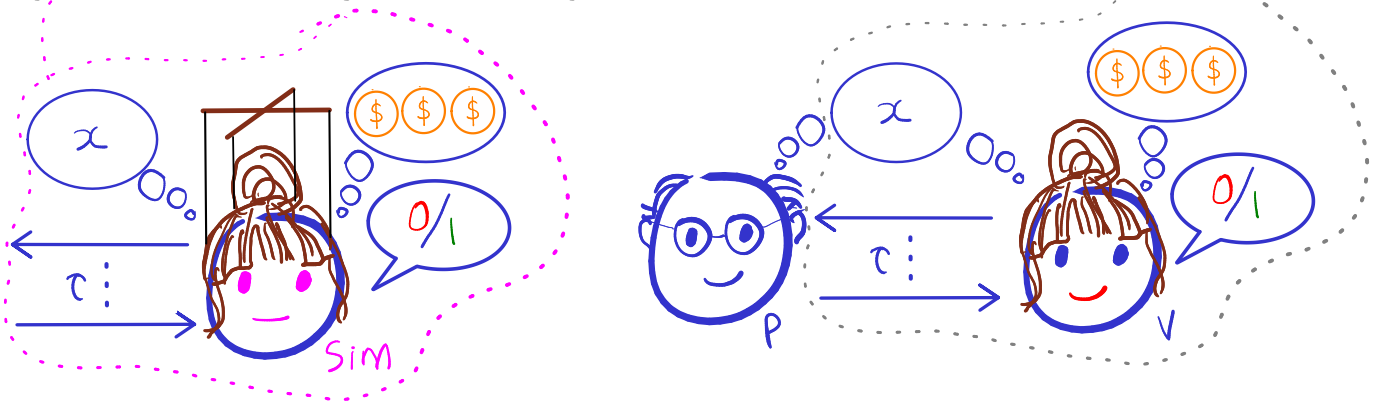

Defintion 2 (Honest-Verifier *Perfect* ZK)

An IP Π is honest-verifier perfect ZK if there exists a PPT simulator Sim *such that for all distinguishers* D *and all* $x \in \mathcal{L}$*, the following is zero*

$$\Pr[\mathsf{D}(\mathsf{View}_{\mathsf{V}}(\langle \mathsf{P}, \mathsf{V} \rangle(x))) = 1] - \Pr[\mathsf{D}(\mathsf{Sim}(x)) = 1]$$

Defining Zero Knowledge via Simulators...

Defintion 2 (~~Honest-Verifier~~ Malicious-verifier *Perfect* ZK)

An IP Π is honest-verifier perfect ZK if ($\forall V^*$) there exists a PPT simulator Sim *such that for all distinguishers* D *and all* $x \in \mathcal{L}$*, the following is zero*

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- Malicious-Verifier ZK: honest verifier $\mathsf{V} \rightarrow$ all verifiers V^*
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Defining Zero Knowledge via Simulators...

Defintion 2 (Honest-Verifier ~~Perfect~~ Computational ZK)

An IP Π is honest-verifier ~~perfect~~ computational ZK if there exists a PPT simulator Sim *such that for ~~all~~ PPT distinguishers* D *and all* $x \in \mathcal{L}$*, the following is ~~zero~~ negligible*

$$\left|\Pr[\mathsf{D}(\mathrm{View}_{\mathsf{V}}(\langle \mathsf{P}, \mathsf{V}\rangle(x))) = 1] - \Pr[\mathsf{D}(\mathsf{Sim}(x)) = 1]\right|$$

- Malicious-Verifier ZK: honest verifier $\mathsf{V} \to$ all verifiers V^*
  - For every V^* there exists a PPT simulator Sim
- Computational ZK: relax
  - all distinguishers $\mathsf{D} \to$ PPT distinguishers
  - zero $\to$ negligible

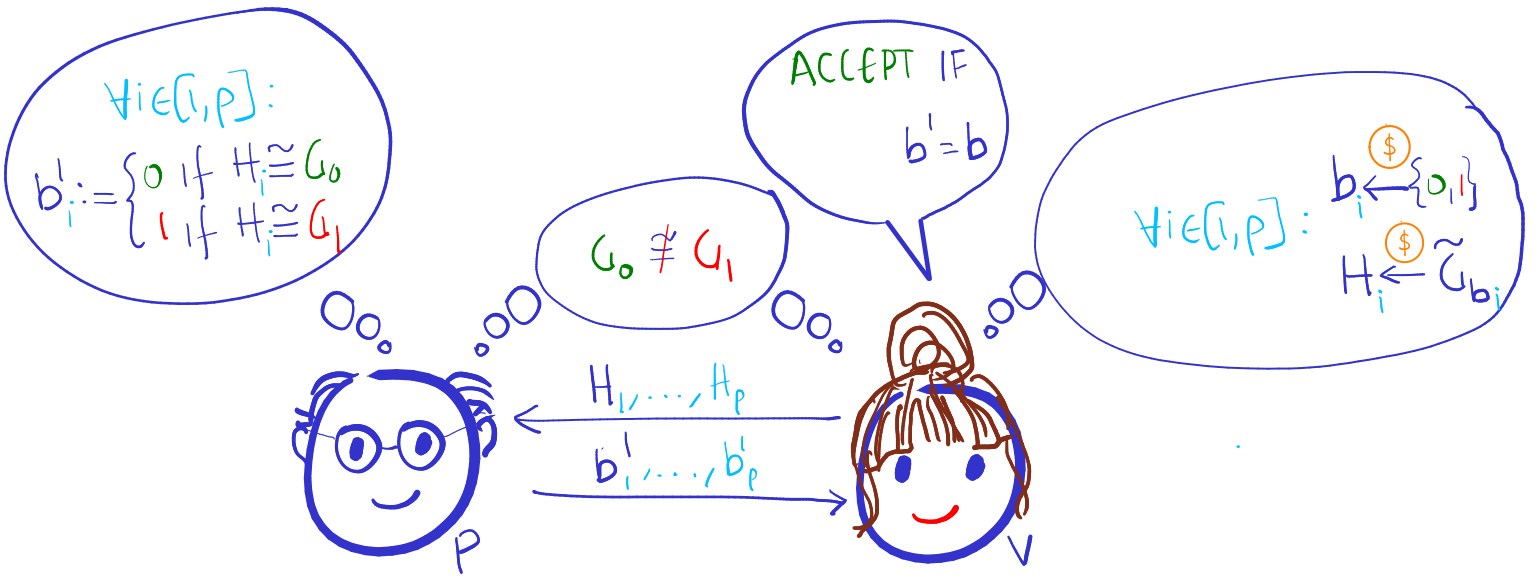
Plan for Today's Lecture

1. Interactive Proof (IP)
2. Zero Knowledge (Interactive) Proof (ZKP)
3. Honest Verifier ZKP for Graph (Non-) Isomorphism

Π_{GNI} is *Honest-Verifier* ZK

Theorem 2

Π_{GNI} is honest-verifier perfect zero-knowledge IP for $\mathcal{L}_{GNI}$



Parallel/sequentially repeat to boost soundness

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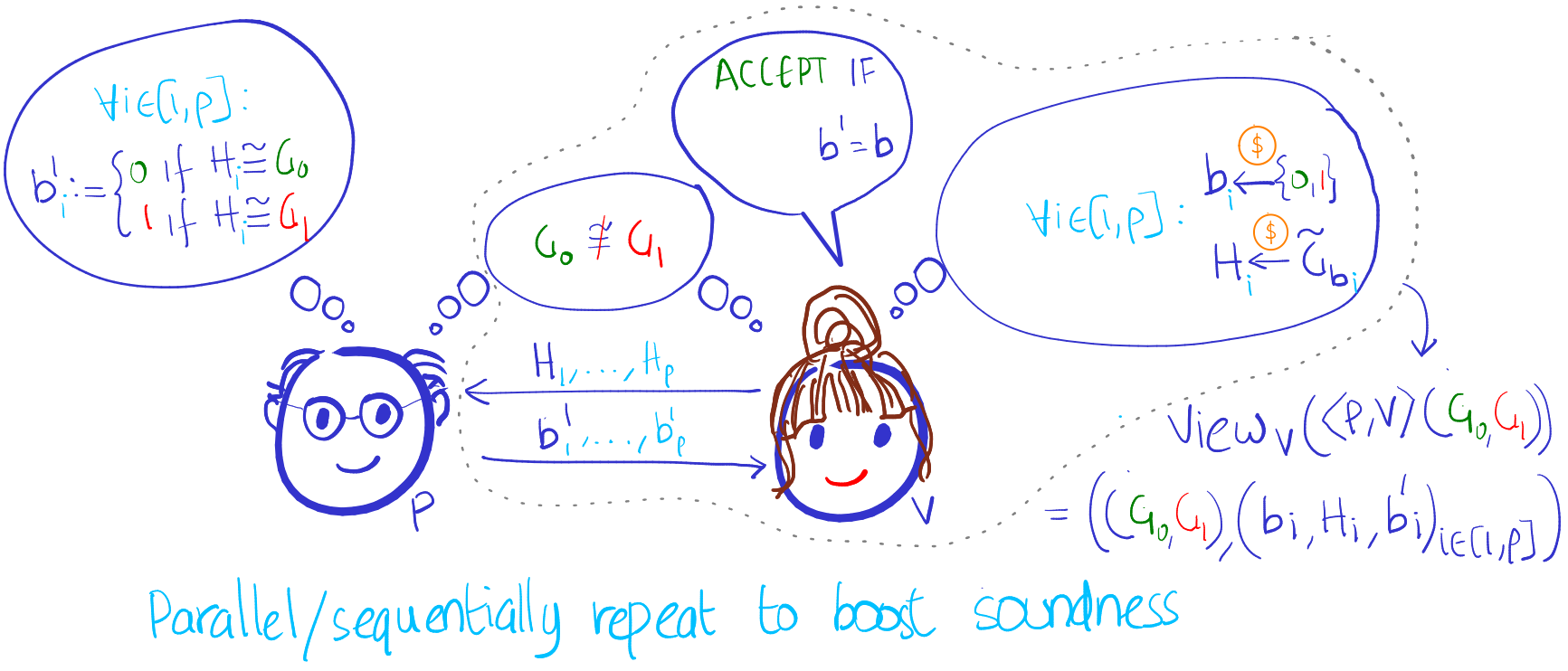

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Proof.

$$\text{view}_V(\langle P, V\rangle(G_0, G_1)) := \left((G_0, G_1), \left(b_i, H_i, b_i'\right)_{i\in[1,\rho]}\right)$$

$\forall\, G_0 \not\cong G_1$:

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Exercise 3

1. *What happens if* V *is malicious and can deviate from protocol?*
2. *Using ideas from* Π_{GNI}*, build honest-verifier ZKP for* $\mathcal{L}_{QNR}$

Honest-Verifier ZKP for GI...

- Idea for ZK:
  1. $G_0 \overset{\pi}{\simeq} G_1 \Rightarrow$ if $G_1 \overset{\sigma}{\simeq} H$ then $G_0 \overset{\sigma\cdot\pi}{\simeq} H$
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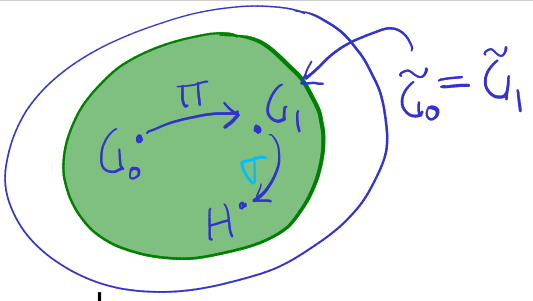

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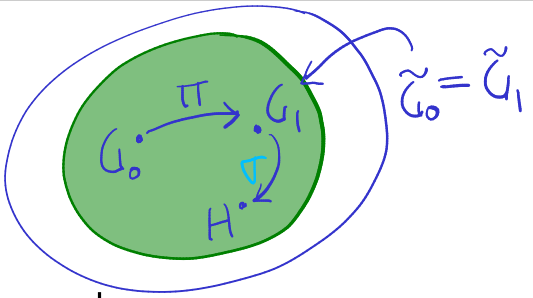

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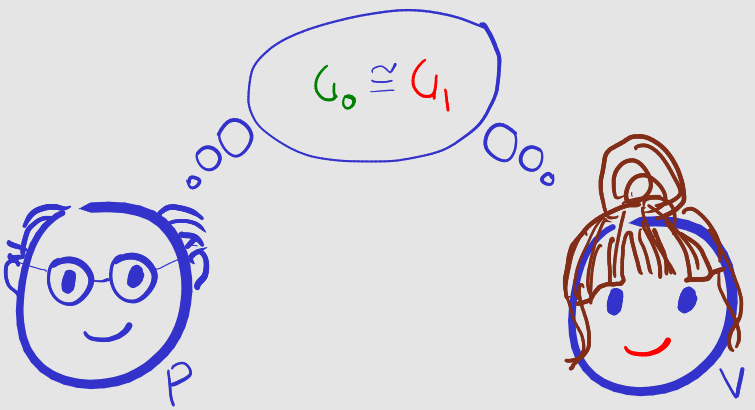
Protocol 2 (Π_{GI}: IP for GI)

$G_0 \simeq G_1$ if $G_1 = H(G_0)$



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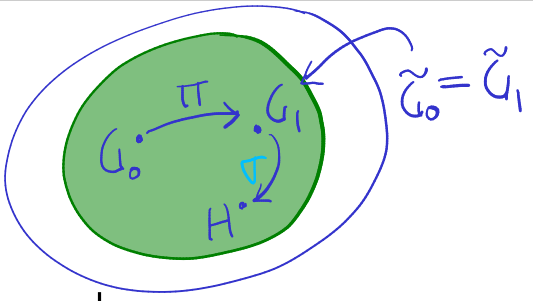

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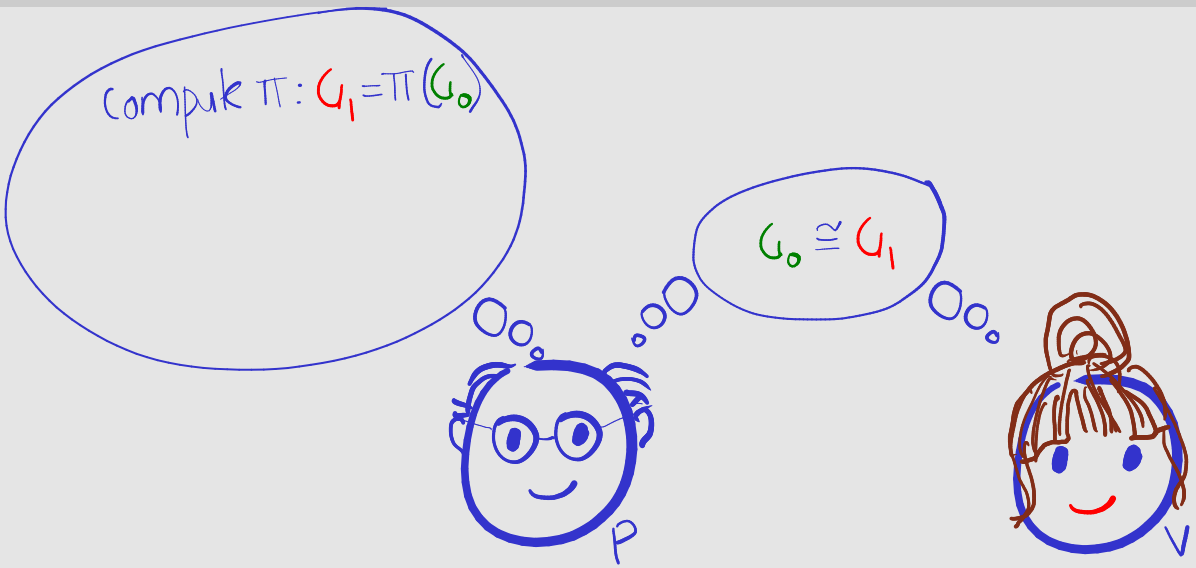

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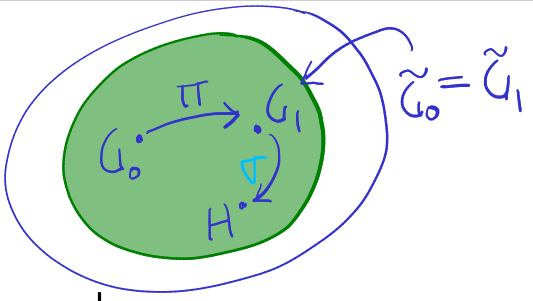

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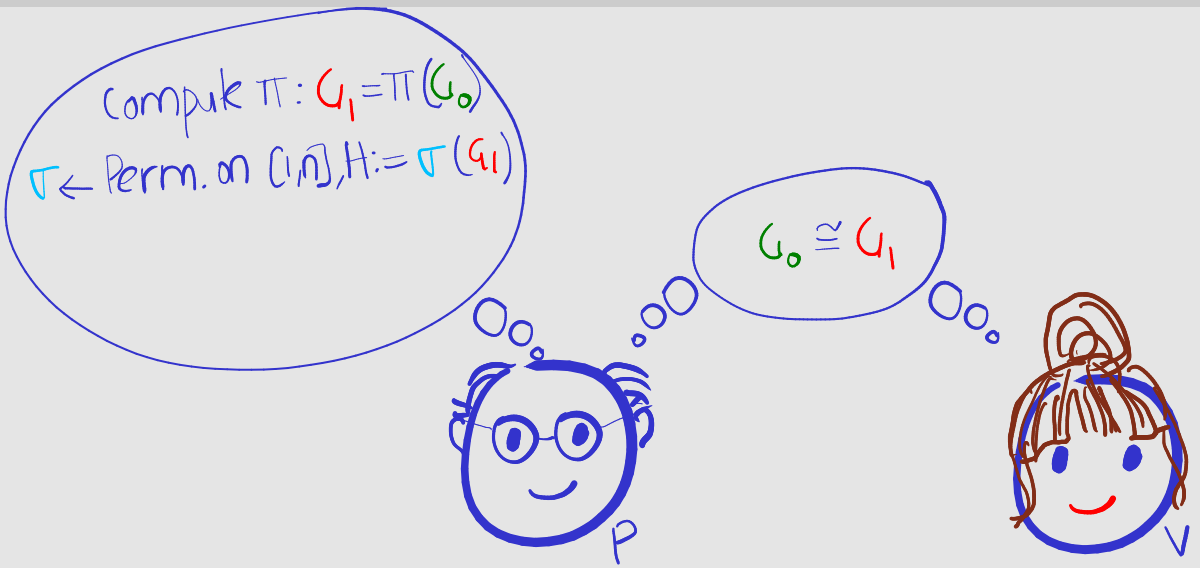

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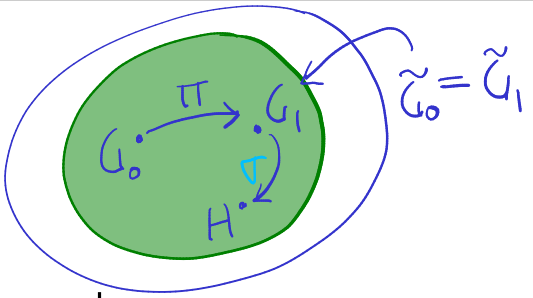

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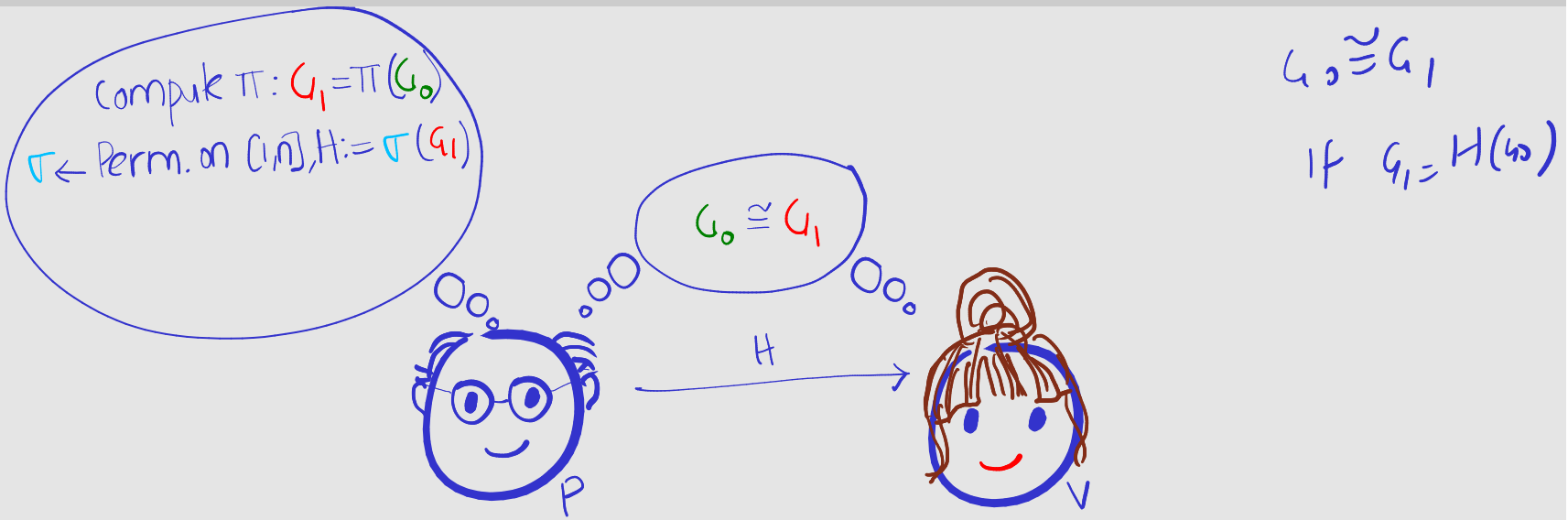

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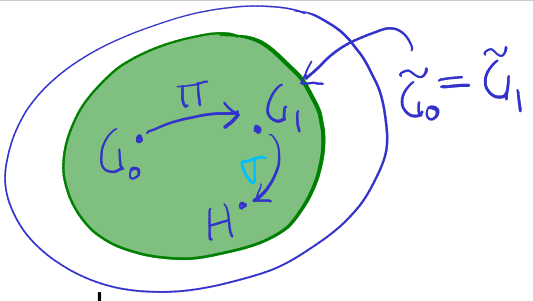

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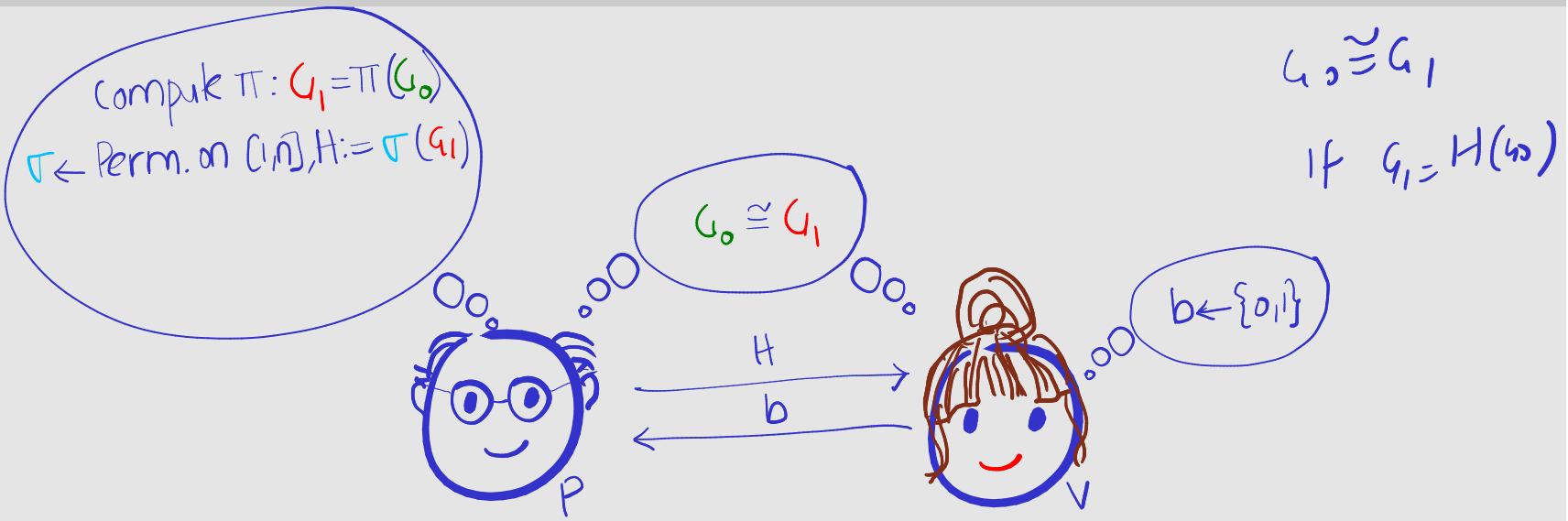

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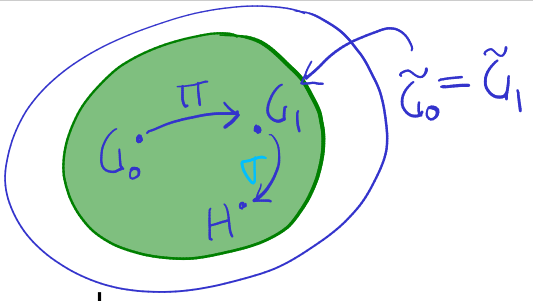

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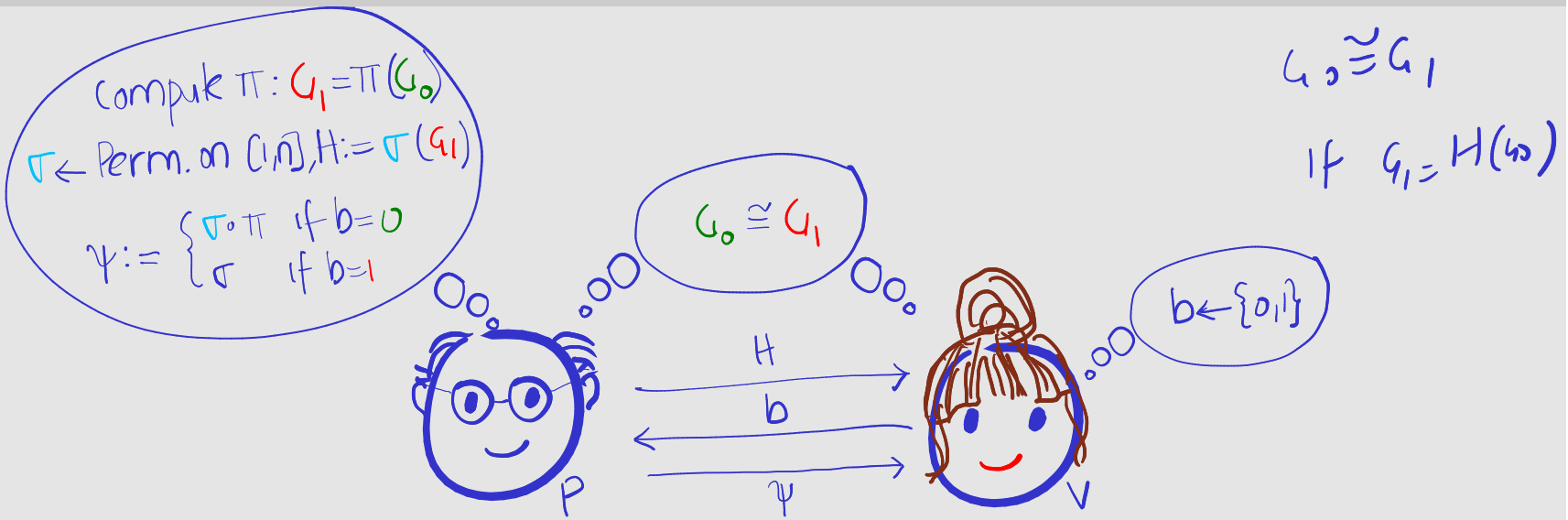

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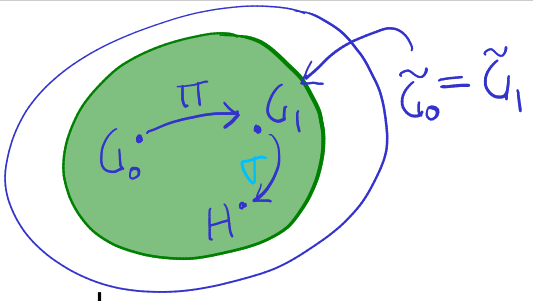

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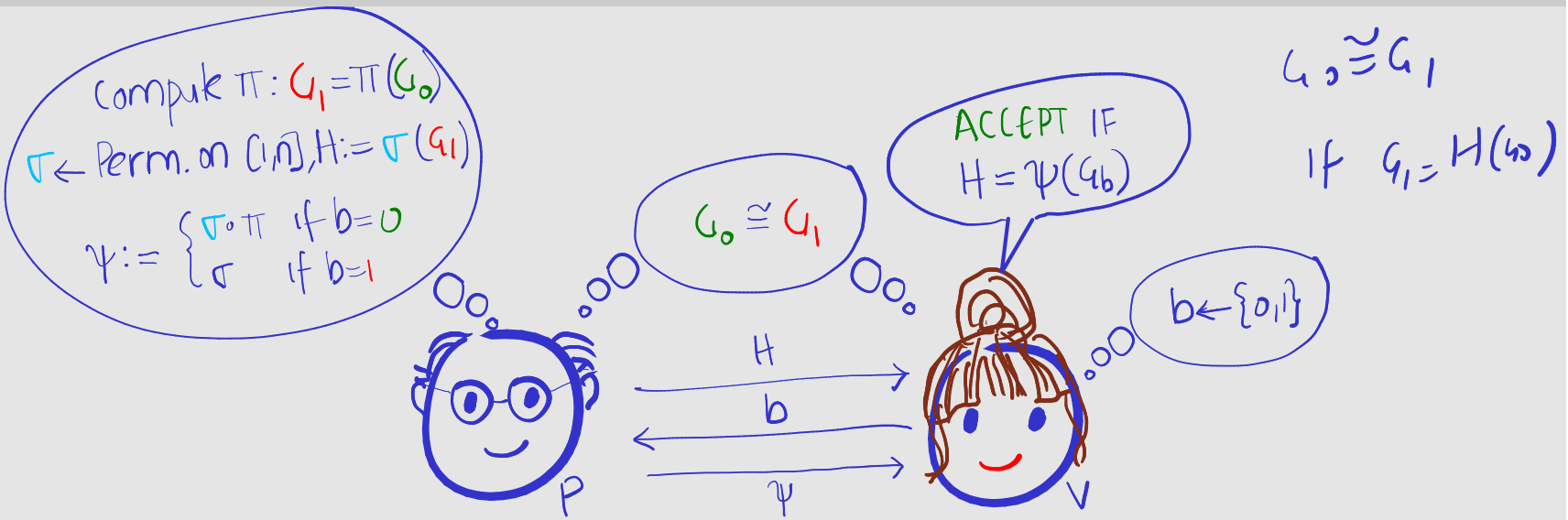

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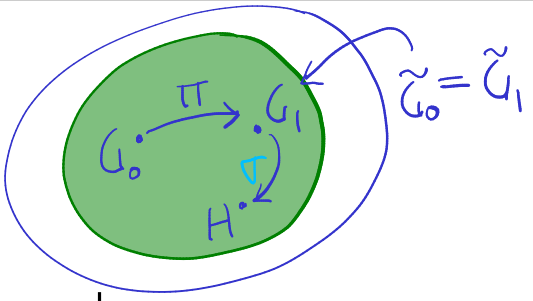

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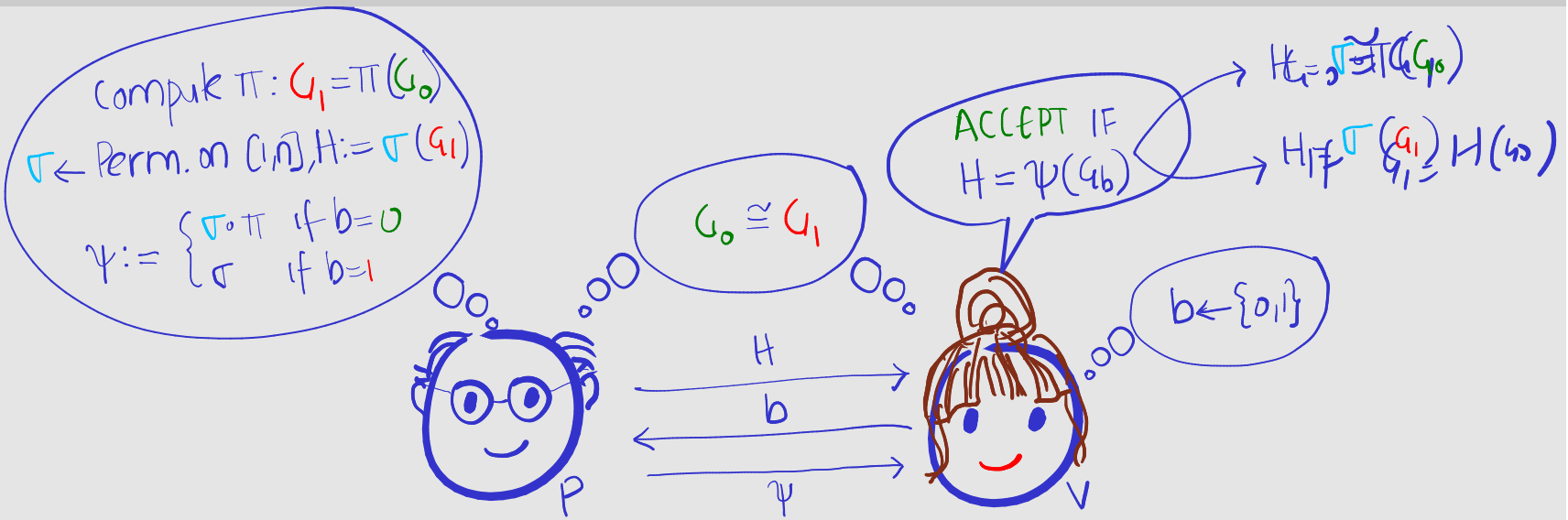

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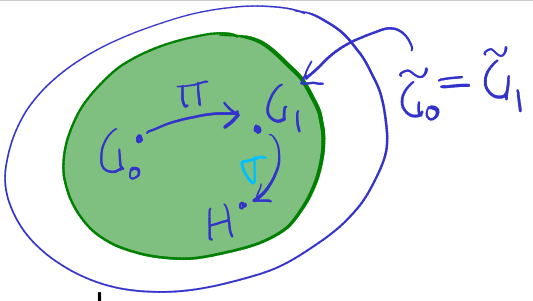

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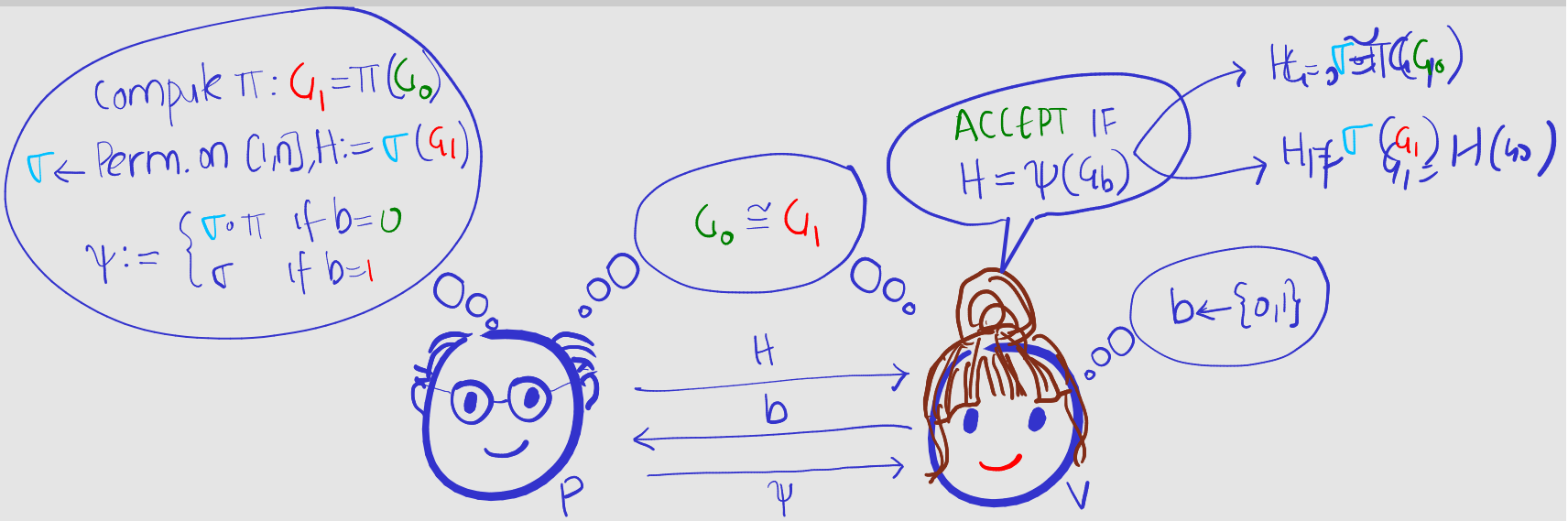

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Honest-Verifier ZKP for GI...

Theorem 3

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Honest-Verifier ZKP for GI...

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Π_{GI} *is honest-verifier perfect zero-knowledge IP for* $\mathcal{L}_{GI}$

Proof.

- Completeness: $G_0 \cong G_1 \Rightarrow$ P can answer both challenges $\Rightarrow$ V always accepts
- Soundness: $G_0 \ncong G_1 \Rightarrow$ for *any* H P^* commits to, $G_0 \cong H$ *and* $G_1 \cong H$ *cannot both hold* $\Rightarrow$ best P^* can do is guess b

Honest-Verifier ZKP for GI...

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- Soundness: $G_0 \ncong G_1 \Rightarrow$ for *any* H P* commits to, $G_0 \cong H$ *and* $G_1 \cong H$ *cannot both hold* $\Rightarrow$ best P* can do is guess b
- Zero knowledge:

$$\text{view}_V(\langle P, V\rangle(G_0, G_1)) := \big((G_0, G_1), (H, b, \psi)\big)$$

$$\forall\, G_0 \overset{\pi}{\cong} G_1 :$$

Honest-Verifier ZKP for GI...

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$\forall\ G_0 \cong G_1$: $\text{view}_V(\langle P,V\rangle(G_0,G_1))$ identically distributed to $\text{Sim}(G_0,G_1)$. □

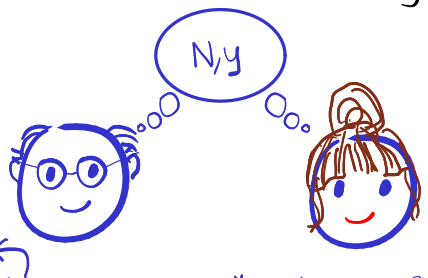
Honest-Verifier ZKP for GI...

Exercise 4

1. *What happens if* V *is malicious and can deviate from protocol?*
2. *Using ideas from* Π_{GI}*, build honest-verifier ZKP for* $\mathcal{L}_{QR}$

Which Languages have ZKPs?

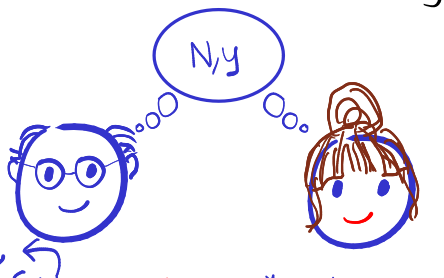
Quadratic residuosity (QR)



$\mathcal{L}_{QR} = \{(N,y) : \exists x \in \mathbb{Z}_N^* \text{ s.t. } y = x^2 \bmod N\}$

✓

Quad. non-residuosity (QNR)



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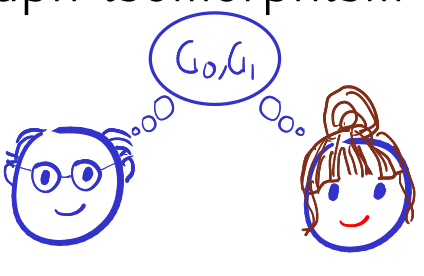
✓

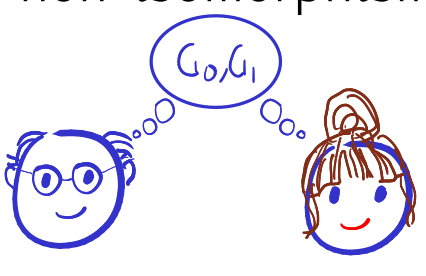
Graph isomorphism (GI)



$\mathcal{L}_{GI} = \{(G_0, G_1) : \exists \text{ permutation } \pi \text{ s.t. } G_1 = \pi(G_0)\}$

✓

Graph non-isomorphism (GNI)



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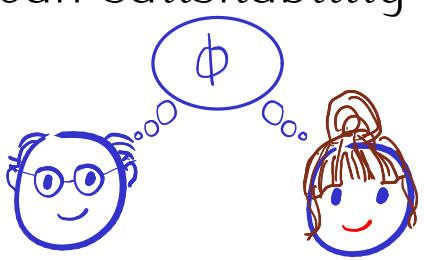
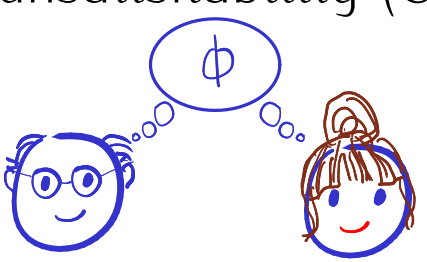
Boolean satisfiability (SAT)



$\mathcal{L}_{SAT} = \{\phi : \exists \text{ assignment } a \text{ s.t. } \phi(a) = 1\}$

Bool. unsatisfiability (UNSAT)



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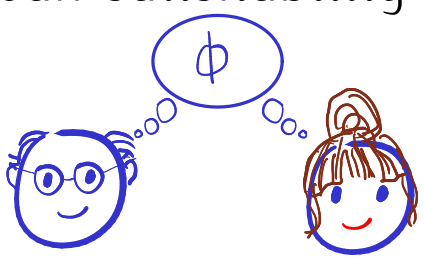

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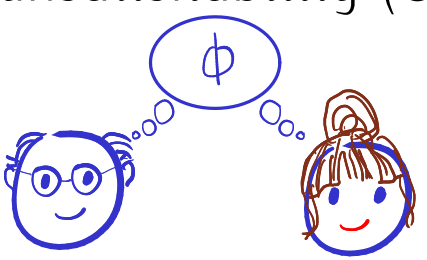

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L15

Which Languages have Computational ZKPs? PSPACE Languages

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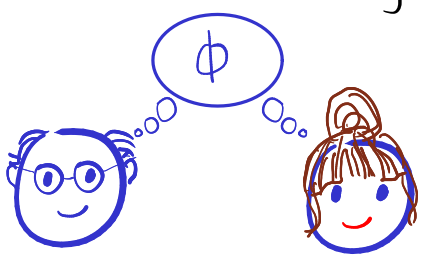
L15



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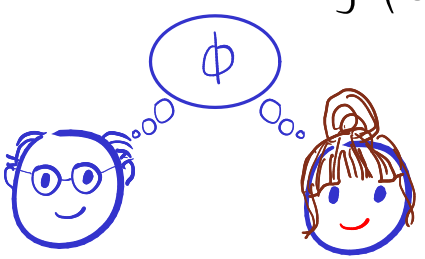

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Are Randomness and Interaction Necessary?

$\leftrightarrows$ Interaction is necessary

Exercise 5

If $\mathcal{L}$ has a non-interactive (i.e, one-message) ZKP then $\mathcal{L} \in$ BPP

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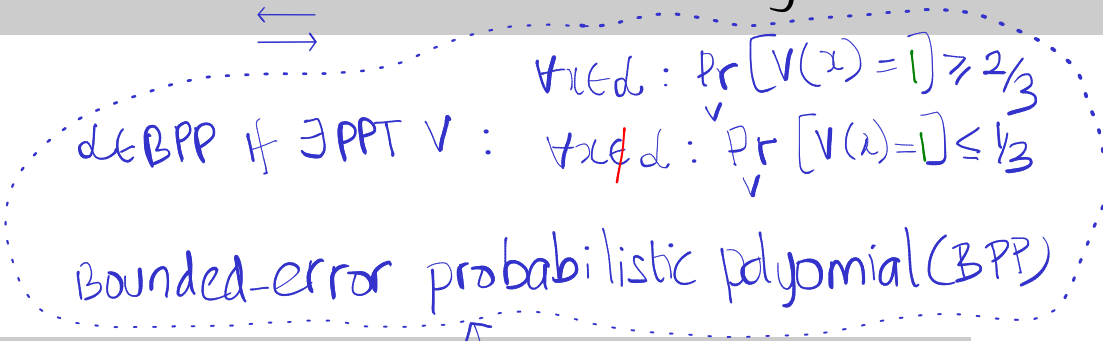

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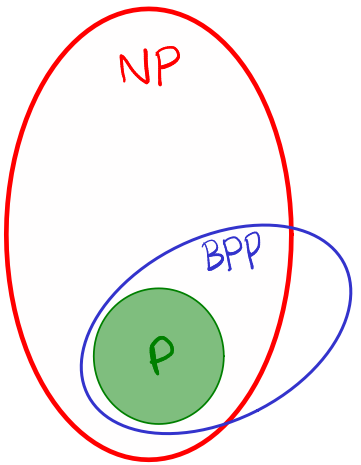

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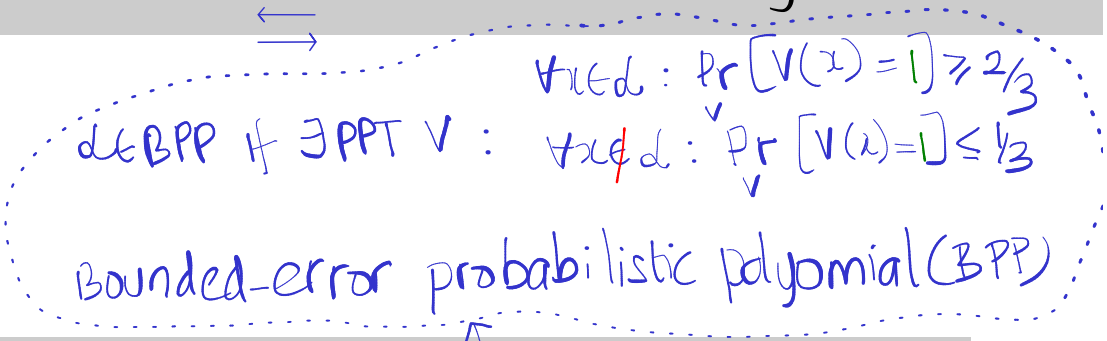

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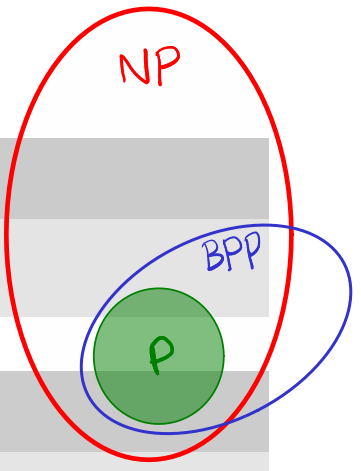
Randomness is necessary

Exercise 6

If $\mathcal{L}$ has an IP with deterministic verifier then $\mathcal{L} \in$ NP



Exercise 7

If $\mathcal{L}$ has an ZKP with deterministic verifier then $\mathcal{L} \in$ BPP

To Recap Today's Lecture

- Traditional "NP" proofs vs *interactive* proofs
  - IP is more powerful: IP for GNI

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  - Modelled "zero knowledge" via simulation paradigm
- *Honest-verifier* ZKP for GNI (HW: QNR) and GI (HW: QR)

Next Lecture

- 27/Sep: Crib session for mid-term exam
- 01/Oct: More ZKP:
  - Malicious-verifier ZKP
  - Computational ZKP for all of NP!
    - New cryptographic primitive: commitment schemes

References

1. [Gol01, Chapter 4] for details of today's lecture
2. [GMR89] for definitional and philosophical discussion on ZK
3. The ZKPs for GI and GNI are taken from [GMR89, GMW91]
4. IP for all of PSPACE is due to [LFKN92, Sha90].
   Computational ZKP for all of PSPACE is due to [GMW91].


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